Dirac Matrices and Lorentz Spinors

Background: In 3D, the spinor $j = \frac{1}{2}$ representation of the Spin(3) rotation group is constructed from the Pauli matrices σ^x, σ^y, and σ^z, which obey both commutation and anticommutation relations

$$[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k \quad \text{and} \quad \{\sigma^i, \sigma^j\} = 2\delta^{ij} \times \mathbf{1}_{2\times 2}. \tag{1}$$

Consequently, the spin matrices

$$\mathbf{S} = -\tfrac{i}{2}\boldsymbol{\sigma} \times \boldsymbol{\sigma} = \tfrac{1}{2}\boldsymbol{\sigma} \tag{2}$$

commute with each other like angular momenta, $[S^i, S^j] = i\epsilon^{ijk}S^k$, so they represent the generators of the rotation group. In this spinor representation, the finite rotations $R(\phi, \mathbf{n})$ are represented by

$$M(R) = \exp\bigl(-i\phi\mathbf{n} \cdot \mathbf{S}\bigr), \tag{3}$$

while the spin matrices themselves transform into each other as components of a 3–vector,

$$M^{-1}(R)S^i M(R) = R^{ij}S^j. \tag{4}$$

In this note, I shall generalize this construction to the *Dirac spinor* representation of the Lorentz symmetry Spin(3,1).

The Dirac Matrices γ^μ generalize the anti-commutation properties of the Pauli matrices σ^i to the $3+1$ Minkowski dimensions:

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \times \mathbf{1}_{4\times 4}. \tag{5}$$

The γ^μ are 4×4 matrices, but there are several different conventions for their specific form. In my class I shall follow the same convention as the Peskin & Schroeder textbook, namely the Weyl convention where in 2×2 block notations

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1}_{2\times 2} \\ \mathbf{1}_{2\times 2} & 0 \end{pmatrix}, \qquad \vec{\gamma} = \begin{pmatrix} 0 & +\vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}. \tag{6}$$

Note that the γ^0 matrix is hermitian while the γ^1, γ^2, and γ^3 matrices are anti-hermitian. Apart from that, the specific forms of the matrices are not important, the Physics follows from the anti-commutation relations (5).

The Lorentz spin matrices generalize $\mathbf{S} = -\frac{i}{2}\boldsymbol{\sigma} \times \boldsymbol{\sigma}$ rather than $\mathbf{S} = \frac{1}{2}\boldsymbol{\sigma}$. In 4D, the vector product becomes the antisymmetric tensor product, so we define

$$S^{\mu\nu} = -S^{\nu\mu} \stackrel{\text{def}}{=} \tfrac{i}{4}[\gamma^\mu, \gamma^\nu]. \tag{7}$$

Thanks to the anti-commutation relations (5) for the γ^μ matrices, the $S^{\mu\nu}$ obey the commutation relations of the Lorentz generators $\hat{J}^{\mu\nu} = -\hat{J}^{\nu\mu}$. Moreover, the commutation relations of the spin matrices $S^{\mu\nu}$ with the Dirac matrices γ^μ are similar to the commutation relations of the $\hat{J}^{\mu\nu}$ with a Lorentz vector such as $\hat{P}^\mu$.

Lemma:

$$[\gamma^\lambda, S^{\mu\nu}] = ig^{\lambda\mu}\gamma^\nu - ig^{\lambda\nu}\gamma^\mu. \tag{8}$$

Proof: Combining the definition (7) of the spin matrices as commutators with the anti-commutation relations (5), we have

$$\gamma^\mu\gamma^\nu = \tfrac{1}{2}\{\gamma^\mu, \gamma^\nu\} + \tfrac{1}{2}[\gamma^\mu, \gamma^\nu] = g^{\mu\nu} \times \mathbf{1}_{4\times 4} - 2iS^{\mu\nu}. \tag{9}$$

Since the unit matrix commutes with everything, we have

$$[X, S^{\mu\nu}] = \tfrac{i}{2}[X, \gamma^\mu\gamma^\nu] \qquad \text{for any matrix } X, \tag{10}$$

and the commutator on the RHS may often be obtained from the Leibniz rules for the commutators or anticommutators:

$$\begin{aligned} [A, BC] &= [A, B]C + B[A, C] = \{A, B\}C - B\{A, C\}, \\ \{A, BC\} &= [A, B]C + B\{A, C\} = \{A, B\}C - B[A, C]. \end{aligned} \tag{11}$$

In particular,

$$[\gamma^\lambda,\gamma^\mu\gamma^\nu] \;=\; \{\gamma^\lambda,\gamma^\mu\}\gamma^\nu \;-\; \gamma^\mu\{\gamma^\lambda,\gamma^\nu\} \;=\; 2g^{\lambda\mu}\gamma^\nu \;-\; 2g^{\lambda\nu}\gamma^\mu \tag{12}$$

and hence

$$[\gamma^\lambda,S^{\mu\nu}] \;=\; \tfrac{i}{2}[\gamma^\lambda,\gamma^\mu\gamma^\nu] \;=\; ig^{\lambda\mu}\gamma^\nu \;-\; ig^{\lambda\nu}\gamma^\mu. \tag{13}$$

Quod erat demonstrandum.

