Finite Multiplets of the Spin(3,1) Group.

In these notes I classify all the finite multiplets of the continuous Lorentz group $SO^+(3, 1)$, or rather of its double-covering group Spin(3, 1). The notes are insterspersed with optional exercises for the students. The solutions to the exercises will appear in a separate page separate page.

I presume you read these notes after finishing your homework#5 and homework#6, so you should be familiar with the Lorents $\hat{\mathbf{J}}$ and $\hat{\mathbf{K}}$ generators and their Dirac spinor representations. In these notes, it's convenient to re-organize the $\hat{\mathbf{J}}$ and $\hat{\mathbf{K}}$ generators into two non-hermitian 3-vectors

$$\hat{\mathbf{J}}_{+} = \frac{1}{2} (\hat{\mathbf{J}} + i\hat{\mathbf{K}}) \text{ and } \hat{\mathbf{J}}_{-} = \frac{1}{2} (\hat{\mathbf{J}} - i\hat{\mathbf{K}}) = \hat{\mathbf{J}}_{+}^{\dagger}.$$
 (1)

1. Show that the two 3-vectors commute with each other, $[\hat{J}^k_+, \hat{J}^\ell_-] = 0$, while the components of each 3-vector satisfy angular momentum commutation relations, $[\hat{J}^k_+, \hat{J}^\ell_+] = i\epsilon^{k\ell m}\hat{J}^m_+$ and $[\hat{J}^k_-, \hat{J}^\ell_-] = i\epsilon^{k\ell m}\hat{J}^m_-$.

By themselves, the 3 \hat{J}_{+}^{k} generate a symmetry group similar to rotations of a 3D space, but since the \hat{J}_{+}^{k} are non-hermitian, the finite irreducible multiplets of this symmetry are non-unitary analytic continuations (to complex "angles") of the ordinary angular momentum multiplets (j) of spin $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$ Likewise, the finite irreducible multiplets of the symmetry group generated by the \hat{J}_{-}^{k} are analytic continuations of the spin-j multiplets of angular momentum. Moreover, the two symmetry groups commute with each other, so the finite irreducible multiplets of the net Lorentz symmetry are tensor products $(j_{+}) \otimes (j_{-})$ of the $\hat{\mathbf{J}}_{+}$ and $\hat{\mathbf{J}}_{-}$ multiplets. In other words, distinct finite irreducible multiplets of the Lorentz symmetry may be labeled by *two* integer or half-integer 'spins' j_{+} and j_{-} , while the states within such a multiplet are $|j_{+}, j_{-}, m_{+}, m_{-}\rangle$ for $m_{+} = -j_{+}, \ldots, +j_{+}$ and $m_{-} =$ $-j_{-}, \ldots, +j_{-}$.

The simplest non-trivial Lorentz multiplets are two inequivalent doublets, the left-handed Weyl spinor **2** and the right-handed Weyl spinor **2**^{*}. The **2** multiplet has $j_+ = \frac{1}{2}$ while $j_- = 0$, hence $\hat{\mathbf{J}}_+$ acts as $\frac{1}{2}\boldsymbol{\sigma}$ while $\hat{\mathbf{J}}_-$ does not act at all, or in terms of the $\hat{\mathbf{J}}$ and $\hat{\mathbf{K}}$ generators $\mathbf{J} = \frac{1}{2}\boldsymbol{\sigma}$ while $\mathbf{K} = -\frac{i}{2}\boldsymbol{\sigma}$. The conjugate $\mathbf{2}^*$ multiplet has $j_- = \frac{1}{2}$ while $j_+ = 0$, hence $\hat{\mathbf{J}}$ acts as $\frac{1}{2}\boldsymbol{\sigma}$ while $\hat{\mathbf{K}}$ acts as $+\frac{i}{2}\boldsymbol{\sigma}$.

- Check that these two doublets are indeed the LH Weyl spinors and the RH Weyl spinor from the homework set#6 (problem 2).
- 3. Check that for finite Lorentz symmetries, the 2×2 matrices M_L and M_R representing them in the LH and the RH Weyl spinor multiplets have determinant = 1.

The complex (but not necessary unitary) 2×2 matrices of unit determinant form a noncompact group called the $SL(2, \mathbb{C})$. This group is isomorphic to the Spin(3, 1), the double cover of the continuous Lorentz group $SO^+(3, 1)$. Just like the SU(2) is isomorphic to the Spin(3), the double cover of the SO(3) rotation group.

For the Spin(3) = SU(2) group, one can construct a multiplet of any spin j from a symmetric tensor product of 2j doublets. This procedure gives us an object $\Phi_{\alpha_1,\ldots,\alpha_{2j}}$ with 2j spinor indices $\alpha_1,\ldots,\alpha_{2j} = 1,2$ that's totally symmetric under permutation of those indices and transforms under an SU(2) symmetry U as

$$\Phi_{\alpha_1,\alpha_2\dots,\alpha_{2j}} \to U^{\beta_1}_{\alpha_1} U^{\beta_2}_{\alpha_2} \cdots U^{\beta_{2j}}_{\alpha_{2j}} \Phi_{\beta_1,\beta_2\dots,\beta_{2j}}.$$
(2)

For integer j, such objects are equivalent to tensors of the SO(3); for example, for j = 2 $\Phi_{\alpha\beta} \equiv \Phi_{\beta\alpha}$ is equivalent to an SO(3) vector $\vec{\Phi}$.

