Correlation Functions in Perturbation Theory

Many aspects of quantum field theory are related to its n-point correlation functions

$$\mathcal{F}_n(x_1,\ldots,x_n) \stackrel{\text{def}}{=} \langle\Omega|\,\mathbf{T}\hat{\Phi}_H(x_1)\cdots\hat{\Phi}_H(x_n)\,|\Omega\rangle \tag{1}$$

— or for theories with multiple fields $\hat{\Phi}^a$,

$$\mathcal{F}_n^{a_1,\ldots,a_n}(x_1,\ldots,x_n) \stackrel{\text{def}}{=} \langle\Omega|\,\mathbf{T}\hat{\Phi}_H^{a_1}(x_1)\cdots\hat{\Phi}_H^{a_n}(x_n)\,|\Omega\rangle\,. \tag{2}$$

Note that all the fields $\hat{\Phi}_H(x)$ here are in the Heisenberg picture so their time dependence involves the complete Hamiltonian $\hat{H}$ of the interacting theory. Likewise, $|\Omega\rangle$ is the ground state of $\hat{H}$, *i.e.* the true physical vacuum of the theory.

In perturbation theory, the correlation functions $\mathcal{F}_n$ of the interacting theory are related to the free theory's correlation functions

$$\langle 0|\,\mathbf{T}\hat{\Phi}_I(x_1)\cdots\hat{\Phi}_I(x_n)\cdots\text{more }\hat{\Phi}_I(z_1)\hat{\Phi}_I(z_2)\cdots|0\rangle\,. \tag{3}$$

involving additional fields $\hat{\Phi}_I(z_1)\hat{\Phi}_I(z_2)\cdots$. Note that in eq. (3) the fields are in the interaction rather than Heisenberg picture, so they evolve with time as free fields according to the free Hamiltonian $\hat{H}_0$. Likewise, $|0\rangle$ is the free theory's vacuum, *i.e.* the ground state of the free Hamiltonian $\hat{H}_0$ rather than the full Hamiltonian $\hat{H}$.

To work out the relation between (1) and (3), we start by formally relating quantum fields in the Heisenberg and the interaction pictures,

$$\hat{\Phi}_H(\mathbf{x},t) \;=\; e^{+i\hat{H}t}\hat{\Phi}_S(\mathbf{x})e^{-i\hat{H}t} \;=\; e^{+i\hat{H}t}e^{-i\hat{H}_0t}\hat{\Phi}_I(\mathbf{x},t)e^{+i\hat{H}_0t}e^{-i\hat{H}t}. \tag{4}$$

We may re-state this relation in terms of evolution operators using a formal expression for the later,

$$\hat{U}_I(t,t_0) \;=\; e^{+i\hat{H}_0t}e^{-i\hat{H}(t-t_0)}e^{-i\hat{H}_0t_0}. \tag{5}$$

Note that this formula applies for both forward and backward evolution, *i.e.* regardless of whether $t > t_0$ or $t < t_0$. In particular,

$$\hat{U}_I(t,0) \;=\; e^{+i\hat{H}_0 t}e^{-i\hat{H}t} \qquad \text{and} \qquad \hat{U}_I(0,t) \;=\; e^{+i\hat{H}t}e^{-i\hat{H}_0 t}, \tag{6}$$

which allows us to re-state eq. (4) as

$$\hat{\Phi}_H(x) \;=\; \hat{U}_I(0,x^0)\hat{\Phi}_I(x)\hat{U}_I(x^0,0). \tag{7}$$

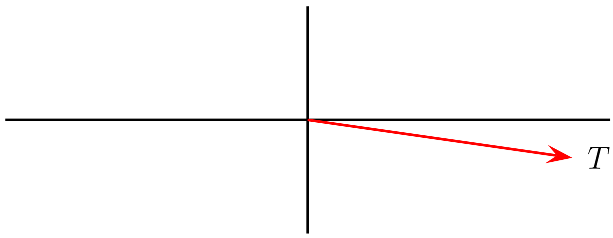
Consequently,

$$\hat{\Phi}_H(x)\hat{\Phi}_H(y) \;=\; \hat{U}_I(0,x^0)\hat{\Phi}_I(x)\hat{U}_I(x^0,y^0)\hat{\Phi}_I(y)\hat{U}_I(y^0,0) \tag{8}$$

because $\hat{U}_I(x^0,0)\hat{U}_I(0,y^0) = \hat{U}_I(x^0,y^0)$, and likewise for n fields

$$\begin{aligned}\hat{\Phi}_H(x_1)\hat{\Phi}_H(x_2)\cdots\hat{\Phi}_H(x_n) \;&=\; \\ &= \hat{U}_I(0,x_1^0)\hat{\Phi}_I(x_1)\hat{U}_I(x_1^0,x_2^0)\hat{\Phi}_I(x_2)\cdots\hat{U}_I(x_{n-1}^0,x_n^0)\hat{\Phi}_I(x_n)\hat{U}_I(x_n^0,0).\end{aligned} \tag{9}$$

Now we need to relate the free vacuum $|0\rangle$ and the true physical vacuum $|\Omega\rangle$. Consider the state $\hat{U}_I(0,-T)\,|0\rangle$ for a complex T, and take the limit of $T \to (+1 - i\epsilon) \times \infty$. That is, $\mathrm{Re}\,T \to +\infty$, $\mathrm{Im}\,T \to -\infty$, but the imaginary part grows slower than the real part. Pictorially, in the complex T plane,



(10)

we go infinitely far to the right at infinitesimally small angle below the real axis.

