Functional Quantization

In quantum mechanics of one or several particles, we may use path integrals to calculate the transition matrix elements as

$$\langle \text{out}| \hat{U}(t_{\text{out}} - t_{\text{in}}) |\text{in}\rangle \; = \; \iint \mathcal{D}[\text{all } x_i(t)] \, \exp(iS[\text{all } x_i(t)]) \times \Psi^*_{\text{out}}(\text{all } x_i \, @t_{\text{out}}) \, \Psi_{\text{in}}((\text{all } x_i \, @t_{\text{in}}) \tag{1}$$

where the paths $x_i(t)$ are independent of each other and are not subject to any boundary conditions, and the action depends on all particles' paths as

$$S[\text{all } x_i(t)] \; = \; \int\limits_{t_{\text{in}}}^{t_{\text{out}}} L \, dt, \qquad L \; = \; \sum_i \frac{m_i}{2} \dot{x}_i^2 \; - \; V(\text{all } x_i). \tag{2}$$

Let's generalize formula (1) to the quantum field theory.

In classical field theory, a finite set of dynamical variables x_i becomes a field $\Phi(\mathbf{x})$ where $\mathbf{x}$ is now a continuous label rather than a dynamical variable of its own. Thus, a set of all the particles' paths $x_i(t)$ becomes a *field configuration* $\Phi(\mathbf{x}, t)$, with action functional

$$S[\Phi(\mathbf{x}, t)] \; = \; \int\limits_{t_{\text{in}}}^{t_{\text{out}}} dt \int d^3\mathbf{x} \, \mathcal{L} \tag{3}$$

where the Lagrangian density is

$$\mathcal{L} \; = \; \tfrac{1}{2}\partial_\mu \Phi \partial^\mu \Phi \; - \; \frac{m^2}{2} \Phi^2 \; - \; \frac{\lambda}{24} \Phi^4 \tag{4}$$

for a scalar field, or something more complicated for other kinds of fields.

In the quantum field theory, the path integral

$$\iint \mathcal{D}[\text{all} x_i(t)] \; \equiv \; \prod_i \iint \mathcal{D}[x_i(t)] \tag{5}$$

generalizes to the *functional integral over the field configurations*

$$\iint \mathcal{D}[\Phi(\mathbf{x}, t)] \tag{6}$$

A careful definition of this functional integral involves discretization of both time and space

to a 4D lattice of $N = TV/a^4$ spacetime points x_n and then taking a zero-lattice-spacing limit

$$\iint \mathcal{D}[\Phi(\mathbf{x}, t)] = \lim_{a \to 0} \text{Normalization\_factor} \times \prod_n \int d\Phi(x_n). \tag{7}$$

I shall address this issue later in class, perhaps next lecture. For now, let's focus on using the functional integral.

The QFT analogue of eq. (1) for the transition matrix element is

$$\langle \text{out}| \hat{U}(t_{\text{out}} - t_{\text{in}}) |\text{in}\rangle = \iint \mathcal{D}[\Phi(\mathbf{x}, t)] \exp(iS[\Phi(\mathbf{x}, t)]) \times \Psi_{\text{out}}^*[\Phi(\mathbf{x}) @ t_{\text{out}}] \Psi_{\text{in}}[\Phi(\mathbf{x}) @ t_{\text{in}}]. \tag{8}$$

Note that in the field theory's Hilbert space, the wave-functions Ψ_{in} and Ψ_{out} depend on the entire 3-space field configuration $\Phi(\mathbf{x})$ for a fixed time $t = t_{\text{in}}$ or $t = t_{\text{out}}$. In terms of such wave-functions, the $\langle \text{out}|\text{in}\rangle$ bracket is given by a 3-space analogue of the function integrals over $\Phi(\mathbf{x})$ at a fixed time,

$$\langle \text{out}|\text{in}\rangle = \iint \mathcal{D}[\Phi(\mathbf{x})] \Psi_{\text{out}}^*[\Phi(\mathbf{x})] \times \Psi_{\text{in}}[\Phi(\mathbf{x})]. \tag{9}$$