Theorem: The $S^{\mu\nu}$ matrices commute with each other like Lorentz generators,

$$\left[S^{\kappa\lambda},S^{\mu\nu}\right] \;=\; ig^{\lambda\mu}S^{\kappa\nu} \;-\; ig^{\lambda\nu}S^{\kappa\mu} \;-\; ig^{\kappa\mu}S^{\lambda\nu} \;+\; ig^{\kappa\nu}S^{\lambda\mu}. \tag{14}$$

Proof: Again, we use the Leibniz rule and eq. (9):

$$\begin{aligned}
\left[\gamma^\kappa\gamma^\lambda,S^{\mu\nu}\right] \;&=\; \gamma^\kappa\left[\gamma^\lambda,S^{\mu\nu}\right] \;+\; \left[\gamma^\kappa,S^{\mu\nu}\right]\gamma^\lambda\\
&=\; \gamma^\kappa\big(ig^{\lambda\mu}\gamma^\nu \;-\; ig^{\lambda\nu}\gamma^\mu\big) \;+\; \big(ig^{\kappa\mu}\gamma^\nu \;-\; ig^{\kappa\nu}\gamma^\mu\big)\gamma^\lambda\\
&=\; ig^{\lambda\mu}\big(\gamma^\kappa\gamma^\nu \;=\; g^{\kappa\nu}-2iS^{\kappa\nu}\big) \;-\; ig^{\lambda\nu}\big(\gamma^\kappa\gamma^\mu \;=\; g^{\kappa\mu}-2iS^{\kappa\mu}\big)\\
&\qquad +\; ig^{\kappa\mu}\big(\gamma^\nu\gamma^\lambda \;=\; g^{\lambda\nu}+2iS^{\lambda\nu}\big) \;-\; ig^{\kappa\nu}\big(\gamma^\mu\gamma^\lambda \;=\; g^{\lambda\mu}+2iS^{\lambda\mu}\big)\\
&=\; 2g^{\lambda\mu}S^{\kappa\nu} \;-\; 2g^{\lambda\nu}S^{\kappa\mu} \;-\; 2g^{\kappa\mu}S^{\lambda\nu} \;+\; 2g^{\kappa\nu}S^{\lambda\mu}
\end{aligned} \tag{15}$$

since all the $\pm ig^{\cdot\cdot}g^{\cdot\cdot}$ cancel each other, hence

$$\left[S^{\kappa\lambda},S^{\mu\nu}\right] \;=\; \tfrac{i}{2}\left[\gamma^\kappa\gamma^\lambda,S^{\mu\nu}\right] \;=\; ig^{\lambda\mu}S^{\kappa\nu} \;-\; ig^{\lambda\nu}S^{\kappa\mu} \;-\; ig^{\kappa\mu}S^{\lambda\nu} \;+\; ig^{\kappa\nu}S^{\lambda\mu}. \tag{16}$$

Quod erat demonstrandum.

In light of this theorem, the $S^{\mu\nu}$ matrices *represent* the Lorentz generators $\hat{J}^{\mu\nu}$ in the 4-component spinor multiplet.

Finite Lorentz transforms:

Any continuous Lorentz transform — a rotation, or a boost, or a product of a boost and a rotation — obtains from exponentiating an infinitesimal symmetry

$$X'^{\mu} \;=\; X^{\mu} \;+\; \epsilon^{\mu\nu}X_{\nu} \tag{17}$$

where the infinitesimal $\epsilon^{\mu\nu}$ matrix is antisymmetric when both indices are raised (or both lowered), $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$. Thus, the $L^{\mu}{}_{\nu}$ matrix of any continuous Lorentz transform is a matrix exponential

$$L^{\mu}{}_{\nu} \;=\; \exp(\Theta)^{\mu}{}_{\nu} \;\equiv\; \delta^{\mu}_{\nu} \;+\; \Theta^{\mu}{}_{\nu} \;+\; \tfrac{1}{2}\Theta^{\mu}{}_{\lambda}\Theta^{\lambda}{}_{\nu} \;+\; \tfrac{1}{6}\Theta^{\mu}{}_{\lambda}\Theta^{\lambda}{}_{\kappa}\Theta^{\kappa}{}_{\nu} \;+\; \cdots \tag{18}$$

of some matrix Θ that becomes antisymmetric when both of its indices are raised or lowered, $\Theta^{\mu\nu} = -\Theta^{\nu\mu}$. Note however that in the matrix exponential (18), the first index of Θ is raised while the second index is lowered, so the antisymmetry condition becomes $(g\Theta)^{\top} = -(g\Theta)$ instead of $\Theta^{\top} = -\Theta$.