Without loss of generality we assume the free theory has zero vacuum energy, thus $\hat{H}_0 |0\rangle = 0$ and hence

$$\hat{U}_I(0,-T)\,|0\rangle \;=\; e^{-i\hat{H}T}e^{+i\hat{H}_0T}\,|0\rangle \;=\; e^{-i\hat{H}T}\,|0\rangle\,. \tag{11}$$

From the interacting theory's point of view, $|0\rangle$ is a superposition of eigenstates $|Q\rangle$ of the full Hamiltonian $\hat{H}$,

$$|0\rangle \;=\; \sum_Q |Q\rangle \times \langle Q|0\rangle \quad\Longrightarrow\quad e^{-i\hat{H}T}\,|0\rangle \;=\; \sum_Q |Q\rangle \times e^{-iTE_Q}\,\langle Q|0\rangle \tag{12}$$

In the $T \to (+1 - i\epsilon)\times\infty$ limit, the second sum here is dominated by the term with the lowest E_Q, so we look for the lowest energy eigenstate $|Q_0\rangle$ with the same quantum numbers as $|0\rangle$ (otherwise, we would have zero overlap $\langle Q_0|0\rangle$). Obviously, such $|Q_0\rangle$ is the physical vacuum $|\Omega\rangle$, so

$$\hat{U}_I(0,-T)\,|0\rangle \xrightarrow[T\to(+1-i\epsilon)\infty]{} |\Omega\rangle \times e^{-iTE_\Omega}\,\langle\Omega|0\rangle \tag{13}$$

and therefore

$$|\Omega\rangle \;=\; \lim_{T\to(+1-i\epsilon)\infty} \hat{U}_I(0,-T)\,|0\rangle \times \frac{e^{+iTE_\Omega}}{\langle\Omega|0\rangle}\,. \tag{14}$$

Likewise,

$$\langle\Omega| \;=\; \lim_{T\to(+1-i\epsilon)\infty} \frac{e^{+iTE_\Omega}}{\langle 0|\Omega\rangle} \times \langle 0|\,\hat{U}_I(+T,0)\,. \tag{15}$$

Combining eqs. (8), (14), and (15), we may now express the two-point function as

$$\langle\Omega|\,\hat{\Phi}_H(x)\hat{\Phi}_H(y)\,|\Omega\rangle \;=\; \lim_{T\to(+1-i\epsilon)\infty} C(T) \times \langle 0|\,\text{Big\_Product}\,|0\rangle \tag{16}$$

where

$$C(T) \;=\; \frac{e^{2iTE_\Omega}}{|\langle 0|\Omega\rangle|^2} \tag{17}$$

is a just a coefficient, and

$$\begin{aligned} \text{Big\_Product} \;&=\; \hat{U}_I(+T,0)\hat{U}_I(0,x^0)\hat{\Phi}_I(x)\hat{U}_I(x^0,y^0)\hat{\Phi}_I(y)\hat{U}_I(y^0,0)\hat{U}_I(0,-T) \\ &=\; \hat{U}_I(+T,x^0)\hat{\Phi}_I(x)\hat{U}_I(x^0,y^0)\hat{\Phi}_I(y)\hat{U}_I(y^0,-T). \end{aligned} \tag{18}$$

For $x^0 > y^0$, the last line here is in proper time order, so if we re-order the operators, the time-orderer $\mathbf{T}$ would put them back where they belong. Thus, using $\mathbf{T}$ to keep track of the operator order, we have

$$\begin{aligned}
\text{Big_Product} &= \mathbf{T}\Big(\hat{U}_I(+T,x^0)\hat{\Phi}_I(x)\hat{U}_I(x^0,y^0)\hat{\Phi}_I(y)\hat{U}_I(y^0,-T)\Big) \\
&= \mathbf{T}\Big(\hat{\Phi}_I(x)\hat{\Phi}_I(y)\times\hat{U}_I(+T,x^0)\hat{U}_I(x^0,y^0)\hat{U}_I(y^0,-T)\Big) \\
&= \mathbf{T}\Big(\hat{\Phi}_I(x)\hat{\Phi}_I(y)\times\hat{U}_I(+T,-T)\Big) \\
&= \mathbf{T}\left(\hat{\Phi}_I(x)\hat{\Phi}_I(y)\times\exp\left(\frac{-i\lambda}{24}\int\limits_{-T}^{+T}\!dt\int\!d^3\mathbf{z}\,\hat{\Phi}_I^4(t,\mathbf{z})\right)\right)
\end{aligned} \tag{19}$$

where the last line follows from the Dyson series for the evolution operator

$$U_I(t_f,t_i) \;=\; \mathbf{T}\text{-exp}\left(-i\int\limits_{t_i}^{t_f}\!dt\,\hat{V}_I(t)\right) \;=\; \mathbf{T}\text{-exp}\left(\frac{-i\lambda}{24}\int\limits_{t_i}^{t_f}\!dt\int\!d^3\mathbf{z}\,\hat{\Phi}_I^4(t,\mathbf{z})\right).$$

Altogether, the two-point correlation function becomes

$$\begin{aligned}
\mathcal{F}_2(x,y) &\stackrel{\text{def}}{=} \langle\Omega|\,\mathbf{T}\hat{\Phi}_H(x)\hat{\Phi}_H(y)\,|\Omega\rangle \\
&= \lim_{T\to(+1-i\epsilon)\infty} C(T)\times\langle 0|\,\mathbf{T}\left(\hat{\Phi}_I(x)\hat{\Phi}_I(y)\times\exp\left(\frac{-i\lambda}{24}\int\!d^4z\,\hat{\Phi}_I^4(z)\right)\right)|0\rangle\,,
\end{aligned} \tag{20}$$

where the spacetime integral has ranges

$$\int\!d^4z \;\equiv\; \int\limits_{-T}^{+T}\!dz^0\int\limits_{\substack{\text{whole}\\ \text{space}}}\!d^3\mathbf{z}\,. \tag{21}$$

Similarly, the n-point correlation functions can be written as

$$\begin{aligned}
\mathcal{F}_n(x_1,\ldots,x_n) &\stackrel{\text{def}}{=} \langle\Omega|\,\mathbf{T}\hat{\Phi}_H(x_1)\cdots\hat{\Phi}_H(x_n)\,|\Omega\rangle \\
&= \lim_{T\to(+1-i\epsilon)\infty} C(T)\times\langle 0|\,\mathbf{T}\left(\hat{\Phi}_I(x_1)\cdots\hat{\Phi}_I(x_n)\times\exp\left(\frac{-i\lambda}{24}\int\!d^4z\,\hat{\Phi}_I^4(z)\right)\right)|0\rangle\,.
\end{aligned} \tag{22}$$

Note that the coefficient $C(T)$ is the same for all the correlations functions (for any n); it's related to the vacuum energy shift according to eq. (17). In particular, for $n=0$ the

