Vacuum Energy and Effective Potentials

Quantum field theories have badly divergent vacuum energies. In perturbation theory, the leading term is the net zero-point energy

$$E_{\text{zero point}} = \sum_{\substack{\text{particle}\\ \text{species}}} \sum_{\mathbf{p},s} E_{\mathbf{p},s} \times \frac{\pm 1}{2} \tag{1}$$

where the $\pm$ sign is $+$ for bosons and $-$ for fermions, while the sub-leading terms follow from the interactions between the quantum fields. However, as long as the vacuum energy remains a constant, it does not have any observable effects besides renormalizing the cosmological constant Λ. Consequently, most of the time we do not pay any attention to the vacuum energy or its divergences.

But sometimes the vacuum energy depends on some parameters of the theory that we may vary, and then ΔE_{vacuum} acts as an effective potential for those parameters. For example, let's put the EM fields in a small cavity, which gives them a discrete spectrum of modes $(\mathbf{p}, s)$ instead of the continuum. Consequently, the EM field in the cavity has a different net zero-point energy density than in infinite space, but the difference is finite,

$$\left.\frac{E_{\text{zero point}}}{\text{volume}}\right|_{\text{cavity}} = \text{divergent\_constant} + \text{finite } \Delta\mathcal{E}(\text{cavity\_size}). \tag{2}$$

This finite difference — called *the Casimir effect* — has observable consequences such as attractive force between two parallel plates at small distances from each other,

$$F_c = -\frac{\pi^2 \hbar c}{240} \times \frac{\text{Area}}{\text{distance}^4}\,, \tag{3}$$

see Wikipedia article for details.

In these notes I would like to focus on different effect — discovered by Sidney Coleman and Eric Weinberg — in which the fields remain in infinite volume but their masses change due to interaction with a non-zero vacuum expectation value $\langle\varphi\rangle$ of some scalar field. The zero point energies of the fields depend on their masses, and the net vacuum energy density acts as an effective potential for the φ field; *it is this effective potential that the VEV $\langle\phi\rangle$ seeks to minimize.*

For example, consider a theory of two scalar fields φ and Φ with classical Lagrangian

$$\mathcal{L} \;=\; \tfrac{1}{2}(\partial_\mu\varphi)^2 \;-\; V_0(\varphi) + \tfrac{1}{2}(\partial_\mu\Phi)^2 \;-\; \frac{m_0^2}{2}\,\Phi^2 \;-\; \frac{g}{4}\,\varphi^2\Phi^2. \tag{4}$$

Suppose $V_0(\varphi) \equiv 0$ so the φ field is massless and does not interact with itself, only with the other field Φ. Classically, the vacuum expectation value $\langle\varphi\rangle$ cannot be determined, so let us allow for a completely general (but x–independent) $\langle\varphi\rangle$ and see how it affects the net zero-point energy of the system.

Since φ does not interact with itself, the shifted field $\delta\varphi(x) = \varphi(x) - \langle\varphi\rangle$ remains massless regardless of the VEV $\langle\varphi\rangle$. On the other hand, the $\langle\varphi\rangle$ does affect the mass of the other field Φ,

$$\mathcal{L} \;\supset\; -\frac{M^2}{2} \times \Phi^2 \quad \text{for} \quad M^2 \;=\; m_0^2 \;+\; \frac{g}{2} \times \langle\varphi\rangle^2\,. \tag{5}$$

Consequently, the zero-point energy of the $\delta\varphi$ field remains constant, but the zero-point energy of Φ depends on the M^2 and hence on the $\langle\varphi\rangle$, so let's calculate this energy and its $\langle\varphi\rangle$–dependence.

In a box of finite volume L^3 particle momenta are discrete and

$$E_{\text{zero point}} \;=\; +\tfrac{1}{2}\sum_{\mathbf{p}} \sqrt{M^2 + \mathbf{p}^2}\,. \tag{6}$$

For large L, the momenta $\mathbf{p}$ have a uniform density $(L/2\pi)^3$ in the momentum space, thus

$$\sum_{\mathbf{p}} \;\xrightarrow[L\to\infty]{}\; L^3 \times \int\frac{d^3\mathbf{p}}{(2\pi)^3}\,, \tag{7}$$

so the *density* of the zero-point energy obtains from the momentum integral

$$\mathcal{E} \;\stackrel{\text{def}}{=}\; \frac{E_{\text{zero point}}}{\text{volume}} \;=\; \frac{1}{2}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\sqrt{M^2 + \mathbf{p}^2}\,. \tag{8}$$