The Dirac spinor representation of the finite Lorentz transform (18) is the 4×4 matrix

$$M_D(L) \;=\; \exp\bigl(-\tfrac{i}{2}\,\Theta_{\alpha\beta}S^{\alpha\beta}\bigr). \tag{19}$$

The group law for such matrices

$$\forall L_1, L_2 \in \mathrm{SO}^{+}(3,1), \quad M_D(L_2L_1) \;=\; M_D(L_2)M_D(L_1) \tag{20}$$

follows automatically from the $S^{\mu\nu}$ satisfying the commutation relations (14) of the Lorentz generators, so I am not going to prove it. Instead, let me show that when the Dirac matrices γ^{μ} are sandwiched between the $M_D(L)$ and its inverse, they transform into each other as components of a Lorentz 4–vector,

$$M_D^{-1}(L)\gamma^{\mu}M_D(L) \;=\; L^{\mu}{}_{\nu}\gamma^{\nu}. \tag{21}$$

This formula makes the Dirac equation transform covariantly under the Lorentz transforms.

Proof: In light of the exponential form (19) of the matrix $M_D(L)$ representing a finite Lorentz transform in the Dirac spinor multiplet, let's use the multiple commutator formula (AKA the Hadamard Lemma): for any 2 matrices F and H,

$$\exp(-F)H\exp(+F) \;=\; H \;+\; \big[H,F\big] \;+\; \tfrac{1}{2}\big[\big[H,F\big],F\big] \;+\; \tfrac{1}{6}\big[\big[\big[H,F\big],F\big],F\big] \;+\; \cdots. \tag{22}$$

In particular, let $H=\gamma^\mu$ while $F=-\frac{i}{2}\,\Theta_{\alpha\beta}S^{\alpha\beta}$ so that $M_D(L)=\exp(+F)$ and $M_D^{-1}(L)=\exp(-F)$. Consequently,

$$M_D^{-1}(L)\gamma^\mu M_D(L) \;=\; \gamma^\mu \;+\; \big[\gamma^\mu,F\big] \;+\; \tfrac{1}{2}\big[\big[\gamma^\mu,F\big],F\big] \;+\; \tfrac{1}{6}\big[\big[\big[\gamma^\mu,F\big],F\big],F\big] \;+\; \cdots \tag{23}$$

where all the multiple commutators turn out to be linear combinations of the Dirac matrices. Indeed, the single commutator here is

$$\big[\gamma^\mu,F\big] \;=\; -\tfrac{i}{2}\Theta_{\alpha\beta}\big[\gamma^\mu,S^{\alpha\beta}\big] \;=\; \tfrac{1}{2}\Theta_{\alpha\beta}\big(g^{\mu\alpha}\gamma^\beta \;-\; g^{\mu\beta}\gamma^\alpha\big) \;=\; \Theta_{\alpha\beta}\,g^{\mu\alpha}\gamma^\beta \;=\; \Theta^\mu{}_\lambda\gamma^\lambda, \tag{24}$$

while the multiple commutators follow by iterating this formula:

$$\big[\big[\gamma^\mu,F\big],F\big] \;=\; \Theta^\mu{}_\lambda\big[\gamma^\lambda,F\big] \;=\; \Theta^\mu{}_\lambda\Theta^\lambda{}_\nu\gamma^\nu, \qquad \big[\big[\big[\gamma^\mu,F\big],F\big],F\big] \;=\; \Theta^\mu{}_\lambda\Theta^\lambda{}_\rho\Theta^\rho{}_\nu\gamma^\nu,\ldots. \tag{25}$$

Combining all these commutators as in eq. (23), we obtain

$$\begin{aligned} M_D^{-1}\gamma^\mu M_D \;&=\; \gamma^\mu \;+\; \big[\gamma^\mu,F\big] \;+\; \tfrac{1}{2}\big[\big[\gamma^\mu,F\big],F\big] \;+\; \tfrac{1}{6}\big[\big[\big[\gamma^\mu,F\big],F\big],F\big] \;+\; \cdots \\ &=\; \gamma^\mu \;+\; \Theta^\mu{}_\nu\gamma^\nu \;+\; \tfrac{1}{2}\,\Theta^\mu{}_\lambda\Theta^\lambda{}_\nu\gamma^\nu \;+\; \tfrac{1}{6}\,\Theta^\mu{}_\lambda\Theta^\lambda{}_\rho\Theta^\rho{}_\nu\gamma^\nu \;+\; \cdots \\ &=\; \Big(\delta^\mu_\nu \;+\; \Theta^\mu{}_\nu \;+\; \tfrac{1}{2}\Theta^\mu{}_\lambda\Theta^\lambda{}_\nu \;+\; \tfrac{1}{6}\Theta^\mu{}_\lambda\Theta^\lambda{}_\rho\Theta^\rho{}_\nu \;+\; \cdots\Big)\gamma^\nu \\ &\equiv\; L^\mu{}_\nu\gamma^\nu. \end{aligned} \tag{26}$$

Quod erat demonstrandum.