$\mathcal{F}_0 = \langle\Omega|\Omega\rangle = 1$, but it's also given by eq. (22), hence

$$\lim_{T\to(+1-i\epsilon)\infty} C(T)\times\langle 0|\,\mathbf{T}\left(\exp\left(\frac{-i\lambda}{24}\int\!d^4z\,\hat{\Phi}_I^4(z)\right)\right)|0\rangle\;=\;1. \tag{23}$$

This allows us to eliminate the $C(T)$ factors from eqs. (22) by taking *ratios* of the free-theory correlation functions,

$$\mathcal{F}_n(x_1,\ldots,x_n)\;=\;\lim_T\frac{\langle 0|\,\mathbf{T}\left(\hat{\Phi}_I(x_1)\cdots\hat{\Phi}_I(x_n)\times\exp\left(\frac{-i\lambda}{24}\int\!d^4z\,\hat{\Phi}_I^4(z)\right)\right)|0\rangle}{\langle 0|\,\mathbf{T}\left(\exp\left(\frac{-i\lambda}{24}\int\!d^4z\,\hat{\Phi}_I^4(z)\right)\right)|0\rangle}. \tag{24}$$

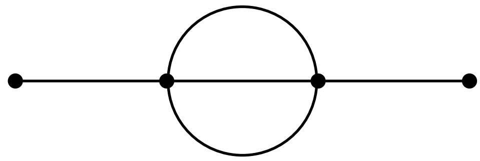
The limit here is $T\to(+1-i\epsilon)\times\infty$, and the T dependence under the limit is implicit in the ranges of the spacetime integrals, *cf.* eq. (21).

In perturbation theory, the vacuum sandwiches in the numerator and the denominator of eq. (24) can be expanded into sums of Feynman diagrams. Indeed, expanding the numerator in a power series in λ, we obtain

$$\begin{aligned}&\langle 0|\mathbf{T}\left(\hat{\Phi}_I(x_1)\cdots\hat{\Phi}_I(x_n)\times\exp\left(\frac{-i\lambda}{24}\int\!d^4z\,\hat{\Phi}_I^4(z)\right)\right)|0\rangle\;=\\ &=\sum_{N=0}^{\infty}\frac{(-i\lambda)^N}{(4!)^N\,N!}\int\!d^4z_1\cdots\!\int\!d^4z_N\;\langle 0|\,\mathbf{T}\hat{\Phi}_I(x_1)\cdots\hat{\Phi}_I(x_n)\times\hat{\Phi}_I^4(z_1)\cdots\hat{\Phi}_I^4(z_N)\,|0\rangle\end{aligned} \tag{25}$$

where each sub-sandwich $\langle 0|\,\mathbf{T}\hat{\Phi}_I(x_1)\cdots\hat{\Phi}_I(x_n)\times\hat{\Phi}_I^4(z_1)\cdots\hat{\Phi}_I^4(z_N)\,|0\rangle$ expands into a big sum of products of $\frac{4N+n}{2}$ Feynman propagators $G_F(x_i-x_j)$, $G_F(x_i-z_j)$, or $G_F(z_i-z_j)$. We have gone through expansion back in November — here are my notes — so let me simply summarize the result in terms of the *Feynman rules for the correlation functions*:

- ⋆ A generic Feynman diagram for the n-point correlation function has n *external vertices* $x_1,\ldots,x_n$ or valence $=1$ plus some number $N=0,1,2,3,\ldots$ of *internal vertices* $z_1,\ldots,Z_N$ of valence $=4$. On the other hand, it has no external lines but only the internal lines between the vertices. Here is an example diagram with 2 external vertices, 2 internal vertices, and 5 internal lines:



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- To evaluate a diagram in coordinate space, first multiply the usual factors:
  - $*$ The free propagator $G_F(z_i - z_j)$ for a line connecting vertices internal z_i and z_j, and likewise for lines connecting an internal vertex z_i to an external vertex x_j, or two external vertices x_i and x_j.
  - $*$ $(-i\lambda)$ factor for each internal vertex.
  - $*$ The combinatorial factor 1/#symmetries of the diagram (including the trivial symmetry).
- Second, integrate $\int d^4z$ over each internal vertex location; the integration range is as in eq. (21). But do not integrate over the external vertices — their location's $x_1, \ldots, x_n$ are the arguments of the n-point correlation function $\mathcal{F}_n(x_1, \ldots, x_n)$.
- To calculate the numerator of eq. (24) to order $\lambda^{N_{\max}}$, sum over *all* diagrams with n external vertices, $N \leq N_{\max}$ internal vertices, and any pattern of lines respecting the valences of all the vertices.

At this point, we are summing over all kinds of diagrams, connected or disconnected, and even the vacuum bubbles are allowed. However, similar to what we had back in November, the vacuum bubbles can be factored out:

$$\sum(\text{all diagrams}) \;=\; \sum\left(\begin{matrix}\text{diagrams without}\\ \text{vacuum bubbles}\end{matrix}\right)\times\sum\left(\begin{matrix}\text{vacuum bubbles}\\ \text{without external vertices}\end{matrix}\right). \tag{27}$$

Moreover, the vacuum bubble factor here is the same for all the free-theory vacuum sandwiches

$$\langle 0|\,\mathbf{T}\left(\hat{\Phi}_I(x_1)\cdots\hat{\Phi}_I(x_n)\times\exp\left(\frac{-i\lambda}{24}\int d^4z\,\hat{\Phi}_I^4(z)\right)\right)|0\rangle$$

in the numerators of eqs. (24) for all the correlation functions, and also in the $n = 0$ sandwich

$$\langle 0|\,\mathbf{T}\left(\exp\left(\frac{-i\lambda}{24}\int d^4z\,\hat{\Phi}_I^4(z)\right)\right)|0\rangle \;=\; \sum\begin{pmatrix}\text{vacuum bubbles}\\ \text{without external vertices}\end{pmatrix} \tag{28}$$

in the all the denominators. This means that the vacuum bubbles simply cancel out from the correlation functions! In other words,

$$\mathcal{F}_n(x_1,\ldots,x_n) \;=\; \sum\begin{pmatrix}\text{Feynman diagrams with}\\ n \text{ external vertices } x_1,\ldots x_n\\ \text{and without vacuum bubbles}\end{pmatrix}. \tag{29}$$

Besides reducing the number of diagrams we need to calculate, the cancellation of the vacuum bubbles leads to another simplification: Instead of evaluating each diagram for a finite T, taking the ratio of two sums of diagrams, and only then taking the $T \to (+1 - i\epsilon)\infty$ limit, we may now take that limit directly for each diagram . In practice, this means integrating each $\int d^4z_i$ over the whole Minkowski spacetime instead of a limited time range from $-T$ to $+T$ as in eq. (21). Consequently, when we Fourier transform the Feynman rules from the coordinate space to the momentum space, we end up with the usual momentum-conservation factors $(2\pi)^4\delta^{(4)}(\pm q_1^{\pm} q_2 \pm q_3 \pm q_4)$ at each internal vertex instead of something much more complicated.