Now, let's go to the Heisenberg picture and consider the matrix element of the quantum field $\hat{\Phi}_H(\mathbf{x}_1, t_1)$ at some point $x_1^\mu = (\mathbf{x}_1, t_1)$. The states and the operators in Heisenberg and Schrödinger pictures are related to each other via the evolution operator $\hat{U}(t_2 - t_1)$, hence

$$\langle \text{out}|\hat{\Phi}_H(\mathbf{x}_1, t_1)|\text{in}\rangle =$$

$$= \langle \text{out}| \hat{U}(t_{\text{out}} - t_1)\hat{\Phi}_S(\mathbf{x}_1)\hat{U}(t_1 - t_{\text{in}}) |\text{in}\rangle$$

$$= \iint \mathcal{D}[\Phi_1(\mathbf{x})] \langle \text{out}| \hat{U}(t_{\text{out}} - t_1) |\Phi_1\rangle \times \Phi_1(\mathbf{x}_1) \times \langle \Phi_1| \hat{U}(t_1 - t_{\text{in}} |\text{in}\rangle$$

$$= \iint \mathcal{D}[\Phi_1(\mathbf{x})] \Phi_1(\mathbf{x}) \times$$

$$\times \int_{\Phi(\mathbf{x}, t_1) = \Phi_1(\mathbf{x})} \mathcal{D}[\Phi(x, t) \text{ for } t_1 < t < t_{\text{out}}] \Psi_{\text{out}}^*[\Phi(\mathbf{x}) @ t_{\text{out}}] \times \exp\big(iS[\Phi] \text{ from } t_1 \text{ to } t_{\text{out}}\big) \times$$

$$\times \int_{\Phi(\mathbf{x}, t_1) = \Phi_1(\mathbf{x})}^{\Phi(\mathbf{x}, t_1) = \Phi_1(\mathbf{x})} \mathcal{D}[\Phi(x, t) \text{ for } t_{\text{in}} < t < t_1] \exp\big(iS[\Phi] \text{ from } t_{\text{in}} \text{ to } t_1\big) \times \Psi_{\text{in}}[\Phi(\mathbf{x}) @ t_{\text{in}}]$$

$$= \iint \mathcal{D}[\Phi(\mathbf{x}, t) \text{ for the whole time}] \Psi_{\text{out}}^*[\Phi(\mathbf{x}) @ t_{\text{out}}] \Psi_{\text{in}}[\Phi(\mathbf{x}) @ t_{\text{in}}] \times$$

$$\times \exp\big(iS[\Phi] \text{ for the whole time}\big) \times \Phi(\mathbf{x}_1, t_1). \tag{10}$$

Likewise, matrix elements of *time-ordered* products of several fields $\hat{\Phi}_H(\mathbf{x}_n, t_n) \cdots \hat{\Phi}_H(\mathbf{x}_1, t_1)$ for $t_n > \cdots > t_1$ are given by functional integrals

$$\langle \text{out}| \, \mathbf{T} \hat{\Phi}_H(\mathbf{x}_n, t_n) \cdots \hat{\Phi}_H(\mathbf{x}_1, t_1) \, |\text{in}\rangle \; =$$
$$= \iint \mathcal{D}[\Phi(\mathbf{x}, t) \text{ for the whole time}] \; \Psi_{\text{out}}^*[\Phi(\mathbf{x}) \, @ t_{\text{out}}] \; \Psi_{\text{in}}[\Phi(\mathbf{x}) \, @ t_{\text{in}}] \times \qquad (11)$$
$$\times \exp\big(iS[\Phi] \text{ for the whole time}\big) \times \Phi(\mathbf{x}_n, t_n) \times \cdots \times \Phi(\mathbf{x}_1, t_1).$$

Now, let's take the time interval between the in and out states to infinity in both directions, or rather let's take the limit

$$t_{\text{out}} \; \to \; +\infty \times (1 - i\epsilon), \quad t_{\text{in}} \; \to \; -\infty \times (1 - i\epsilon). \qquad (12)$$

As we saw back in January (*cf.* class notes), in this limit

$$\langle \text{out}| \, e^{-i\hat{H}t_{\text{out}}} \; \to \; \langle \Omega| \times \text{c-number factor},$$
$$e^{+i\hat{H}t_{\text{in}}} \, |\text{in}\rangle \; \to \; |\Omega\rangle \times \text{c-number factor}, \qquad (13)$$

where $|\Omega\rangle$ is the vacuum state of the quantum field theory. Therefore, we may obtain the *correlation functions of n quantum fields*

$$G_n(x_1, x_2, \ldots, x_n) \; = \; \langle \Omega| \, \mathbf{T} \hat{\Phi}_H(x_1) \cdots \hat{\Phi}_H(x_n) \, |\Omega\rangle \qquad (14)$$

from the functional integrals

$$G_n(x_1, x_2, \ldots, x_n) \; = \; \text{const} \times \iint \mathcal{D}[\Phi(x)] \, \exp\left(iS[\Phi] = i\int d^4x \, \mathcal{L}\right) \times \Phi(x_1) \times \cdots \times \Phi(x_n).$$
$$(15)$$