Dirac Equation and Dirac Spinor Fields

History:

Originally, the Klein–Gordon equation was thought to be the relativistic version of the Schrödinger equation — that is, an equation for the wave function $\psi(\mathbf{x},t)$ for one relativistic particle. But pretty soon this interpretation run into trouble with bad probabilities (negative, or >1) when a particle travels through high potential barriers or deep potential wells. There were also troubles with relativistic causality, and a few other things.

Paul Adrien Maurice Dirac had thought that the source of all those troubles was the ugly form of relativistic Hamiltonian $\hat{H}=\sqrt{\hat{\mathbf{p}}^2+m^2}$ in the coordinate basis, and that he could solve all the problems with the Klein-Gordon equation by rewriting the Hamiltonian as a first-order differential operator

$$\hat{H} \;=\; \hat{\mathbf{p}}\cdot\vec{\alpha} \;+\; m\beta \quad\Longrightarrow\quad \text{Dirac equation}\quad i\frac{\partial\psi}{\partial t} \;=\; -i\vec{\alpha}\cdot\nabla\psi \;+\; m\beta\psi \tag{27}$$

where $\alpha_1,\alpha_2,\alpha_3,\beta$ are matrices acting on a multi-component wave function. Specifically, all four of these matrices are Hermitian, square to 1, and anticommute with each other,

$$\{\alpha_i,\alpha_j\} \;=\; 2\delta_{ij}\,, \quad \{\alpha_i,\beta\} \;=\; 0, \quad \beta^2 \;=\; 1. \tag{28}$$

Consequently

$$\left(\vec{\alpha}\cdot\hat{\mathbf{p}}\right)^2 \;=\; \alpha_i\alpha_j\times\hat{p}_i\hat{p}_j \;=\; \tfrac{1}{2}\{\alpha_i,\alpha_j\}\times\hat{p}_i\hat{p}_j \;=\; \delta_{ij}\times\hat{p}_i\hat{p}_j \;=\; \hat{\mathbf{p}}^2, \tag{29}$$

and therefore

$$\hat{H}^2_{\text{Dirac}} \;=\; \Big(\vec{\alpha}\cdot\hat{\mathbf{p}}+\beta m\Big)^2 \;=\; \left(\vec{\alpha}\cdot\hat{\mathbf{p}}\right)^2 \;+\; \{\alpha_i,\beta\}\times\hat{p}_i m \;+\; \beta^2\times m^2 \;=\; \hat{\mathbf{p}^2} \;+\; 0 \;+\; m^2. \tag{30}$$

This, the Dirac Hamiltonian squares to $\hat{\mathbf{p}}^2+m^2$, as it should for the relativistic particle.

The Dirac equation (27) turned out to be a much better description of a relativistic electron (which has spin $=\frac{1}{2}$) than the Klein–Gordon equation. However, it did not resolve the troubles with relativistic causality or bad probabilities for electrons going through big potential differences $e\Delta\Phi > 2m_ec^2$. Those problems are not solvable in the context of a relativistic single-particle quantum mechanics but only in the quantum field theory.

Modern point of view:

Today, we interpret the Dirac equation as the equation of motion for a Dirac spinor field $\Psi(x)$, comprising 4 complex component fields $\Psi_\alpha(x)$ arranged in a column vector

$$\Psi(x) \;=\; \begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \\ \Psi_3(x) \\ \Psi_4(x) \end{pmatrix}, \tag{31}$$

and transforming under the continuous Lorentz symmetries $x'^\mu = L^\mu_{\ \nu} x^\nu$ according to

$$\Psi'(x') \;=\; M_D(L)\Psi(x). \tag{32}$$

The classical Euler–Lagrange equation of motion for the spinor field is the Dirac equation

$$i\frac{\partial}{\partial t}\Psi \;+\; i\vec{\alpha}\cdot\nabla\Psi \;-\; m\beta\Psi \;=\; 0. \tag{33}$$

To recast this equation in a Lorentz-covariant form, let

$$\beta \;=\; \gamma^0, \quad \alpha^i \;=\; \gamma^0\gamma^i\,; \tag{34}$$

it is easy to see that if the γ^μ matrices obey the anticommutation relations (5) then the $\vec{\alpha}$ and β matrices obey the relations (28) and vice verse. Now let's multiply the whole LHS of the Dirac equation (33) by the $\beta = \gamma^0$:

$$0 \;=\; \gamma^0\Big(i\partial_0 \;+\; i\gamma^0\vec{\gamma}\cdot\nabla \;-\; m\gamma^0\Big)\Psi(x) \;=\; \Big(i\gamma^0\partial_0 \;+\; i\gamma^i\partial_i \;-\; m)\Psi(x), \tag{35}$$

and hence

$$(i\gamma^\mu\partial_\mu \;-\; m)\Psi(x) \;=\; 0. \tag{36}$$

As expected from $\hat{H}^2_{\text{Dirac}} = \hat{\mathbf{p}}^2 + m^2$, the Dirac equation for the spinor field implies the Klein–Gordon equation for each component $\Psi_\alpha(x)$. Indeed, if $\Psi(x)$ obey the Dirac equation, then obviously