So here are the *momentum-space Feynman rules for the correlation functions:*

- Since all the lines are internal, assign a variable momentum q_i^μ to each line and specify the direction of this momentum flow (from which vertex to which vertex).

∗ Each line carries a propagator $\dfrac{i}{q^2 - m^2 + i0}$.

∗ Each external vertex x carries a factor e^{+iqx} or e^{-iqx}, depending on whether the momentum q flows into or out from the vertex.

∗ Each internal vertex carries factor $(-i\lambda) \times (2\pi)^4\delta^{(4)}(\pm q_1^{\pm} q_2 \pm q_3 \pm q_4)$.

∗ Overall combinatorial factor 1/#symmetries for the whole diagram.

- Multiply all these factors together, then integrate over all the momenta q_i^μ.

For example, the diagram (26) evaluates to

$$
\begin{aligned}
\mathcal{F}_2(x,y)\ \supset\ & \frac{1}{6}\int\frac{d^4q_1}{(2\pi)^4}\cdots\int\frac{d^4q_5}{(2\pi)^4}\prod_{i=1}^{5}\frac{i}{q_i^2-m^2+i\epsilon}\times e^{-iq_1x}\times e^{+iq_2y}\times\\
&\qquad\times(-i\lambda)(2\pi)^4\delta^{(4)}(q_1-q_3-q_4-q_5)\times\\
&\qquad\times(-i\lambda)(2\pi)^4\delta^{(4)}(q_3+q_4+q_5-q_2)\\
=\ & \frac{-i\lambda^2}{6}\int\frac{d^4q_1}{(2\pi)^4}\,e^{-iq_1(x-y)}\times\left(\frac{1}{q_1^2-m^2+i\epsilon}\right)^2\times\\
&\qquad\times\iint\frac{d^4q_3\,d^4q_4}{(2\pi)^8}\,\frac{1}{q_3^2-m^2+i\epsilon}\times\frac{1}{q_4^2-m^2+i\epsilon}\times\\
&\qquad\qquad\times\frac{1}{(q_5=q_1-q_3-q_4)^2-m^2+i\epsilon}
\end{aligned}
\tag{30}
$$

Note: as defined in eq. (1), the correlation functions $\mathcal{F}_n(x_1,\ldots,x_n)$ obtain by summing *all* Feynman diagrams without vacuum bubbles, *cf.* eq. (29). Both the connected and the disconnected diagrams are included, as long as each connected part of a disconnected diagram has some external vertices. However, the disconnected diagrams' contributions can be resummed in terms of correlation functions of fewer fields. Indeed, let's define the *connected correlation functions*

$$
\mathcal{F}_n^{\text{conn}}(x_1,\ldots,x_n)\ =\ \sum\begin{pmatrix}\textit{connected}\ \text{Feynman diagrams}\\ \text{with } n \text{ external vertices}\end{pmatrix}. \tag{31}
$$

Then the original $\mathcal{F}_n$ functions can be obtained from these via *cluster expansion:*

$$
\begin{aligned}
\mathcal{F}_2(x,y)\ &=\ \mathcal{F}_2^{\text{conn}}(x,y),\\
\mathcal{F}_4(x,y,x,w)\ &=\ \mathcal{F}_4^{\text{conn}}(x,y,z,w)\ +\ \mathcal{F}_2^{\text{conn}}(x,y)\times\mathcal{F}_2^{\text{conn}}(z,w)\\
&\qquad+\ \mathcal{F}_2^{\text{conn}}(x,z)\times\mathcal{F}_2^{\text{conn}}(y,w)\ +\ \mathcal{F}_2^{\text{conn}}(x,w)\times\mathcal{F}_2^{\text{conn}}(y,z),\\
\mathcal{F}_6(x,y,x,u,v,w)\ &=\ \mathcal{F}_6^{\text{conn}}(x,y,z,u,v,w)\\
&\qquad+\ \Bigl(\mathcal{F}_2^{\text{conn}}(x,y)\times\mathcal{F}_4^{\text{conn}}(z,u,v,w)\ +\ \text{permutations}\Bigr)\\
&\qquad+\ \Bigl(\mathcal{F}_2^{\text{conn}}(x,y)\times\mathcal{F}_2^{\text{conn}}(z,u)\times\mathcal{F}_2^{\text{conn}}(v,w)\ +\ \text{permutations}\Bigr),\\
&\textit{etc., etc.}
\end{aligned}
\tag{32}
$$

The *connected* 4-point, 6-point, *etc.*, correlation functions are related to the scattering amplitudes via the LSZ reduction formula — named after Harry Lehmann, Kurt Symanzik, and

Wolfhart Zimmermann, — see $7.2 of the Peskin and Schroeder textbook for the details. I shall explain the LSZ reduction formula later in class.