Note that the action integral $S = \int d^4x \, \mathcal{L}$ for each field configuration $\Phi(x)$ is taken over the whole Minkowski space time, so it's Lorentz invariant, and the correlations functions (15) are also Lorentz invariant.[⋆]

⋆ Strictly speaking, the limit (12) implies that all the time coordinates x^0 of all fields in (15) live on the slightly tilted time axis that runs from $-\infty \times (1 - i\epsilon)$ to $+\infty \times (1 - i\epsilon)$. Consequently, the action has a complexified version of the Lorentz symmetry, but after analytic continuation of all the correlation functions back to the real axis, we recover the usual Lorentz symmetry $SO(3, 1)$. Alternatively, we may analytically continue to the imaginary times; this gives us $SO(4)$ symmetry in the Euclidean spacetime.

We do not know the constant factor in front of the functional integral (15) and we do not know the overall normalization of the functional integral. But we can get rid of both of these unknown factors by taking ratios of correlation functions. In particular, using

$$G_0() \;=\; \langle\Omega|\Omega\rangle \;=\; 1 \tag{16}$$

we can write all the other correlation functions as normalization-independent ratios

$$G_n(x_1, x_2, \ldots, x_n) \;=\; \frac{\langle\Omega|\, \mathbf{T}\hat{\Phi}_H(x_1)\cdots\hat{\Phi}_H(x_n)\,|\Omega\rangle}{\langle\Omega|\Omega\rangle}$$

$$=\; \frac{\displaystyle\iint\mathcal{D}[\Phi(x)]\,\exp\left(i\int d^4x\,\mathcal{L}\right)\times\Phi(x_1)\times\cdots\times\Phi(x_n)}{\displaystyle\iint\mathcal{D}[\Phi(x)]\,\exp\left(i\int d^4x\,\mathcal{L}\right)}\,. \tag{17}$$

In a few pages, we shall learn how to calculate such ratios in perturbation theory and re-derive the Feynman rules. But before we do the perturbation theory, let's see how functional integrals work for the free scalar field.

Functional Integrals for the Free Scalar Field

The free scalar field has a quadratic action functional

$$S[\Phi(x)] \;=\; \int d^4x\left(\mathcal{L} \;=\; \tfrac12\partial_\mu\Phi\partial^\mu\Phi \;-\; \tfrac12 m^2\Phi^2\right) \;=\; -\frac12\int d^4x\,\Phi(x)\,(\partial^2 + m^2)\Phi(x). \tag{18}$$

Hence, the functional integral

$$Z \;=\; \iint\mathcal{D}[\Phi(x)]\,\exp\bigl(iS[\Phi(x)]\bigr) \tag{19}$$

is a kind of a Gaussian integral, generalization of the ordinary Gaussian integral

$$Z(A) \;=\; \int d\xi_1\int d\xi_2\cdots\int d\xi_N\,\exp\left(-\frac12\sum_{j,k=1}^N A_{jk}\xi_j\xi_k\right) \tag{20}$$

to a continuous family of variables, $\{\xi_j\}\to\mathcal{D}[\Phi(x)]$.

Lemma 1: The ordinary Gaussian integral (20) evaluates to

$$Z(A) = \frac{(2\pi)^{N/2}}{\sqrt{\det(A)}}.$$ (21)

Note: A_{jk} is a symmetric $N \times N$ matrix, real or complex. To make the integral (20) converge, the real part $\mathrm{Re}\, A_{jk}$ of the A matrix should be positive definite.

Proof: Any symmetric matrix can be diagonalized, thus

$$\sum_{jk} A_{jk}\xi_j\xi_k = \sum_{k} B_k\eta_k^2$$ (22)

where $\eta_1, \ldots, \eta_N$ are some independent linear combinations of the $\xi_1, \ldots, \xi_N$. Changing integration variables from the ξ_k to η_k carries a Jacobian

$$J = \left| \det\left(\frac{\partial \xi_j}{\partial \eta_k} \right) \right|;$$ (23)

by linearity of the $\xi \to \eta$ transform, this Jacobian is constant. Therefore,

$$
\begin{aligned}
Z &= J \times \int d^N \eta \exp\left(-\frac{1}{2} \sum_k B_k \eta_k^2 \right) \\
&= J \times \prod_{k=1}^{N} \int d\eta_k \exp\left(-\frac{B_k}{2} \eta_k^2 \right) \\
&= J \times \prod_{k=1}^{N} \sqrt{\frac{2\pi}{B_k}} = \frac{(2\pi)^{N/2} \times J}{\sqrt{\det(B)}}
\end{aligned}
$$ (24)