We are interested in the dependence of this energy density on the VEV $\langle\varphi\rangle$, so let's focus on the difference

$$\Delta\mathcal{E}(\langle\varphi\rangle) \;\stackrel{\text{def}}{=}\; \mathcal{E}(\langle\varphi\rangle) \;-\; \mathcal{E}(0) \;=\; \frac{1}{2}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\Big(\sqrt{M^2(\langle\varphi\rangle) + \mathbf{p}^2} \;-\; \sqrt{m_0^2 + \mathbf{p}^2}\Big). \tag{9}$$

It is this difference that acts as the effective potential for the φ field.

Before we try to evaluate the integral (9), let us re-write as a 4D momentum integral. The difference of any two energies E_1 and E_2 can be written as

$$\begin{aligned}
E_2 \;-\; E_1 \;&=\; \int\limits_{E_1^2}^{E_2^2} \frac{dE^2}{2E} \;=\; \int\limits_{E_1^2}^{E_2^2} dE^2 \times \int\limits_{-\infty}^{+\infty} \frac{dp_4}{2\pi}\, \frac{1}{E^2+p_4^2} \;=\; \int\limits_{-\infty}^{+\infty} \frac{dp_4}{2\pi} \int\limits_{E_2^2}^{E_1^2} dE^2\, \frac{1}{E^2+p_4^2} \\
&=\; \int\limits_{-\infty}^{+\infty} \frac{dp_4}{2\pi}\, \log\frac{E_2^2+p_4^2}{E_1^2+p_4^2}
\end{aligned}$$

Plugging this formula into eq. (9) gives us

$$\Delta\mathcal{E} \;=\; \frac{1}{2}\!\int\!\frac{d^3\mathbf{p}}{(2\pi)^3} \int\!\frac{dp_4}{2\pi}\, \log\frac{M^2+\mathbf{p}^2+p_4^2}{m_0^2+\mathbf{p}^2+p_4^2} \;=\; \frac{1}{2}\!\int\!\frac{d^4p_E}{(2\pi)^4}\, \log\frac{M^2+p_E^2}{m_0^2+p_E^4} \tag{10}$$

where $p_E^\mu = (\mathbf{p}, p_4)$ acts as a 4D Euclidean momentum. To clarify the physical meaning of this formula, let's expand the logarithm in the integrand into powers of

$$M^2 \;-\; m_0^2 \;=\; \frac{g\,\langle\varphi\rangle^2}{2} \tag{11}$$

and hence into powers of $\langle\varphi\rangle$,

$$\log\frac{M^2+p_E^2}{m_0^2+p_E^2} \;=\; \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \times \frac{(\frac{g}{2}\,\langle\varphi\rangle^2)^n}{(p_E^2+m_0^2)^n}\,. \tag{12}$$

Consequently,

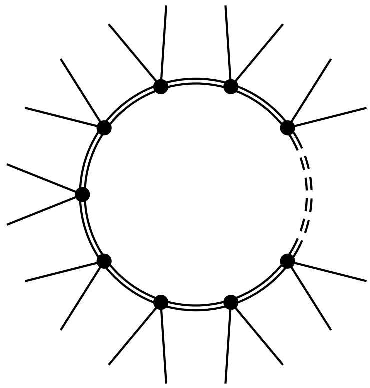
$$\begin{aligned}
\Delta\mathcal{E} \;&=\; \frac{1}{2}\!\int\!\frac{d^4p_E}{(2\pi)^4} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \times \frac{(\frac{g}{2}\,\langle\varphi\rangle^2)^n}{(p_E^2+m_0^2)^n} \\
&=\; \sum_{n=1}^{\infty} \frac{\langle\varphi\rangle^{2n}}{2n\times 2^n} \times \int\!\frac{d^4p_E}{(2\pi)^4}\, \frac{(-1)^{n-1}g^n}{(p_E^2+m_0^2)^n}
\end{aligned} \tag{13}$$

or in terms of Minkowski 4–momentum p^μ,

$$\Delta\mathcal{E} \;=\; \sum_{n=1}^{\infty} \frac{i\,\langle\varphi\rangle^{2n}}{2n\times 2^n} \times \int\!\frac{d^4p}{(2\pi)^4}\, (-ig)^n \times \left(\frac{i}{p^2-m_0^2+i0}\right)^{n}. \tag{14}$$