The 2–Point Correlation Function

Meanwhile, let us focus on the 2-point correlation function $\mathcal{F}_2(x-y)$, which is related to the renormalization of the particle mass and the strength of the quantum field. For $x^0 > y^0$ we have

$$\begin{aligned}\mathcal{F}_2(x-y) &\stackrel{\text{def}}{=} \langle\Omega|\,\mathbf{T}\hat{\Phi}_H(x)\hat{\Phi}_H(y)\,|\Omega\rangle \;=\; \langle\Omega|\,\hat{\Phi}_H(x)\hat{\Phi}_H(y)\,|\Omega\rangle \\ &= \sum_{|\Psi\rangle} \langle\Omega|\,\hat{\Phi}_H(x)\,|\Psi\rangle \times \langle\Psi|\,\hat{\Phi}_H(y)\,|\Omega\rangle\end{aligned} \tag{33}$$

where the sum is over all the quantum states $|\Psi\rangle$ of the theory. Or rather, over all quantum states which can be created by the action of the quantum field $\hat{\Phi}(y)$ on the vacuum state. In the free theory, such states would be limited to the one-particle states with different momenta, but the interacting field $\hat{\Phi}_H(y)$ may also create a three-particle state, or a five-particle state, *etc.*, *etc.* In a more general theory, the quantum states $|\Psi\rangle$ which could be created by the action of some quantum field $\hat{\varphi}(y)$ on the vacuum include all the multi-particle states which have the same net conserved quantum numbers as a single naive quantum of the field $\hat{\varphi}(y)$. For example, in QED, the states $\hat{A}^\mu(y)\,|\Omega\rangle$ created by the EM field acting on the vacuum include the one-photon states, the three-photon states, *etc.*, but also the electron-positron states — including both the un-bound two-particle states and the hydrogen atom-like bound states, — as well as the states including one or more e^-e^+ pairs and several photons. In other words, all the quantum states which can get mixed with a single-photon state by the QED interactions.

For simplicity, let me keep the states $|\Psi\rangle$ in eq. (33) completely generic. As to their quantum numbers, let me separate the net energy-momentum p^μ of all the particles involved from all the other quantum numbers which I'll denote by the lower-case ψ, thus $|\Psi\rangle = |\psi, p^\mu\rangle$. Note: for the single-particle and bound states, the spectrum of ψ is discrete, while for the un-bound multi-particle states the spectrum of ψ is continuous since ψ includes the relative velocities of the several particles. As to the spectrum of the net momentum p^μ, it spans the positive-energy mass shell for the mass which depends on ψ, thus

$$\text{any } \mathbf{p}, \quad p^0 \;=\; +\sqrt{\mathbf{p}^2 + M^2(\psi)} \tag{34}$$

where $M(\psi)$ is the *invariant mass* of the state $|\psi;p\rangle$. Altogether, in terms of the $|\Psi\rangle = |\psi;p\rangle$ eq. (33) becomes

$$\mathcal{F}_2(x-y) \;=\; \sum_\psi \int \frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E(\mathbf{p},M(\psi))} \times \langle\Omega|\,\hat\Phi_H(x)\,|\psi,p\rangle \times \langle\psi,p|\,\hat\Phi_H(y)\,|\Omega\rangle\,. \tag{35}$$

Next, consider the x–dependence of the matrix element $\langle\Omega|\,\hat\Phi_H(x)\,|\psi,p\rangle$ and the y–dependence of the $\langle\psi,p|\,\hat\Phi_H(y)\,|\Omega\rangle$. The quantum field theory has transpational symmetry in all 4 dimensions of spacetime, and the net energy-momentum operator $\hat P^\mu$ is the generator of this symmetry. In the Heisenberg picture of the theory, this means

$$\hat\Phi_H(x+a) \;=\; \exp(+ia_\mu\hat P^\mu)\,\hat\Phi_H(x)\,\exp(-ia_\mu\hat P^\mu) \tag{36}$$

and in particular

$$\hat\Phi_H(x) \;=\; \exp(+ix_\mu\hat P^\mu)\,\hat\Phi_H(0)\,\exp(-ix_\mu\hat P^\mu). \tag{37}$$

At the same time, the states $\langle\Omega|$ and $|\psi,p\rangle$ are eigenstates of the net energy-momentum operators: the vacuum $\langle\Omega|$ has $P=0$ while the state $|\psi,p\rangle$ has $P=p$. Consequently,

$$\langle\omega|\exp(ix_\mu\hat P^\mu) \;=\; \langle\Omega| \quad\text{while}\quad \exp(-ix_\mu\hat P^\mu)\,|\psi,p\rangle \;=\; e^{-ix_\mu p^\mu} \times |\psi,p\rangle\,, \tag{38}$$

and therefore

$$\langle\Omega|\,\hat\Phi_H(x)\,|\psi,p\rangle \;=\; \langle\Omega|\exp(ix_\mu\hat P^\mu)\,\hat\Phi_H(0)\,\exp(-ix_\mu\hat P^\mu)\,|\psi,p\rangle \;=\; e^{-ix_\mu p^\mu} \times \langle\Omega|\,\hat\Phi_H(0)\,|\psi,p\rangle\,. \tag{39}$$

Similarly,

$$\langle\psi,p|\,\hat\Phi_H(y)\,|\Omega\rangle \;=\; \langle\psi,p|\exp(iy_\mu\hat P^\mu)\,\hat\Phi_H(0)\,\exp(-iy_\mu\hat P^\mu)\,|\Omega\rangle \;=\; e^{+iy_\mu p^\mu} \times \langle\psi,p|\,\hat\Phi_H(0)\,|\Omega\rangle\,. \tag{40}$$

Combining these two formulae, we have

$$\begin{aligned}
\langle\Omega|\,\hat\Phi_H(x)\,|\psi,p\rangle\times\langle\psi,p|\,\hat\Phi_H(y)\,|\Omega\rangle\ &=\ e^{-ipx+ipy}\times\langle\Omega|\,\hat\Phi_H(0)\,|\psi,p\rangle\,\langle\psi,p|\,\hat\Phi_H(0)\,|\Omega\rangle\\
&=\ e^{-ip(x-y)}\times\Big|\langle\psi,p|\,\hat\Phi_H(0)\,|\Omega\rangle\Big|^2
\end{aligned}\tag{41}$$

where only the $e^{-ip(x-y)}$ factor depends on the x and y coordinates. Moreover, it's the only factor depending on the total momentum p! Indeed, the state $\hat\Phi_H(0)\,|\Omega\rangle$ is invariant under orthochronous Lorentz symmetries, hence the matrix element

$$\langle\psi,p|\,\hat\Phi_H(0)\,|\Omega\rangle\quad\text{is the same for all }\mathbf{p}\in\text{the mass shell.}\tag{42}$$

Renaming this $\mathbf{p}$-independent matrix element as simply $\langle\psi|\,\hat\Phi_H(0)\,|\Omega\rangle$, we have

$$\langle\Omega|\,\hat\Phi_H(x)\,|\psi,p\rangle\times\langle\psi,p|\,\hat\Phi_H(y)\,|\Omega\rangle\ =\ \Big|\langle\psi|\,\hat\Phi_H(0)\,|\Omega\rangle\Big|^2\times e^{-ip(x-y)}.\tag{43}$$