where B is the diagonal matrix $B_{jk} = \delta_{jk} \times B_k$. This matrix is related to A according to

$$B = \left(\frac{\partial \xi}{\partial \eta} \right)^\top A \left(\frac{\partial \xi}{\partial \eta} \right),$$ (25)

hence

$$\det(B) = \det(A) \times \det^2\left(\frac{\partial \xi}{\partial \eta} \right) = \det(A) \times J^2$$ (26)

and therefore

$$Z = (2\pi)^{N/2} \times \frac{J}{\sqrt{\det(B)}} = \frac{(2\pi)^{N/2}}{\sqrt{\det(A)}}.$$ (21)

Lemma 2 concerns the ratios of Gaussian integrals for the same matrix A:

$$\frac{\int d^N\xi \, \exp\left(-\frac{1}{2}\sum_{ij} A_{ij}\xi_i\xi_j\right) \times \xi_k\xi_\ell}{\int d^N\xi \, \exp\left(-\frac{1}{2}\sum_{ij} A_{ij}\xi_i\xi_j\right)} \;=\; \left(A^{-1}\right)_{k\ell}. \tag{27}$$

Proof:

$$\int d^N\xi \exp\left(-\frac{1}{2}\sum_{ij} A_{ij}\xi_i\xi_j\right) \times \xi_k\xi_\ell \;=\; -2\frac{\partial}{\partial A_{k\ell}}\int d^N\xi \exp\left(-\frac{1}{2}\sum_{ij} A_{ij}\xi_i\xi_j\right), \tag{28}$$

hence

$$
\begin{aligned}
\frac{\int d^N\xi \, \exp\left(-\frac{1}{2}\sum_{ij} A_{ij}\xi_i\xi_j\right) \times \xi_k\xi_\ell}{\int d^N\xi \, \exp\left(-\frac{1}{2}\sum_{ij} A_{ij}\xi_i\xi_j\right)} \;&=\; -2\frac{\partial}{\partial A_{k\ell}}\log\left[\int d^N\xi \exp\left(-\frac{1}{2}\sum_{ij} A_{ij}\xi_i\xi_j\right)\right] \\
&=\; -2\frac{\partial}{\partial A_{k\ell}}\log\frac{(2\pi)^{N/2}}{\sqrt{\det(A)}} \\
&=\; +\frac{\partial}{\partial A_{k\ell}}\log\det(A) \\
&=\; \left(A^{-1}\right)_{k\ell}.
\end{aligned}
\tag{29}
$$

Both lemmas apply to the Gaussian functional integrals such as (19). For example, consider the free scalar propagator $G_2(x,y)$. Eq. (17) gives us a formula for this propagator in terms of a ratio of two Gaussian functional integrals

$$G_2(x,y) \;=\; \frac{\iiint \mathcal{D}[\Phi(x)] \, \exp\left(-\frac{i}{2}\int d^4x \, \Phi(\partial^2+m^2)\Phi\right) \times \Phi(x)\Phi(y)}{\iiint \mathcal{D}[\Phi(x)] \, \exp\left(-\frac{i}{2}\int d^4x \, \Phi(\partial^2+m^2)\Phi\right)}. \tag{30}$$

The role of the matrix A here is played by the differential operator $i(\partial^2+m^2)$. This operator is purely anti-hermitian, which corresponds to a purely imaginary matrix A. To make the Gaussian integrals converge, we add an infinitesimal real part, $A_{ij} \to A_{ij} + \epsilon\delta_{ij}$, or $i(\partial^2 + m^2) \to i(\partial^2+m^2) + \epsilon$. Consequently, lemma 2 tells us that

$$G_2(x,y) \;=\; \langle x|\frac{1}{i(\partial^2+m^2)+\epsilon}|y\rangle \;=\; \int \frac{d^4p}{(2\pi)^4}\frac{ie^{-ip(x-y)}}{p^2-m^2+i\epsilon}. \tag{31}$$

Note that the rule $A \to A+\epsilon$ we use to make the Gaussina integral converge for an imaginary (or anhti-hermitian) matrix A immediately leads us to the Feynman propagator (31) rather than some other Green's function.