For the Lorentz group Spin(3, 1) we have a similar situation — any multiplet can be constructed by tensoring together a bunch of two-component spinors of the $SL(2, \mathbb{C})$. But unlike the SU(2), the $SL(2, \mathbb{C})$ has two different spinors $2 \not\cong 2^*$ transforming under different rules. Notationally, we shall distinguish them by different index types: the un-dotted Greek indices belong to spinor that transform according to $M \in SL(2, \mathbb{C})$ while the dotted Greek indices belong to spinors that transform according to M^* :

$$(\psi_L)\alpha \rightarrow M^{\beta}_{\alpha}(\psi_L)_{\beta} \cong (\sigma_2\psi_R)_{\dot{\gamma}} \rightarrow M^{*\dot{\delta}}_{\dot{\gamma}}(\sigma_2\psi_R)_{\dot{\delta}}, \quad M \in SL(2, \mathbf{C}).$$
 (3)

Combining such spinors to make a multiplet with 'spins' j_+ and j_- , we make an object $\Phi_{\alpha_1,\ldots,\alpha_{(2j_+)};\dot{\gamma}_1,\ldots,\dot{\gamma}_{(2j_-)}}$ with $2j_+$ un-dotted indices and $2j_-$ dotted indices. Φ_{\ldots} is totally

symmetric under permutations of the un-dotted indices with each other or dotted indices with each other, but there is no symmetry between dotted and un-dotted indices. Under an $SL(2, \mathbb{C})$ symmetry M, the un-dotted indices transform according to M while the dotted indices transform according to the M^* , thus

$$\Phi_{\alpha_{1},\dots,\alpha_{(2j_{+})};\dot{\gamma}_{1},\dots,\dot{\gamma}_{(2j_{-})}} \to M_{\alpha_{1}}^{\beta_{1}}\cdots M_{\alpha_{(2j_{+})}}^{\beta_{(2j_{+})}} \times M_{\dot{\gamma}_{1}}^{*M\dot{\delta}_{1}}\cdots M_{\dot{\gamma}_{(2j_{-})}}^{*M\dot{\delta}_{(2j_{-})}}\cdots \times \Phi_{\beta_{1},\dots,\beta_{(2j_{+})};\dot{\delta}_{1},\dots,\dot{\delta}_{(2j_{-})}}.$$
(4)

Of particular importance among such multi-spinors is the bi-spinor $V_{\alpha\dot{\gamma}}$ with $j_+ = j_- = \frac{1}{2}$ — it is equivalent to the Lorentz vector V^{μ} . The map between bi-spinors and Lorentz vectors involves four hermitian 2×2 matrices $\sigma_{\mu} = (1, \boldsymbol{\sigma})$. In $SL(2, \mathbf{C})$ terms, each σ_{μ} matrix has one dotted and one un-dotted index, thus $(\sigma_{\mu})_{\alpha\dot{\gamma}}$. Using the σ_{μ} , we may re-cast any Lorentz vector V^{μ} as a matrix

$$V^{\mu} \rightarrow V^{\mu} \sigma_{\mu} = V^{0} + \mathbf{V} \cdot \boldsymbol{\sigma} \tag{5}$$

an hence as a $\left(\frac{1}{2}, \frac{1}{2}\right)$ bi-spinor

$$V_{\alpha\dot{\gamma}} = \left(V^{\mu}\sigma_{\mu}\right)_{\alpha\dot{\gamma}} = V^{0}\delta_{\alpha\dot{\gamma}} + \mathbf{V}\cdot\boldsymbol{\sigma}_{\alpha\dot{\gamma}}.$$
 (6)

Under an $SL(2, \mathbb{C})$ symmetry, the bi-spinor transforms as

$$V_{\alpha\dot{\gamma}} \rightarrow V_{\alpha\dot{\gamma}}' = M_{\alpha}^{\beta} M_{\dot{\gamma}}^{*\dot{\delta}} V_{\beta\dot{\delta}}, \qquad (7)$$

or in matrix form,

$$V^{\mu}\sigma_{\mu} \rightarrow V^{\prime\mu}\sigma_{\mu} = M \left(V^{\mu}\sigma_{\mu} \right) M^{\dagger}.$$
(8)

Since the four matrices σ_{μ} form a complete basis of 2 × 2 matrices, eq. (8) defines a linear transform $V'^{\mu} = L^{\mu}_{\ \nu}(M)V^{\nu}$.

- 4. Prove that for any SL(2, C) matrix M, the transform L^μ_ν(M) defined by eq. (8) is real (real V^μ for real V^μ), Lorentzian (preserves V[']_μV^μ = V_μV^μ) and orthochronous. Hint: prove and use det(V_μσ^μ) = V_μV^μ.
 - * For extra challenge, show that this transform is proper, det(L) = +1.

- 5. Verify that this $SL(2, \mathbb{C}) \to SO^+(3, 1)$ map respects the group law, $L(M_2M_1) = L(M_2)L(M_1)$.
- 6. Show that for the L(M) defined by eq. (8), the LH Weyl spinor representation of L(M)is $M_L(L) = M$ while the RH Weyl spinor representation is $\overline{M} = \sigma_2 M^* \sigma_2$.

In general, any (j_+, j_-) multiplet of the $SL(2, \mathbb{C})$ with integer net spin $j_+ + j_-$ is equivalent to some kind of a Lorentz tensor. (Here, we include the scalar and the vector among the tensors.) For example, the (1, 1) multiplet is equivalent to a symmetric, traceless 2-index tensor $T^{\mu\nu} = +T^{\nu\mu}$, $T^{\mu}_{\mu} = 0$. For $j_+ \neq j_-$ the representation is complex, but one can make a real tensor by combining two multiplets with opposite j_+ and j_- , for example the (1,0) and the (0,1) multiplets are together equivalent to the antisymmetric 2-index tensor $F^{\mu\nu} = -F^{\nu\mu}$.

7. Verify the above examples.

Hint: For any kind of angular momentum, $(j = \frac{1}{2}) \otimes (j = \frac{1}{2}) = (j = 1) \oplus (j = 0).$