Consequently, eq. (35) for the two-point correlation function becomes

$$\begin{aligned}
\mathcal{F}_2(x-y)\ &=\ \sum_\psi\Big|\langle\psi|\,\hat\Phi_H(0)\,|\Omega\rangle\Big|^2\times\int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2p^0}\,e^{-ip(x-y)}\Big|^{p^0=+\sqrt{\mathbf{p}^2+M^2(\psi)}}\\
&=\ \sum_\psi\Big|\langle\psi|\,\hat\Phi_H(0)\,|\Omega\rangle\Big|^2\times D(x-y;M(\psi)).
\end{aligned}\tag{44}$$

Now remember that eq. (44) follows from eq. (33), which obtains only for $x^0>y^0$. In the opposite case of $x^0<y^0$, we have

$$\begin{aligned}
\mathcal{F}_2(x-y)\ &=\ \langle\Omega|\,\mathbf{T}\hat\Phi_H(x)\hat\Phi_H(y)\,|\Omega\rangle\ =\ \langle\Omega|\,\hat\Phi_H(y)\hat\Phi_H(x)\,|\Omega\rangle\\
&=\ \sum_\psi\int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E(\mathbf{p},M(\psi))}\,\langle\Omega|\,\hat\Phi_H(y)\,|\psi,p\rangle\times\langle\psi,p|\,\hat\Phi_H(x)\,|\Omega\rangle\,,
\end{aligned}\tag{45}$$

similar to eq. (35) but with x and y exchanging thjeir roles. Consequently, proceeding exactly as above, we obtain

$$\mathcal{F}_2(x-y) \;=\; \sum_\psi \left|\langle\psi|\,\hat{\Phi}_H(0)\,|\Omega\rangle\right|^2 \times D(y-x;M(\psi)). \tag{46}$$

Altogether, for any time order of the x^0 and the y^0, we have

$$\begin{aligned}
\mathcal{F}_2(x-y) \;&=\; \sum_\psi \left|\langle\psi|\,\hat{\Phi}_H(0)\,|\Omega\rangle\right|^2 \times \begin{cases} D(x-y;M(\psi)) & \text{for } x^0>y^0, \\ D(y-x;M(\psi)) & \text{for } x^0<y^0, \end{cases} \\
&=\; \sum_\psi \left|\langle\psi|\,\hat{\Phi}_H(0)\,|\Omega\rangle\right|^2 \times G_F(x-y;M(\psi)) \\
&=\; \sum_\psi \left|\langle\psi|\,\hat{\Phi}_H(0)\,|\Omega\rangle\right|^2 \times \int\frac{d^4p}{(2\pi)^4}\,\frac{ie^{-ip(x-y)}}{p^2-M^2(\psi)+i\epsilon}\,.
\end{aligned} \tag{47}$$

Eq. (47) is usually written as the Källén–Lehmann spectral representation:

$$\mathcal{F}_2(x-y) \;=\; \int\limits_0^\infty \frac{dm^2}{2\pi}\,\rho(m^2)\times\int\frac{d^4p}{(2\pi)^4}\,\frac{ie^{-ip(x-y)}}{p^2-m^2+i\epsilon}\,. \tag{48}$$

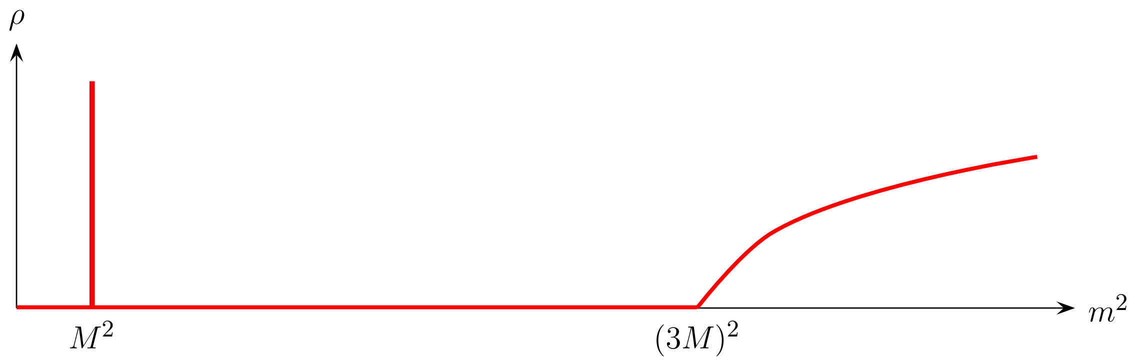
where

$$\rho(m^2) \;\stackrel{\text{def}}{=}\; \sum_\psi \left|\langle\psi|\,\hat{\Phi}_H(0)\,|\Omega\rangle\right|^2 \times (2\pi)\delta(M^2(\psi)-m^2) \tag{49}$$

is the *spectral density function.* Here are some of its key features:

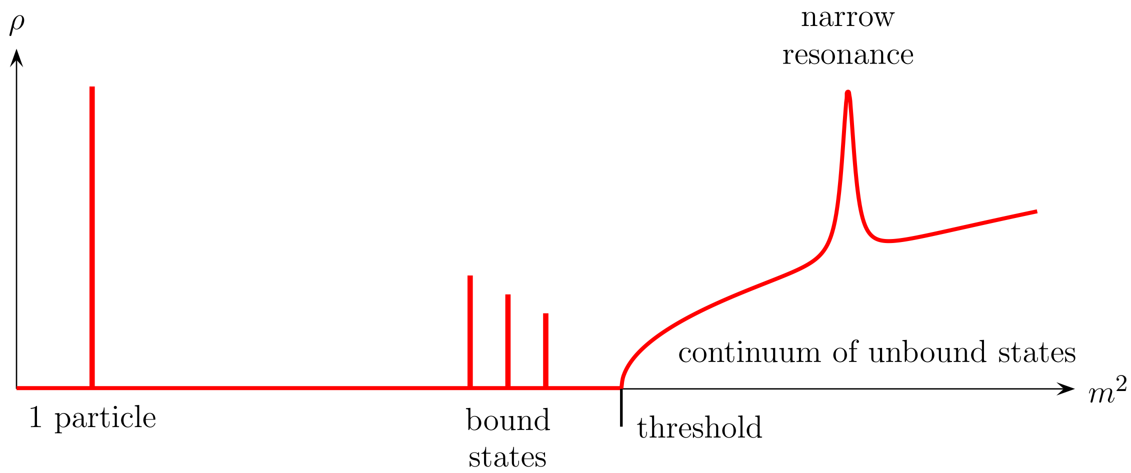
- In any QFT, for any quantum field, the spectral density function is real and non-negative, $\rho(m^2)\geq 0$ at all m^2.
- In the free field theory, $\rho(m^2)=2\pi\delta(m^2-M^2)$ where M is the particle's mass.