$$\big(-i\gamma^\nu\partial_\nu\,-\,m\big)\times\big(i\gamma^\mu\partial_\mu\,-\,m\big)\Psi(x)\;=\;0, \tag{37}$$

but the differential operator on the LHS is equal to the Klein–Gordon $m^2+\partial^2$ times a unit matrix:

$$\big(-i\gamma^\nu\partial_\nu-m\big)\big(i\gamma^\mu\partial_\mu-m\big)\;=\;m^2+\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu=\;m^2+\tfrac{1}{2}\{\gamma^\mu,\gamma^\nu\}\partial_\nu\partial_\mu\;=\;m^2+g^{\mu\nu}\partial_\nu\partial_\mu\,. \tag{38}$$

The Dirac equation (36) **transforms covariantly under the Lorentz symmetries** — its LHS transforms exactly like the spinor field itself.
Proof: Note that since the Lorentz symmetries involve the x^μ coordinates as well as the spinor field components, the LHS of the Dirac equation becomes

$$\big(i\gamma^\mu\partial'_\mu\,-\,m\big)\Psi'(x') \tag{39}$$

where

$$\partial'_\mu\;\equiv\;\frac{\partial}{\partial x'^\mu}\;=\;\frac{\partial x^\nu}{\partial x'^\mu}\times\frac{\partial}{\partial x^\nu}\;=\;\big(L^{-1}\big)_\mu^{\ \nu}\times\partial_\nu\,. \tag{40}$$

Consequently,

$$\partial'_\mu\Psi'(x')\;=\;\big(L^{-1}\big)_\mu^{\ \nu}\times M_D(L)\,\partial_\nu\Psi(x) \tag{41}$$

and hence

$$\gamma^\mu\partial'_\mu\Psi'(x')\;=\;\big(L^{-1}\big)_\mu^{\ \nu}\times\gamma^\mu M_D(L)\,\partial_\nu\Psi(x). \tag{42}$$

But according to eq. (23),

$$\begin{aligned} M_D^{-1}(L)\gamma^\mu M_D(L)\;=\;L^\mu_{\ \nu}\gamma^\nu\quad&\Longrightarrow\quad\gamma^\mu M_D(L)\;=\;L^\mu_{\ \nu}\times M_D(L)\gamma^\nu\\ &\Longrightarrow\quad\big(L^{-1}\big)_\mu^{\ \nu}\times\gamma^\mu M_D(L)\;=\;M_D(L)\gamma^\nu, \end{aligned} \tag{43}$$

so

$$\gamma^\mu\partial'_\mu\Psi'(x')\;=\;M_D(L)\times\gamma^\nu\partial_\nu\Psi(x). \tag{44}$$

Altogether,

$$\big(i\gamma^\mu\partial_\mu\,-\,m\big)\Psi(x)\;\xrightarrow[\text{Lorentz}]{}\;\big(i\gamma^\mu\partial'_\mu\,-\,m\big)\Psi'(x')\;=\;M_D(L)\times\big(i\gamma^\mu\partial_\mu\,-\,m\big)\Psi(x), \tag{45}$$

which proves the covariance of the Dirac equation. *Quod erat demonstrandum.*

Dirac Lagrangian

The Dirac equation is a first-order differential equation, so to obtain it as an Euler–Lagrange equation, we need a Lagrangian which is linear rather than quadratic in the spinor field's derivatives. Thus, we want

$$\mathcal{L} \;=\; \overline{\Psi}\times\bigl(i\gamma^\mu\partial_\mu\,-\,m\bigr)\Psi \tag{46}$$

where $\overline{\Psi}(x)$ is some kind of a conjugate field to the $\Psi(x)$. Since Ψ is a complex field, we treat Ψ and $\overline{\Psi}$ as linearly-independent from each other, so the Euler–Lagrange equation for the $\overline{\Psi}$ immediately gives us the Dirac equation for the $\Psi(x)$ field,

$$0\;=\;\frac{\partial\mathcal{L}}{\partial\overline{\Psi}}\,-\,\partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\overline{\Psi}}\;=\;\bigl(i\gamma^\nu\partial_\nu\,-\,m\bigr)\Psi\,-\,\partial_\mu\bigl(0\bigr). \tag{47}$$

To keep the action $S=\int d^4x\mathcal{L}$ Lorentz-invariant, the Lagrangian (46) should transform as a Lorentz scalar, $\mathcal{L}'(x')=\mathcal{L}(x)$. In light of eq. (19) for the $\Psi(x)$ field and covariance (45) of the Dirac equation, the conjugate field $\overline{\Psi}(x)$ should transform according to

$$\overline{\Psi}'(x')\;=\;\overline{\Psi}(x)\times M_D^{-1}(L)\quad\Longrightarrow\quad\mathcal{L}'(x')\;=\;\mathcal{L}(x). \tag{48}$$

Note that the $M_D(L)$ matrix is generally not unitary, so the inverse matrix $M_D^{-1}(L)$ in eq. (48) is different from the hermitian conjugate $M_D^\dagger(L)$. Consequently, the conjugate field $\overline{\Psi}(x)$ cannot be identified with the hermitian conjugate field $\Psi^\dagger(x)$, since the latter transforms to

$$\Psi'^\dagger(x')\;=\;\Psi^\dagger(x)\times M_D^\dagger(L)\;\neq\Psi^\dagger(x)\times M_D^{-1}(L). \tag{49}$$

Instead of the hermitian conjugate, we are going to use the Dirac conjugate spinor, see below.