To obtain correlation functions G_n for $n > 2$ we need analogues of Lemma 2 for Gaussian integrals involving more than two ξs outside the exponential. For example, for four ξ's we have

$$\frac{\int d^N \xi \, \exp\left(-\frac{1}{2}\sum_{ij} A_{ij}\xi_i\xi_j\right) \times \xi_k\xi_\ell\xi_m\xi_n}{\int d^N \xi \, \exp\left(-\frac{1}{2}\sum_{ij} A_{ij}\xi_i\xi_j\right)} \; = \tag{32}$$

$$= \; \left(A^{-1}\right)_{k\ell} \times \left(A^{-1}\right)_{mn} \; + \; \left(A^{-1}\right)_{km} \times \left(A^{-1}\right)_{\ell n} \; + \; \left(A^{-1}\right)_{kn} \times \left(A^{-1}\right)_{\ell m}.$$

Applying this formula to the Gaussian functional integral for the free scalar fields, we obtain for the 4–point function

$$G_4(x,y,z,w) \; = \; G_2(x-y) \times G_2(z-w) \; + \; G_2(x-z) \times G_2(y-w)$$
$$+ \; G_2(x-w) \times G_2(y-z)$$

$$= \; \left(\begin{array}{c}\bullet\!\!-\!\!\bullet \\ \bullet\!\!-\!\!\bullet\end{array}\right) \; + \; \left(\begin{array}{cc}\bullet & \bullet \\ | & | \\ \bullet & \bullet\end{array}\right) \; + \; \left(\bigtimes\right). \tag{33}$$

Likewise, for higher even numbers of ξs,

$$\frac{\int d^N \xi \, \exp\left(-\frac{1}{2}\sum_{ij} A_{ij}\xi_i\xi_j\right) \times \xi_{(1)}\cdots\xi_{(N)}}{\int d^N \xi \, \exp\left(-\frac{1}{2}\sum_{ij} A_{ij}\xi_i\xi_j\right)} \; = \; \sum_{\text{pairings}}\left(\prod_{\text{pairs}} \left(A^{-1}\right)_{\text{index pair}}\right) \tag{34}$$

and similarly

$$G_N^{\text{free}}(x_1,\dots,x_N) \; = \; \frac{\iiint \mathcal{D}[\Phi(x)] \, \exp\left(-\frac{i}{2}\int d^4x \, \Phi(\partial^2 + m^2)\Phi\right) \times \Phi(x_1)\cdots\Phi(x_N)}{\iiint \mathcal{D}[\Phi(x)] \, \exp\left(-\frac{i}{2}\int d^4x \, \Phi(\partial^2 + m^2)\Phi\right)}$$

$$= \; \sum_{\text{pairings}}\left(\prod_{\text{pairs}} \bullet\!\!-\!\!\bullet\right). \tag{35}$$

Feynman Rules for the Interacting Field

A self-interacting scalar field has action

$$S[\Phi(x)] \;=\; S_{\text{free}}[\Phi(x)] \;-\; \frac{\lambda}{4!}\int d^4x\,\Phi^4(x). \tag{36}$$

Expanding *the exponential* e^{iS} in powers of the coupling λ, we have

$$\exp\big(iS[\Phi(x)]\big) \;=\; \exp\big(iS_{\text{free}}[\Phi(x)]\big) \times \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-i\lambda}{4!}\int d^4x\,\Phi^4(x)\right)^n. \tag{37}$$

This expansion gives us the perturbation theory for the functional integral of the interacting field,

$$\int\!\!\!\int \mathcal{D}[\Phi(x)]\,\exp\big(iS[\Phi(x)]\big) \times \Phi(x_1)\cdots\Phi(x_k) \;=\;$$

$$= \int\!\!\!\int \mathcal{D}[\Phi(x)]\,\exp\left(iS_{\text{free}}[\Phi(x)] = \frac{i}{2}\int d^4x\,\Phi(\partial^2+m^2)\Phi\right) \times$$

$$\times\,\Phi(x_1)\cdots\Phi(x_k)\times\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-i\lambda}{4!}\int d^4x\,\Phi^4(x)\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{(-i\lambda)^n}{n!\,(4!)^n}\int d^4z_1\cdots\int d^4z_n$$

$$\int\!\!\!\int \mathcal{D}[\Phi(x)]\,\exp\left(\frac{i}{2}\int d^4x\,\Phi(\partial^2+m^2)\Phi\right)\times\Phi(x_1)\cdots\Phi(x_k)\times\Phi^4(z_1)\cdots\Phi^4(z_n)$$

$$\langle\!\langle\text{using eq. (35) for the free-field functional integrals}\rangle\!\rangle$$

$$= \left[\int\!\!\!\int \mathcal{D}[\Phi(x)]\,\exp\left(\frac{i}{2}\int d^4x\,\Phi(\partial^2+m^2)\Phi\right)\right]\times$$