- In the interacting $\lambda\Phi^4$ theory, the spectral density function has both a delta-spike at $m^2 = M^2$ and a smooth continuum above the 3-particle threshold,



  ⋆ Note: the M^2 position of the delta-spike is the mass^2 of the physical particle rather than the bare mass^2 in the Feynman rules of the perturbation theory. Likewise, the continuum begins at $(3M)^2$, which is the threshold for the mass^2 for the physical 3-particle states.

- In a general quantum field theory, the spectral density functions get contributions from several kind of states: single particle, bound states of several particles, unbound states, unstable resonances, *etc.*, *etc.* The single-particle states and the bound states give rise to the delta-spikes of the spectral density function, the un-bound multi-particle states give rise to the continuum starting at the threshold (the minimal invariant mass^2 of the unbond state), while the resonances give rise to narrow peaks on top of the continuum. Schematically,



Analytic Behavior

Now let's translate all these features of the spectral density function $\rho(m^2)$ into the analytic behavior of the two-point function $\mathcal{F}_2(x-y)$, or rather of its Fourier transform

$$\mathcal{F}_2(p) \;=\; \int d^4x\, e^{ip_\mu x^\mu}\, \mathcal{F}_2(x-0) \;=\; \int\limits_0^\infty \frac{dm^2}{2\pi}\, \rho(m^2) \times \frac{i}{p^2-m^2+i\epsilon}\,. \tag{50}$$

In the $\lambda\Phi^4$ theory, the one-particle state contributes the delta-spike to the spectral density function, while the multi-particle unbound states give rise to the smooth continuum, thus

$$\rho(m^2) \;=\; Z \times 2\pi\delta(m^2 - M^2_{\text{particle}}) \;+\; \text{smooth}, \tag{51}$$

where

$$Z \;=\; \Big|\langle 1\,\text{particle}|\,\hat{\Phi}_H(0)\,|\Omega\rangle\Big|^2 \;>\; 0. \tag{52}$$

In other words, $\sqrt{Z}$ is the strength with which the quantum field $\hat{\Phi}$ creates single particles from the vacuum. In the free theory $Z=1$ but in the interacting theory it is subject to quantum corrections.

Plugging the delta-spike (51) into eq. (50) for the two-point function, we immedietaley see that it has a pole at $p^2 = M^2_{\text{particle}}$ with residue Z,

$$\mathcal{F}_2(p^2) \;=\; \frac{iZ}{p^2 - M^2_{\text{particle}} + i\epsilon} \;+\; \text{smooth}(p^2). \tag{53}$$

Conversely, if we find — from the perturbation theory, or by any other means — that the two-point function has a pole at $p^2 = M^2$ with residue Z, then the spectral density function has a delta-splike just like in eq. (51), which means that the pole position M^2 is precisely the physical mass of the particle!

In perturbation theory, the Feynman vertices use the bare coupling λ_{bare} which is different from the physical coupling λ_{phys} of the theory; likewise, the Feynman propagators use the bare mass m_{bare} wich is different from the physical mass of the particle. To relate the bare mass to the physical mass, we should use the perturbation theory to calculate the two-point correlation function $\mathcal{F}_2(p^2)$. That two-point function should have a pole, generally at $M^2_{\rm pole} \neq m^2_{\rm bare}$. It is that pole mass^2 $M^2_{\rm pole}$ which should be identified with the physical mass^2 of the particle! In other words, we should get the pole mass^2 as a perturbative expansion

$$M^2_{\rm pole} \;=\; m^2_{\rm bare} \;+\; \text{loop corrections} \;=\; f(m^2_{\rm bare}, \lambda_{\rm bare}, \Lambda_{UV}), \tag{54}$$

and then we should identify $M^2_{\rm pole} = M^2_{\rm particle}$ and solve the equation

$$f(m^2_{\rm bare}; \text{other stuff}) \;=\; M^2_{\rm particle} \tag{55}$$

for the $m^2_{\rm bare}$. Next week, I shall explain how this works in some detail.

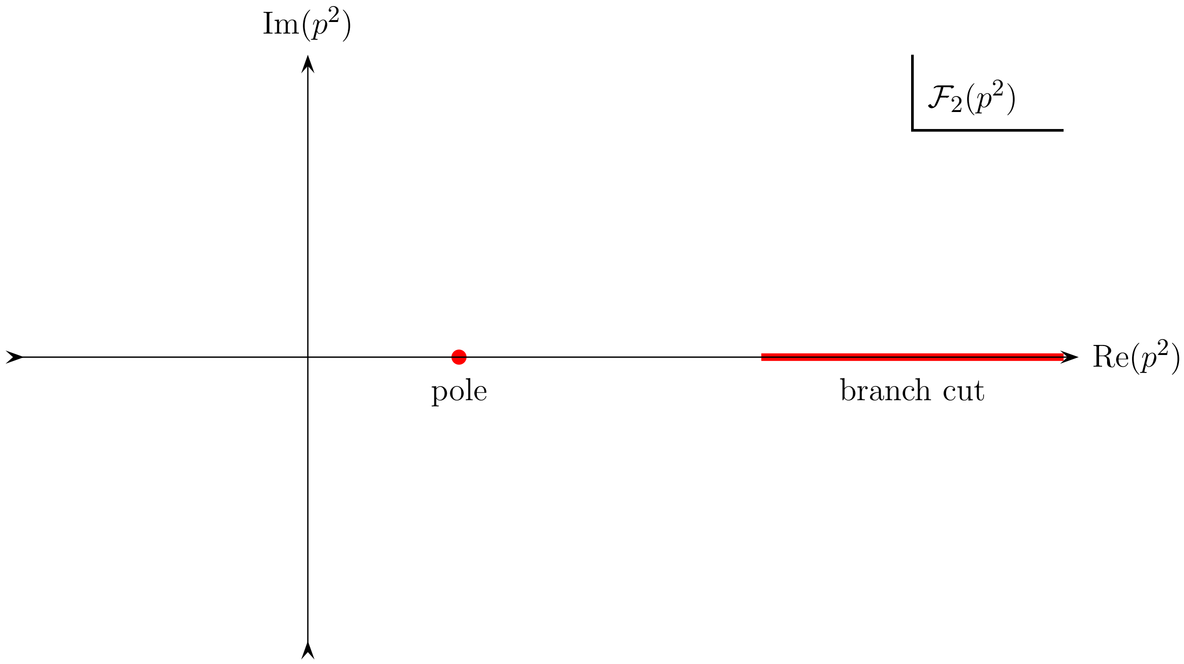
Meanwhile, let's consider the un-bound states contribution to the two-point function. In the integral

