# Expansion of Free Relativistic Fields into Creation and Annihilation Operators

Similar to the scalar field I have discussed in class, any free relativistic field $\widehat{\Psi}_\aleph(x)$ — where $\aleph$ stands for a vector, tensor, or spinor index or milti-index — can be expanded into creation and annihilation operators:

$$\widehat{\Psi}_\aleph(x) \;=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_\mathbf{k}}\sum_\lambda \Bigl(e^{-ikx}U_\aleph(\mathbf{k},\lambda)\,\hat a_{\mathbf{k},\lambda} \;+\; e^{+ikx}V_\aleph(\mathbf{k},\lambda)\,\hat a^\dagger_{\mathbf{k},\lambda}\Bigr)^{k^0=+\omega_\mathbf{k}} \tag{1}$$

for a real (hermitian) quantum field, or

$$\begin{aligned} \widehat{\Psi}_\aleph(x) \;&=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_\mathbf{k}}\sum_\lambda \Bigl(e^{-ikx}U_\aleph(\mathbf{k},\lambda)\,\hat a_{\mathbf{k},\lambda} \;+\; e^{+ikx}V_\aleph(\mathbf{k},\lambda)\,\hat b^\dagger_{\mathbf{k},\lambda}\Bigr)^{k^0=+\omega_\mathbf{k}}, \\ \widehat{\Psi}^\dagger_\aleph(x) \;&=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_\mathbf{k}}\sum_\lambda \Bigl(e^{-ikx}V^*_\aleph(\mathbf{k},\lambda)\,\hat b_{\mathbf{k},\lambda} \;+\; e^{+ikx}U^*_\aleph(\mathbf{k},\lambda)\,\hat a^\dagger_{\mathbf{k},\lambda}\Bigr)^{k^0=+\omega_\mathbf{k}} \end{aligned} \tag{2}$$

for a complex field and its hermitian conjugate. In all these formulae:

- The quantum fields in the Heisenberg picture of QM $\Longrightarrow$ time-dependent, but the creation/annihilation operators $\hat a_{\mathbf{k},\lambda}$, $\hat a^\dagger_{\mathbf{k},\lambda}$, *etc.*, are in the Schrödinger picture.
- $kx \equiv k_\mu x^\mu = \omega_k t - \mathbf{k}\cdot\mathbf{x}$ for $\omega_k = +\sqrt{\mathbf{k}^2+m^2}$.
- The $U_\aleph(\mathbf{k},\lambda)$ and $V_\aleph(\mathbf{k},\lambda)$ are the coefficients of the plane-wave plane-wave solutions of the classical field equations,

$$\Psi_\aleph(x) \;=\; e^{-ikx}\times U_\aleph(\mathbf{k},\lambda) \quad\text{and}\quad \Psi_\aleph(x) \;=\; e^{+ikx}\times V_\aleph(\mathbf{k},\lambda) \quad\text{for } k^0 = +\omega_\mathbf{k}\,, \tag{3}$$

  where $\lambda$ labels the *polarizations* — *i.e.*, independent solutions for the same $k^\mu$.
- For the bosonic fields

$$\bigl[\hat a_{\mathbf{k},\lambda},\hat a^\dagger_{\mathbf{k}',\lambda'}\bigr] \;=\; \bigl[\hat b_{\mathbf{k},\lambda},\hat b^\dagger_{\mathbf{k}',\lambda'}\bigr] \;=\; \delta_{\lambda\lambda'}\times 2\omega_\mathbf{k}(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}') \tag{4}$$

  while all other pairs of creation or annihilation operators commute with each other. For the fermionic fields

$$\bigl\{\hat a_{\mathbf{k},\lambda},\hat a^\dagger_{\mathbf{k}',\lambda'}\bigr\} \;=\; \bigl\{\hat b_{\mathbf{k},\lambda},\hat b^\dagger_{\mathbf{k}',\lambda'}\bigr\} \;=\; \delta_{\lambda\lambda'}\times 2\omega_\mathbf{k}(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}') \tag{5}$$

  while all other pairs of creation or annihilation operators anti-commute with each other.