Dirac conjugates:

Let Ψ be a 4-component Dirac spinor and Γ be any 4×4 matrix; we *define* their Dirac conjugates according to

$$\overline{\Psi}\;=\;\Psi^\dagger\times\gamma^0,\quad\overline{\Gamma}\;=\;\gamma^0\times\Gamma^\dagger\times\gamma^0. \tag{50}$$

Thanks to $\gamma^0\gamma^0\;=\;1$, the Dirac conjugates behave similarly to hermitian conjugates or transposed matrices:

- For a a product of 2 matrices, $\overline{(\Gamma_1 \times \Gamma_2)} = \overline{\Gamma}_2 \times \overline{\Gamma}_1$.
- Likewise, for a matrix and a spinor, $\overline{(\Gamma \times \Psi)} = \overline{\Psi} \times \overline{\Gamma}$.
- The Dirac conjugate of a complex number is its complex conjugate, $\overline{(c \times \mathbf{1})} \; = \; c^* \times \mathbf{1}$.
- For any two spinors Ψ_1 and Ψ_2 and any matrix Γ, $\overline{\Psi}_1 \overline{\Gamma} \Psi_2 \; = \; \big(\overline{\Psi}_2 \Gamma \Psi_1\big)^*$.
    - The Dirac spinor fields are fermionic, so they anticommute with each other, even in the classical limit. Nevertheless, $\big(\Psi_\alpha^\dagger \Psi_\beta\big)^\dagger \; = \; +\Psi_\beta^\dagger \Psi_\alpha$, and therefore for any matrix Γ, $\overline{\Psi}_1 \overline{\Gamma} \Psi_2 \; = \; +\big(\overline{\Psi}_2 \Gamma \Psi_1\big)^*$.

The point of the Dirac conjugation (50) is that it works similarly for all four Dirac matrices γ^μ,

$$\overline{\gamma^\mu} \; = \; +\gamma^\mu. \tag{51}$$

<u>Proof</u>: For $\mu = 0$, the γ^0 is hermitian, hence

$$\overline{\gamma^0} \; = \; \gamma^0 (\gamma^0)^\dagger \gamma^0 \; = \; +\gamma^0 \gamma^0 \gamma^0 \; = \; +\gamma^0. \tag{52}$$

For $\mu = i = 1, 2, 3$, the γ^i are anti-hermitian and also anticommute with the γ^0, hence

$$\overline{\gamma^i} \; = \; \gamma^0 (\gamma^i)^\dagger \gamma^0 \; = \; -\gamma^0 \gamma^i \gamma^0 \; = \; +\gamma^0 \gamma^0 \gamma^i \; = \; +\gamma^i. \tag{53}$$

Corollary: *The Dirac conjugate of the matrix*

$$M_D(L) \; = \; \exp\big(-\tfrac{i}{2} \Theta_{\mu\nu} S^{\mu\nu}\big) \tag{19}$$

representing any continuous Lorentz symmetry $L = \exp(\Theta)$ is the inverse matrix

$$\overline{M}_D(L) \; = \; M_D^{-1}(L) \; = \; \exp\big(+\tfrac{i}{2} \Theta_{\mu\nu} S^{\mu\nu}\big). \tag{54}$$

<u>Proof</u>: Let

$$X \; = \; -\tfrac{i}{2} \Theta_{\mu\nu} S^{\mu\nu} \; = \; +\tfrac{1}{8} \Theta_{\mu\nu} [\gamma^\mu, \gamma^\nu] \; = \; +\tfrac{1}{4} \Theta_{\mu\nu} \gamma^\mu \gamma^\nu \tag{55}$$

for some real antisymmetric Lorentz parameters $\Theta_{\mu\nu} = -\Theta_{\nu\mu}$. The Dirac conjugate of the

X matrix is

$$\overline{X} \;=\; \overline{\tfrac{1}{4}\Theta_{\mu\nu}\gamma^\mu\gamma^\nu} \;=\; \tfrac{1}{4}\Theta^*_{\mu\nu}\overline{\gamma}^\nu\overline{\gamma}^\mu \;=\; \tfrac{1}{4}\Theta_{\mu\nu}\gamma^\nu\gamma^\mu \;=\; \tfrac{1}{4}\Theta_{\nu\mu}\gamma^\mu\gamma^\nu \;=\; -\tfrac{1}{4}\Theta_{\mu\nu}\gamma^\mu\gamma^\nu \;=\; -X. \tag{56}$$