$$\mathcal{F}_2(p^2) \;=\; \int\limits_0^\infty \frac{dm^2}{2\pi}\,\rho(m^2)\times\frac{i}{p^2-m^2+i\epsilon}\,, \tag{50}$$

a smooth positive $\rho(m^2)$ above the threshold $m^2_{\rm thr} = 9M^2$ gives rise to the branch cut running from the threshold to $+\infty$ along the real axis. Indeed, for $p^2 >$ the threshold so that $\rho(m^2 = p^2)$ is positive and smooth, we have

$$\begin{aligned}\mathcal{F}_2(p^2+i\epsilon) \;-\; \mathcal{F}_2(p^2-i\epsilon) \;&=\; \frac{1}{2\pi i}\int\limits_0^\infty \frac{dm^2\,\rho(m^2)}{m^2-p^2-i\epsilon} \;-\; \frac{1}{2\pi i}\int\limits_0^\infty \frac{dm^2\,\rho(m^2)}{m^2-p^2+i\epsilon} \\ &=\; \frac{1}{2\pi i}\oint\limits_{\substack{\text{circle}\\ \text{around } p^2}} \frac{dm^2\,\rho(m^2)}{m^2-p^2} \;=\; \rho(p^2).\end{aligned} \tag{56}$$

Thus, the 2-point function has a discontinuity across the real axis between $p^2+i\epsilon$ and $p^2-i\epsilon$, which means a branch cut on its Riemann surface:



Let's take a closer look at the real axis of this Riemann surface. For the real and negative (spacelike) p^2, the two-point function, or rather the

$$i\mathcal{F}_2(p^2) \;=\; \int\limits_0^{+infty} \frac{dm^2}{2\pi}\, \frac{\rho(m^2)}{m^2 - p^2} \tag{57}$$

is real, positive, and single-valued. As we continue to the positive (timelike) p^2 but stay below the threshold, the $i\mathcal{F}_2(p^2)$ remain real and single-valued. But once we cross over the threshold, the integral (57) includes the singularity at $m^2 = p^2$, which gives rise to the branch cut. In this regime,

$$\text{(above the threshold)} \quad i\mathcal{F}_2(p^2 \pm i\epsilon) \;=\; \text{real} \;\pm\; i\,\frac{\rho(p^2)}{2}\,. \tag{58}$$

So what should we do for real p^2 above the threshold? The $i\epsilon$ in the denominator under the integral (50) gives the answer*: we should shift such real p^2 upward in the complex plane,

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⋆ The $p^2 - m^2 + i\epsilon$ in the denominator of eq. (50) stems from the similar denominator in the Källén–Lehmann representation (48), which in turn comes from the Feynman propagator $G_F(x - y; M(\psi))$ for the scalar field, *cf.* eq. (47).

$p^2 \to p^2 + i\epsilon$, and evaluate the 2-point function for $p^2 + i\epsilon$. In other words, *the physical 'bank' of the branch cut is the upper bank.*

More generally, the Riemann surface of the 2-point function $\mathcal{F}_2(p^2)$ has the physical sheet and an infinite series of the un-physical sheets. The physical sheet begins on the upper bank of the branch cut and extends counterclockwise to the negative real axis and back to the positive axis. *On this physical sheet, the $\mathcal{F}_2(p^2)$ no off-axis poles. Instead, all the poles are at real positive p^2 and correspond to physical stable particles (or bound states).*

However, the two-point function may have additional off-axis poles on the un-physical sheet of the Riemann surface beyond the branch cut. Such poles — if any — corresponds to the unstable particles or resonances. Specifically:

- First, we define the $\mathcal{F}(p^2) \stackrel{\text{def}}{=} \mathcal{F}_2(p^2 + i\epsilon)$ along the upper — physical — bank of the branch cut.

- Second, we analytically continue this function to complex p^2. For positive $\mathrm{Im}(p^2)$ this continuation takes us to the physical sheet of the Riemann surface, while for the negative $\mathrm{Im}(p^2)$ it takes us to the unphysical sheet below the branch cut.

- It is on this un-physical sheet that the two-point function may have an off-axis pole, or perhaps several poles. For example, suppose it has a pole at $p^2 = M^2 - iM\Gamma$. Mathematically, this means we start with $p^2 = M^2 + i\epsilon$, analytically continue from positive $\mathrm{Im}\, p^2$ to negative $\mathrm{Im}\, p^2$, and only then hit the pole at $\mathrm{Im}\, p^2 = -M\Gamma$.

- Suppose Γ is small so the pole on the un-physical sheet is close to the real axis. Then for the real p^2 near M^2, the two-point function is dominated by that pole,

$$\text{for } p^2 \approx M^2, \quad \mathcal{F}(p^2) \;=\; \frac{iZ}{p^2 - M^2 + iM\Gamma} \;+\; \mathrm{smooth}(p^2). \tag{59}$$

  This is the *Breit–Wigner resonance*.

- Physically, such a resonance corresponds to an un-stable particle. By the optical theorem, the rensonance's width Γ equals to the net decay rate of the unstable particle, including alll possibel decay products. In other words, $1/\Gamma$ is the average lifetime of the unstable particle.