For each n, the momentum integral here evaluates the 1-loop Feynman diagram with $2n$

external φ legs carrying zero momenta,



$$= \int \frac{d^4p}{(2\pi)^4} (-ig)^n \times \left(\frac{i}{p^2 - m_0^2 + i0} \right)^n \tag{15}$$

where the internal propagators belong to the massive Φ field. This gives us a formula for the $\Delta\mathcal{E}$ in terms of Feynman diagrams.

Note that the combinatoric factors in eq. (14) are different from the Feynman rules for scattering amplitudes. In scattering amplitudes, all the external legs are treated as distinct, and we should add up diagrams related by non-trivial permutations of the external legs, for example

$$\frac{(2n)!}{2n \times 2^n}$$

diagrams that look line (15). But in eq. (14) the net combinatoric factor is

$$\frac{1}{2n \times 2^n}\,,$$

which corresponds to a single diagram with $(2n) \times 2^n$ symmetries; in terms of the diagram (15), this means treating the external legs as identical — especially since they all carry the same momentum $p = 0$.

Feynman Rules for the Effective Potential

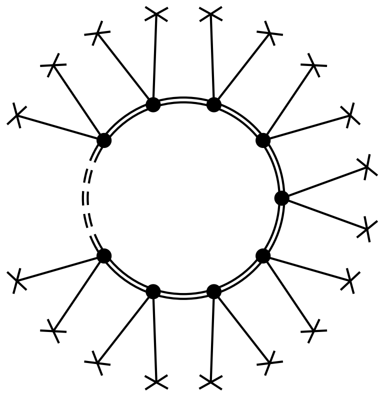
In light of the above, let me formulate the Feynman rules for the effective potentials $V_{\text{eff}}(\langle\varphi\rangle)$. Besides the usual vertices, propagators, and loops, there are also *vacuum legs*

$$\text{(vertex)} \longrightarrow\!\!\times \quad = \langle\varphi\rangle$$

Physically, the vacuum legs correspond to insertions of the scalar VEVs into the Lagrangian and hence into the amplitudes, so the vacuum legs (also called the vacuum-insertion lines) belong only to the scalar species with $\langle\varphi\rangle \neq 0$. Although graphically the vacuum legs may look like the external lines, they do not correspond to any incoming or outgoing particles, so the momentum flowing through a vacuum leg is always zero. Combinatorically, symmetries of a diagram may permute the vacuum legs belonging to VEVs of the same scalar field φ.

With these Feynman rules, eq. (14) becomes simply

$$\Delta\mathcal{E}_{\text{zero point}} \;=\; i\sum_{n=1}^{\infty} \tag{16}$$



For the general quantum field theories — which may have multiple scalar, vector, and fermionic fields, and several scalars may have non-zero VEVs — the effective potential also follows from Feynman diagrams with vacuum legs (but no other kinds of external legs),

$$V_{\text{eff}}(\langle\varphi\rangle) \;=\; i\sum \text{ all 1PI vacuum diagrams.} \tag{17}$$

Note: this sum includes diagrams with any number of loops, $L = 0, 1, 2, \ldots$.

⋆ In theories with $V_0(\varphi) \neq 0$, this expansion begins with tree diagrams. At the tree level,

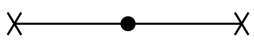
$$V^{\text{tree}}_{\text{eff}}(\langle\varphi\rangle) \;=\; \text{classical } V_0(\varphi). \tag{18}$$

Indeed, suppose the φ field has a classical potential

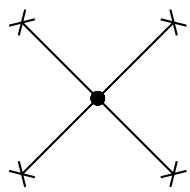
$$V_0(\varphi) \;=\; \frac{m^2_\varphi}{2}\times\varphi^2 \;+\; \frac{\lambda_\varphi}{24}\times\varphi^4, \tag{19}$$

then at the tree level

$$-iV^{\text{tree}}_{\text{eff}} \;=\;$$



$$+$$



$$= \frac{-im^2_\varphi}{2}\times\langle\varphi\rangle^2 \;+\; \frac{-i\lambda_\varphi}{24}\times\langle\varphi\rangle^4 \;\equiv\; -iV_0(\langle\varphi\rangle). \tag{20}$$