Consequently,

$$\overline{X^2} \;=\; \overline{X}\times\overline{X} \;=\; +X^2, \quad \overline{X^3} \;=\; \overline{X\times X^2} \;=\; \overline{X^2}\times\overline{X} \;=\; -X^3, \quad \ldots, \quad \overline{X^n} \;=\; (-X)^n, \tag{57}$$

and hence

$$\overline{\exp(X)} \;=\; \sum_n \frac{1}{n!}\,\overline{X^n} \;=\; \sum_n \frac{1}{n!}\,(-X)^n \;=\; \exp(-X). \tag{58}$$

In light of eq. (55), this means

$$\overline{\exp\bigl(-\tfrac{i}{2}\Theta_{\mu\nu}S^{\mu\nu}\bigr)} \;=\; \exp\bigl(+\tfrac{i}{2}\Theta_{\mu\nu}S^{\mu\nu}\bigr), \tag{59}$$

that is,

$$\overline{M}_D(L) \;=\; M_D^{-1}(L). \tag{60}$$

Quod erat demonstrandum.

Back to the Dirac Lagrangian:
Thanks to the theorem (60), the conjugate field $\overline{\Psi}(x)$ in the Lagrangian (46) is simply the Dirac conjugate of the Dirac spinor field $\Psi(x)$,

$$\overline{\Psi}(x) \;=\; \Psi^\dagger(x)\times\gamma^0, \tag{61}$$

which transforms under Lorentz symmetries as

$$\overline{\Psi}'(x') \;=\; \overline{\Psi'(x')} \;=\; \overline{M_D(L)\times\Psi(x)} \;=\; \overline{\Psi}(x)\times\overline{M}_D(x) \;=\; \overline{\Psi}(x)\times M_D^{-1}(L). \tag{62}$$

Consequently, the Dirac Lagrangian

$$\mathcal{L} \;=\; \overline{\Psi}\times\bigl(i\gamma^\mu\partial_\mu \,-\, m\bigr)\Psi \;=\; \Psi^\dagger\gamma^0\times\bigl(i\gamma^\mu\partial_\mu \,-\, m\bigr)\Psi \tag{46}$$

is a Lorentz scalar and the action is Lorentz invariant.

Hamiltonian for the Dirac Field

Canonical quantization of the Dirac spinor field $\Psi(x)$ — just like any other field — begins with the classical Hamiltonian formalism. Let's start with the canonical conjugate fields,

$$\Pi_\alpha \;=\; \frac{\partial\mathcal{L}}{\partial(\partial_0\Psi_\alpha)} \;=\; \left(i\overline{\Psi}\gamma^0\right)_\alpha \;=\; i\Psi^\dagger_\alpha \tag{63}$$

— the canonical conjugate to the Dirac spinor field $\Psi(x)$ is simply its hermitian conjugate $\Psi^\dagger(x)$. This is similar to what we had for the non-relativistic field, and it happens for the same reason — the Lagrangian which is linear in the time derivative.

In the non-relativistic field theory, the conjugacy relation (63) in the classical theory lead to the equal-time commutation relations in the quantum theory,

$$\left[\hat\psi(\mathbf{x},t),\hat\psi(\mathbf{y},t)\right] \;=\; 0,\quad \left[\hat\psi^\dagger(\mathbf{x},t),\hat\psi^\dagger(\mathbf{y},t)\right] \;=\; 0,\quad \left[\hat\psi(\mathbf{x},t),\hat\psi^\dagger(\mathbf{y},t)\right] \;=\; \delta^{(3)}(\mathbf{x}-\mathbf{y}). \tag{64}$$

However, the Dirac spinor field describes spin $=\frac{1}{2}$ particles — like electrons, protons, or neutrons — which are fermions rather than bosons. So instead of the commutations relations (64), the spinor fields obey the *equal-time anti-commutation relations*

$$\begin{aligned}
\left\{\hat\Psi_\alpha(\mathbf{x},t),\hat\Psi_\beta(\mathbf{y},t)\right\} \;&=\; 0,\\
\left\{\hat\Psi^\dagger_\alpha(\mathbf{x},t),\hat\Psi^\dagger_\beta(\mathbf{y},t)\right\} \;&=\; 0,\\
\left\{\hat\Psi_\alpha(\mathbf{x},t),\hat\Psi^\dagger_\beta(\mathbf{y},t)\right\} \;&=\; \delta_{\alpha\beta}\delta^{(3)}(\mathbf{x}-\mathbf{y}).
\end{aligned} \tag{65}$$