$$\times\sum_{n=0}^{\infty}\frac{(-i\lambda)^n}{n!\,(4!)^n}\int d^4z_1\cdots\int d^4z_n\sum_{\text{pairings}}\left(\text{products of }\frac{k+4n}{2}\text{ free propagators}\right)$$

$$= \left[\int\!\!\!\int \mathcal{D}[\Phi(x)]\,\exp\left(\frac{i}{2}\int d^4x\,\Phi(\partial^2+m^2)\Phi\right)\right]\times$$

$$\times\sum_{n=0}^{\infty}\sum\Big(\text{Feynman diagrams with }k\text{ external and }n\text{ internal vertices}\Big). \tag{38}$$

This is how the Feynman rules emerge from the functional integral formalism.

At this stage, we are summing over all the Feynman diagrams, connected or disconnected, and even the vacuum bubbles are allowed. Also, the functional integral (38) carries an overall factor

$$Z_0 \;=\; \iint \mathcal{D}[\Phi(x)] \, \exp\left(\frac{i}{2}\int d^4x \, \Phi(\partial^2 + m^2)\Phi\right) \tag{39}$$

which multiplies the whole perturbation series. Fortunately, this pesky factor — as well as all the vacuum bubbles — cancel out when we divide eq. (38) by a similar functional integral for $k = 0$ to get the correlation function G_k. Indeed,

$$
\begin{aligned}
G_k(x_1, \ldots, x_k) \;&=\; \frac{\iint \mathcal{D}[\Phi(x)] \, \exp\big(iS[\Phi(x)]\big) \times \Phi(x_1) \cdots \Phi(x_k)}{\iint \mathcal{D}[\Phi(x)] \, \exp\big(iS[\Phi(x)]\big)} \\[2mm]
&=\; \frac{\cancel{Z_0} \times \sum \text{all Feynman diagrams with } k \text{ external vertices}}{\cancel{Z_0} \times \sum \text{all Feynman diagrams without external vertices}} \\[2mm]
&=\; \sum \text{Feynman diagrams with } k \text{ external vertices and no vacuum bubbles.}
\end{aligned}
\tag{40}
$$

Generating Functional

In the functional integral formalism there is a compact way of summarizing all the *connected* correlation functions in terms of a single generating functional. Let's add to the scalar field's Lagrangian $\mathcal{L}$ a linear source term,

$$\mathcal{L}(\Phi, \partial\Phi) \;\to\; \mathcal{L}(\Phi, \partial\Phi) \;+\; J \times \Phi \tag{41}$$

where $J(x)$ is an arbitrary function of x. Now let's calculate the partition function for $\Phi(x)$ as a functional of the source $J(x)$,

$$
\begin{aligned}
Z[J(x)] \;&=\; \iint \mathcal{D}[\Phi(x)] \, \exp\left(i\int d^4x(\mathcal{L} + J\Phi)\right) \\[2mm]
&=\; \iint \mathcal{D}[\Phi(x)] \, \exp\big(iS[\Phi(x)]\big) \times \exp\left(i\int d^4x \, J(x)\Phi(x)\right).
\end{aligned}
\tag{42}
$$

To make use of this partition function, we need functional (*i.e.*, variational) derivatives $\delta/\delta J(x)$. Here is how they work:

$$\frac{\delta}{\delta J(x)}\int J(y)\Phi(y)d^4y \;=\; \Phi(x), \tag{43}$$

$$\frac{\delta}{\delta J(x)} \int \partial_\mu J(y) \times A^\mu(y) d^4 y = -\partial_\mu A^\mu(x), \tag{44}$$

$$\frac{\delta}{\delta J(x)} \exp\left(i \int J(y)\Phi(y) d^4 y\right) = i\Phi(x) \times \exp\left(i \int J(y)\Phi(y) d^4 y\right), \tag{45}$$

etc., *etc.* Applying these derivatives to the partition function (42), we have

$$\frac{\delta}{\delta J(x)} Z[J] = \iint \mathcal{D}[\Phi(y)] \exp\left(i \int d^4 y (\mathcal{L} + J\Phi)\right) \times i\Phi(x) \tag{46}$$

and similarly

$$\frac{\delta}{\delta J(x_1)} \frac{\delta}{\delta J(x_2)} \cdots \frac{\delta}{\delta J(x_k)} Z[J] = \iint \mathcal{D}[\Phi(y)] \exp\left(i \int d^4 y (\mathcal{L} + J\Phi)\right) \times i^k \Phi(x_1)\Phi(x_2)\cdots\Phi(x_k). \tag{47}$$