- $\star$ The one-loop vacuum diagrams such as (16) comprise the zero-point energy density $\mathcal{E}$ of the fields whose propagators run around the loop, or rather the $\Delta\mathcal{E}$ due to masses of those fields being affected by their couplings to the VEVs. However, at this level of approximation we ignore any interactions between the quantum fields except for their couplings to the $\langle\varphi\rangle$.
- $\star$ Interactions between the quantum fields lead to Hamiltonian terms of the form

$$\begin{aligned} \hat{H} \supset\ & \hat{a}\hat{a}\hat{a}, \quad \hat{a}^\dagger\hat{a}\hat{a}, \quad \hat{a}^\dagger\hat{a}^\dagger\hat{a}, \quad \hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger, \\ & \hat{a}\hat{a}\hat{a}\hat{a}, \quad \hat{a}^\dagger\hat{a}\hat{a}\hat{a}, \quad \hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}, \quad \hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\hat{a}, \quad \hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger. \end{aligned} \tag{21}$$

  These terms affect the ground state energy of the theory at second and higher orders of the perturbation theory. In the Feynman diagram language, these higher-order corrections correspond to the two-loop and multi-loop diagrams in the sum (17).

One Loop Calculation

Now that we have the general rules, lets actually calculate the one-loop effective potential for our example of a scalar field Φ whose M^2 (5) depends on the VEV of another scalar φ.

Instead of separately evaluating the one-loop diagrams with different numbers of vacuum legs, it's more convenient to add them up before integrating over the loop momentum. In other words, let's go back from the one-loop formula (16) to eq. (10) for the zero-point energy of the Φ field, thus

$$^{\rm net}V_{\rm eff}^{1\,\rm loop}(\langle\varphi\rangle) \;=\; \Delta\mathcal{E} \;=\; \frac{1}{2}\int\!\frac{d^4p_E}{(2\pi)^4}\,\log\frac{M^2+p_E^2}{m_0^2+p_E^2}\,. \tag{22}$$

The momentum integral here diverges quadratically, but we may reduce the divergence by taking derivatives WRT M^2. Indeed,

$$\begin{aligned} F(M^2) &= \int\!\frac{d^4p_E}{(2\pi)^4}\,\log\frac{M^2+p_E^2}{m_0^2+p_E^2} && \text{diverges as }\Lambda^2, \\ \frac{dF}{dM^2} &= \int\!\frac{d^4p_E}{(2\pi)^4}\,\frac{1}{M^2+p_E^2} && \text{diverges as }\Lambda^2, \\ \frac{d^2F}{(dM^2)^2} &= \int\!\frac{d^4p_E}{(2\pi)^4}\,\frac{-1}{(M^2+p_E^2)^2} && \text{diverges as }\log\Lambda, \\ \frac{d^3F}{(dM^2)^3} &= \int\!\frac{d^4p_E}{(2\pi)^4}\,\frac{2}{(M^2+p_E^2)^3} && \text{converges.} \end{aligned} \tag{23}$$

Specifically,

$$\frac{d^3F}{(dM^2)^3} \;=\; \int\!\frac{d^4p_E}{(2\pi)^4}\,\frac{2}{(M^2+p_E^2)^3} \;=\; \frac{1}{16\pi^2}\int\limits_0^\infty \frac{2p_E^2\,dp_E^2}{(M^2+p_E^2)^3} \;=\; \frac{1}{16\pi^2}\times\frac{1}{M^2} \tag{24}$$

and hence

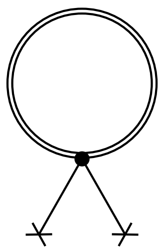
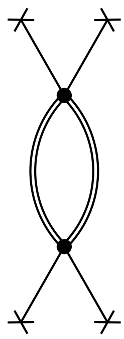
$$F(M^2) \;=\; \frac{M^4}{32\pi^2}\times\log\frac{M^2}{m_0^2} +\; A\times(M^4-m_0^2) \;+\; B\times(M^2-m_0^2) \tag{25}$$

for some divergent constants A and B. Consequently,

$$V_{\rm eff}^{1\,\rm loop}(\langle\varphi\rangle) \;=\; \frac{1}{64\pi^2}\times\left(m_0^2+\frac{g\,\langle\varphi\rangle^2}{2}\right)^2\times\log\left(m_0^2+\frac{g\,\langle\varphi\rangle^2}{2}\right)+\frac{a}{24}\times\langle\varphi\rangle^4+\frac{b}{2}\times\langle\varphi\rangle^2. \tag{26}$$

for some related constants a and b.