Next, the classical Hamiltonian obtains as

$$\begin{aligned}
H \;&=\; \int\! d^3\mathbf{x}\,\mathcal{H}(\mathbf{x}),\\
\mathcal{H} \;&=\; i\Psi^\dagger\partial_0\Psi \;-\; \mathcal{L}\\
&=\; i\Psi^\dagger\partial_0\Psi \;-\; \Psi^\dagger\gamma^0\big(i\gamma^0\partial_0 \,+\, i\vec\gamma\cdot\nabla \,-\, m\big)\Psi\\
&=\; \Psi^\dagger\big(-i\gamma^0\vec\gamma\cdot\nabla \,+\, \gamma^0 m\big)\Psi
\end{aligned} \tag{66}$$

where the terms involving the time derivative ∂_0 cancel out. Consequently, the Hamiltonian operator of the quantum field theory is

$$\hat{H} = \int d^3\mathbf{x}\, \hat{\Psi}^\dagger(\mathbf{x}) \big(-i\gamma^0\vec{\gamma}\cdot\nabla + \gamma^0 m\big) \hat{\Psi}(\mathbf{x}). \tag{67}$$

Note that the derivative operator $(-i\gamma^0\vec{\gamma}\cdot\nabla + \gamma^0 m)$ in this formula is precisely the 1-particle Dirac Hamiltonian (27). This is very similar to what we had for the quantum non-relativistic fields,

$$\hat{H} = \int d^3\mathbf{x}\, \hat{\psi}^\dagger(\mathbf{x}) \left(\frac{-1}{2M}\nabla^2 + V(\mathbf{x})\right) \hat{\psi}(\mathbf{x}), \tag{68}$$

except for a different differential operator, Schrödinger instead of Dirac.

In the Heisenberg picture, the quantum Dirac field obeys the Dirac equation. To see how this works, we start with the Heisenberg equation

$$i\frac{\partial}{\partial t}\hat{\Psi}_\alpha(\mathbf{x},t) = \big[\hat{\Psi}_\alpha(\mathbf{x},t), \hat{H}\big] = \int d^3\mathbf{y}\, \big[\hat{\Psi}_\alpha(\mathbf{x},t), \hat{\mathcal{H}}(\mathbf{y},t)\big], \tag{69}$$

and then evaluate the last commutator using the anti-commutation relations (65) and the Leibniz rules (11). Indeed, let's use the Leibniz rule

$$[A, BC] = \{A, B\}C - B\{A, C\} \tag{70}$$

for

$$\begin{aligned} A &= \hat{\Psi}_\alpha(\mathbf{x},t), \\ B &= \hat{\Psi}^\dagger_\beta(\mathbf{y},t), \\ C &= \big(-i\gamma^0\vec{\gamma}\cdot\nabla + \gamma^0 m\big)_{\beta\gamma} \hat{\Psi}_\gamma(\mathbf{y},t), \end{aligned} \tag{71}$$

so that $BC = \hat{\mathcal{H}}(\mathbf{y},t)$. For the A, B, C at hand,

$$\{A, B\} = \delta_{\alpha\beta}\delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{72}$$

while

$$\{A, C\} = \big(-i\gamma^0\vec{\gamma}\cdot\nabla_y + \gamma^0 m\big)_{\beta\gamma} \{\hat{\Psi}_\alpha(\mathbf{x},t), \hat{\Psi}_\gamma(\mathbf{y},t)\} = (\text{diff.op.}) \times 0 = 0. \tag{73}$$

Consequently

$$\begin{aligned}
\big[\hat{\Psi}_\alpha(\mathbf{x},t),\hat{\mathcal{H}}(\mathbf{y},t)\big] \;&\equiv\; [A,BC] \\
&=\; \{A,B\}\times C \;-\; B\times\{A,C\} \\
&=\; \delta_{\alpha\beta}\delta^{(3)}(\mathbf{x}-\mathbf{y})\times\big(-i\gamma^0\vec{\gamma}\cdot\nabla\,+\,\gamma^0 m\big)_{\beta\gamma}\hat{\Psi}_\gamma(\mathbf{y},t) \\
&\qquad\quad -\;0,
\end{aligned} \tag{74}$$

hence

$$\begin{aligned}
\big[\hat{\Psi}_\alpha(\mathbf{x},t),\hat{H}\big] \;&=\int d^3\mathbf{y}\,\delta^{(3)}(\mathbf{x}-\mathbf{y})\times\big(-i\gamma^0\vec{\gamma}\cdot\nabla\,+\,\gamma^0 m\big)_{\alpha\gamma}\hat{\Psi}_\gamma(\mathbf{y},t) \\
&=\;\big(-i\gamma^0\vec{\gamma}\cdot\nabla\,+\,\gamma^0 m\big)_{\alpha\gamma}\hat{\Psi}_\gamma(\mathbf{x},t),
\end{aligned} \tag{75}$$

and therefore

$$i\partial_0\hat{\Psi}(\mathbf{x},t) \;=\; \big(-i\gamma^0\vec{\gamma}\cdot\nabla\,+\,\gamma^0 m\big)\hat{\Psi}(\mathbf{x},t). \tag{76}$$

Or if you prefer,

$$\big(i\gamma^\mu\partial_\mu\,-\,m\big)\hat{\Psi}(x) \;=\; 0. \tag{77}$$