Thus, the partition function (42) *generates* all the functional integrals of the scalar theory:

$$\forall k, \quad \iint \mathcal{D}[\Phi(y)] \exp\left(i \int d^4 y \mathcal{L}\right) \times \Phi(x_1)\cdots\Phi(x_k) = (-i)^k \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_k)} Z[J]\bigg|_{J(y)\equiv 0}. \tag{48}$$

Consequently, all the correlation functions (40) follows from the derivatives of the $Z[J]$ according to

$$G_k(x_1, \ldots, x_k) = \frac{(-i)^k}{Z[J]} \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_k)} Z[J]\bigg|_{J(y)\equiv 0}. \tag{49}$$

In terms of the Feynman diagrams, the correlation functions (49) contain both connected and disconnected graphs; only the vacuum bubbles are excluded. To obtain the connected correlation functions

$$G_k^{\text{conn}}(x_1, \ldots, x_k) = \sum \text{connected Feynman diagrams only} \tag{50}$$

we should take the $\delta/\delta J(x)$ derivatives of $\log Z[J]$ rather than $Z[J]$ itself,

$$G_k^{\text{conn}}(x_1, \ldots, x_k) = (-i)^k \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_k)} \log Z[J(y)]\bigg|_{J(y)\equiv 0}. \tag{51}$$

In other words, $\log Z[J(x)]$ is the *generating functional* for for all the connected correlation functions of the quantum field theory.

Proof: For the sake of generality, let's allow for $G_1(x) = \langle \Phi(x) \rangle \neq 0$. For $k = 1$, $G_1^{\mathrm{conn}} \equiv G_1$ and eq. (51) is equivalent to eq. (49). For $k = 2$ we use

$$\frac{\delta^2 \log Z[J]}{\delta J(x)\, \delta J(y)} = \frac{1}{Z} \frac{\delta^2 Z}{\delta J(x)\, \delta J(y)} - \frac{1}{Z} \frac{\delta Z}{\delta J(x)} \times \frac{1}{Z} \frac{\delta Z}{\delta J(y)} . \tag{52}$$

Let's multiply this formula by $(-i)^2$, set $J \equiv 0$ after taking the derivatives, and apply eqs. (49) to the right hand side; then

$$(-i)^2 \frac{\delta^2 \log Z[J]}{\delta J(x)\, \delta J(y)}\bigg|_{J\equiv 0} = G_2(x,y) - G_1(x) \times G_1(y) = G_2^{\mathrm{conn}}(x,y). \tag{53}$$

The $k = 1$ and $k = 2$ cases serve as a base of a proof by induction in k. Now we need to prove that is eqs. (51) hold true for all $\ell < k$ that they also hold true for k itself.

A k-point correlation function has a cluster expansion in terms of connected correlation functions,

$$G_k(x_1, \ldots, x_k) = G_k^{\mathrm{conn}}(x_1, \ldots, x_k) + \sum_{\substack{\text{cluster} \\ \text{decompositions}}} \left(\prod_{\text{clusters}} G^{\mathrm{conn}}(\text{cluster}) \right) \tag{54}$$

where each cluster is proper subset of $(x_1, \ldots, x_n)$ and we sum over decompositions of the whole set $(x_1, \ldots, x_n)$ into a union of disjoint clusters. For example, for $k = 3$

$$G_3(x,y,z) = G_3^{\mathrm{conn}}(x,y,z) + G_1(x) \times G_1(y) \times G_1(z)$$
$$+ G_2^{\mathrm{conn}}(x,y) \times G_1(z) + G_2^{\mathrm{conn}}(x,z) \times G_1(y)$$
$$+ G_2^{\mathrm{conn}}(y,z) \times G_1(x)$$

$$\tag{55}$$

There is a similar expansion of derivatives of Z in terms of derivatives of $\log Z$,

$$\frac{1}{Z} \frac{\delta^k Z}{\delta J(x_1) \cdots \delta J(x_k)} = \frac{\delta^k \log Z}{\delta J(x_1) \cdots \delta J(x_k)} + \sum_{\substack{\text{cluster} \\ \text{decompositions}}} \left(\prod_{\substack{(x_{(1)}, \ldots, x_{(\ell)})}}^{\text{clusters}} \frac{\delta^\ell \log Z}{\delta J(x_{(1)}) \cdots \delta J(x_{(\ell)})} \right),$$

$$\tag{56}$$

Now let's set $J(y) \equiv 0$ after taking all the derivatives in this expansion. On the left hand side, this gives us $i^k G_k(x_1, \ldots, x_k)$ by eq. (49). On the right hand side, for each cluster

$$\frac{\delta^\ell \log Z}{\delta J(x_{(1)}) \cdots \delta J(x_{(\ell)})} = i^\ell G_\ell^{\text{conn}}(x_{(1)}, \ldots, x_{(\ell)}) \tag{57}$$

by the induction assumption. Altogether, we have

$$G_n(x_1, \ldots, x_n) = \left.\frac{(-i)^k \delta^k \log Z}{\delta J(x_1) \cdots \delta J(x_k)}\right|_{J \equiv 0} + \sum_{\substack{\text{cluster} \\ \text{decompositions}}} \left(\prod_{\text{cluster}} G^{\text{conn}}(\text{cluster})\right), \tag{58}$$

and comparing this formula to eq. (54) we immediately see that

$$G_n^{\text{conn}}(x_1, \ldots, x_n) = \left.\frac{(-i)^k \delta^k \log Z}{\delta J(x_1) \cdots \delta J(x_k)}\right|_{J \equiv 0}, \tag{51}$$

quod erat demonstrandum.

We may summarize all the equations (51) in a single formula which makes manifest the role of $\log Z$ as a *generating functional* for the connected correlation functions,

$$\log Z[J(x)] = \log Z[0] + \sum_{k=1}^{\infty} \frac{i^k}{k!} \int d^4 x_1 \cdots \int d^4 x_k \, G_k^{\text{conn}}(x_1, \ldots, x_k) \times J(x_1) \cdots J(x_k). \tag{59}$$

To see how this works, consider the free theory as an example. A free scalar field has only one connected correlation function, namely $G_2^{\text{conn}}(x - y) = G_F(x - y)$; all the other G_k^{conn} require interactions and vanish for the free field. Hence, the partition function of the free field should have form

$$\log Z^{\text{free}}[J(x)] = \log Z^{\text{free}}[0] + \frac{i^2}{2} \int d^4 \int d^4 y \, G_F(x - y) \times J(x) J(y) + \text{nothing else.} \tag{60}$$

To check this formula, let's calculate the partition function. For the free field Φ,

$$\mathcal{L}^{\text{free}} + J\Phi = -\tfrac{1}{2}\Phi(\partial^2 + m^2)\Phi + J\Phi = -\tfrac{1}{2}\Phi'(\partial^2 + m^2)\Phi' + \tfrac{1}{2}J(\partial^2 + m^2)^{-1}J \tag{61}$$

(up to a total derivative) where $\Phi' = \Phi - (\partial^2 + m^2)^{-1} J$. In other words,

$$S^{\text{free}}[\Phi(x), J(x)] \overset{\text{def}}{=} \int (\mathcal{L} + J\Phi) d^4x \;=\; S^{\text{free}}[\Phi'(x), \cancel{J}] + \frac{i}{2}\int d^4 \int d^4y\, J(x) G_F(x-y) J(y) \quad (62)$$

where

$$\Phi'(x) \;=\; \Phi(x) \;-\; i\int d^4y\, G_F(x-y) J(y). \qquad (63)$$

In the path integral, $\mathcal{D}[\Phi(x)] = \mathcal{D}[\Phi'(x)]$, hence

$$\begin{aligned}
Z^{\text{free}}[J(x)] \;&=\; \iint \mathcal{D}[\Phi(x)]\, \exp\Big(iS^{\text{free}}[\Phi(x), J(x)]\Big) \\
&=\; \iint \mathcal{D}[\Phi'(x)]\, \exp\Big(iS^{\text{free}}[\Phi'(x), \cancel{J}]\Big) \times \exp\Big(-\frac{1}{2}\int d^4x \int d^4y\, J(x) G_F(x-y) J(y)\Big) \\
&=\; \exp\Big(-\frac{1}{2}\int d^4x \int d^4y\, J(x) G_F(x-y) J(y)\Big) \times \iint \mathcal{D}[\Phi'(x)]\, \exp\Big(iS^{\text{free}}[\Phi'(x), \cancel{J}]\Big) \\
&=\; \exp\Big(-\frac{1}{2}\int d^4x \int d^4y\, J(x) G_F(x-y) J(y)\Big) \times Z^{\text{free}}[0].
\end{aligned}$$

$$(64)$$

Taking the log of both sides of this formula gives us eq. (60), which confirms that $\log Z^{\text{free}}[J]$ is indeed the generating functional for the correlation functions of the free field.