To calculate these constants we go back to the Feynman diagram expansion (16) and note that only the one-propagator and two-propagator diagrams suffer from the UV divergence. Thus, we identify

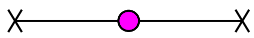
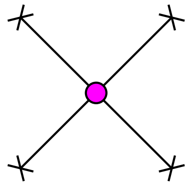
$\frac{b}{2}\times\langle\varphi\rangle^2 \;=$  and $\frac{a}{24}\times\langle\varphi\rangle^4 \;=$  (27)

and hence

$$b \;=\; \frac{g}{32\pi^2}\left(\Lambda^2\;-\;m_0^2\,\log\frac{\Lambda^2}{m_0^2}\;+\;\text{finite}\right),\qquad a\;=\;-\frac{3g^2}{32\pi^2}\left(\log\frac{\Lambda^2}{m_0^2}\;+\;\text{finite}\right). \tag{28}$$

Note that the diagrams (27) look exactly like the diagrams that renormalize the mass^2 and the self-coupling λ_φ of the field φ, so their divergences must be canceled by the counterterms

$\delta^m_{(\varphi)}$ and $\delta^\lambda_{(\varphi)}$. The same counterterms also appear in the effective potential as

$$V_{\text{eff}} \supset =$$  $$+$$  $$= \delta^m_{(\varphi)} \times \frac{\langle\varphi\rangle^2}{2} + \delta^\lambda_{(\varphi)} \times \frac{\langle\varphi\rangle^4}{24}, \quad (29)$$

so they cancel the divergences of the a and b. Working through the finite parts of the counterterms (never mind the details), we finally arrive at the *Coleman–Weinberg effective potential* for the two-scalar model.

$$\begin{aligned} V_{\text{eff}}(\langle\varphi\rangle) \ &= \ \frac{1}{64\pi^2} \times \left(m_0^2 + \frac{g\langle\varphi\rangle^2}{2}\right)^2 \times \log\left(m_0^2 + \frac{g\langle\varphi\rangle^2}{2}\right) \\ &\quad + \left(\frac{\lambda_\varphi}{24} - \frac{3g^2}{512\pi^2}\right) \times \langle\varphi\rangle^4 + \left(\frac{m_\varphi^2}{2} - \frac{gm_0^2}{128\pi^2}\right) \times \langle\varphi\rangle^2 . \end{aligned} \tag{30}$$

In a general quantum field theory, the scalar VEV $\langle\varphi\rangle$ affects the masses of many particles of different spins, including scalars, vectors, spinors, *etc.* The general form of the Coleman–Weinberg effective potentials for such theories is

$$V \ = \ \sum_{\substack{\text{massive}\\ \text{particles}}} (2S+1)(-1)^{2S} \times \frac{M^4(\langle\varphi\rangle)}{64\pi^2} \times \log\frac{M^2(\langle\varphi\rangle)}{\mu^2} \ + \ \frac{\tilde{\lambda}_\varphi\langle\varphi\rangle^4}{24} \ + \ \frac{\tilde{m}_\varphi^2\langle\varphi\rangle^2}{2} \tag{31}$$

where S is the particle's spin and $M(\langle\varphi\rangle)$ is its VEV-dependent mass. Also, $\tilde{\lambda}_\varphi$ is the classical self-coupling of the scalar field φ plus a finite $O(g^2)$ correction, and likewise for the scalar mass $\tilde{m}_\varphi^2$. Finally, μ is an arbitrary mass scale for measuring particles' masses; a redefinition of μ can be canceled by a suitable redefinition of the $\tilde{\lambda}_\varphi$.

Coleman–Weinberg Effect

In the ground state of a quantum field theory, the vacuum expectation values $\langle\varphi\rangle$ of scalar fields minimize the net energy density of the theory. Presumably there are no field gradients in the vacuum states, so the net energy density is the *effective potential* $V_{\text{eff}}(\langle\varphi\rangle)$, including the classical potential as well as the quantum corrections from the loop diagrams. And sometimes, the minima of this effective potential has different symmetries than what we would expect from the classical potential.

As an example, consider a massless charged scalar field ϕ coupled to the EM field A^μ,

$$\mathcal{L}_{\text{phys}} \;=\; -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \;+\; D_\mu\phi^* D^\mu\phi \;-\; \frac{\lambda}{4}\,\phi^{*2}\phi^2. \tag{32}$$

The classical scalar potential $V_{\text{cl}} = \frac{\lambda}{4}(\phi^*\phi)^2$ has a unique minimum at $\phi = 0$, so we expect the $U(1)$ gauge symmetry of the theory to remain unbroken. However, we shall see in a moment that the Coleman–Weinberg effective potential can lead to $\langle\phi\rangle \neq 0$ and hence to Higgsing of the gauge symmetry.

To see how this works, suppose that ϕ somehow acquires a non-zero VEV $\langle\phi\rangle = h/\sqrt{2}$. Consequently, the photon gets a mass $M_\gamma = h \times e$, the imaginary component of the complex scalar field becomes the longitudinal polarization of the photon, while the real part of the scalar field gives rise to the physical Higgs particle of mass $M_H = h \times \sqrt{\lambda/2}$. Plugging these masses into the Coleman–Weinberg effective potential, we get

$$V(h) \;=\; \frac{\lambda h^4}{16} \;+\; 3 \times \frac{e^4h^4}{64\pi^2} \times \log\frac{(eh)^2}{\mu^2} \;+\; \frac{(\lambda^2/4)h^4}{64\pi^2} \times \log\frac{(\lambda/2)h^2}{\mu^2}\,. \tag{33}$$

Now suppose $\lambda \sim e^4 \ll e^2$. In this case, $M_\gamma \gg M_H$ and the dominant contribution to the Coleman–Weinberg potential comes form the photon's mass. Consequently,

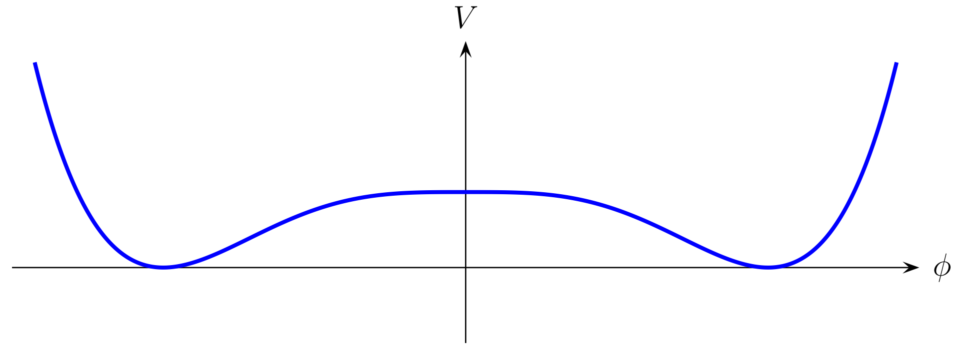
$$V(h) \;\approx\; \frac{\lambda h^4}{16} \;+\; 3 \times \frac{e^4h^4}{64\pi^2} \times \log\frac{(eh)^2}{\mu^2} \;=\; \frac{3\alpha^2}{4} \times h^4 \times \left(\log\frac{e^2h^2}{\mu^2} \;+\; \frac{\lambda}{12\alpha^2}\right), \tag{34}$$

which we may rewrite as

$$V(h) \;\approx\; \frac{3\alpha^2}{4} \times h^4 \times \left(\log\frac{h^2}{v^2} \;-\; \frac{1}{2}\right) \tag{35}$$

$$\text{for} \quad v \;=\; \frac{\mu}{e} \times \exp\left(-\frac{1}{4} \;-\; \frac{\lambda}{24\alpha^2}\right). \tag{36}$$

The pictures below show this potential a function of complex $\langle\phi\rangle$. Here is the profile along the real-ϕ axis:



Note *local maximum* instead of minimum at $\phi = 0$, while the minima lie at $\phi = \pm v/\sqrt{2}$. In the complex ϕ plane, the there is a whole ring of minima at

$$|\,\langle\phi\rangle\,|^2 \;=\; \frac{v^2}{2} \;>\; 0. \tag{37}$$

Indeed, here is the 3D picture of $V(\phi)$: