1. Consider once again the massive vector field $\hat{A}^\mu(x)$. In the previous homework (set#4, problem 2), you (should have) expanded the free vector field into the creation and annihilation operators multiplied by the plane-waves according to

$$\hat{A}^\mu(x) = \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_\mathbf{k}}\sum_\lambda\Big(e^{-ikx}\times f^\mu_{\mathbf{k},\lambda}\times\hat{a}_{\mathbf{k},\lambda} + e^{+ikx}\times f^{*\mu}_{\mathbf{k},\lambda}\times\hat{a}^\dagger_{\mathbf{k},\lambda}\Big)^{k^0=+\omega_\mathbf{k}}. \tag{1}$$

The λ here labels the independent polarizations of a vector particle (for example, the helicities $\lambda=-1,0,+1$), while $f^\mu_{\mathbf{k},\lambda}$ are the polarization vectors obeying

$$k_\mu f^\mu_{\mathbf{k},\lambda} = 0, \quad g_{\mu\nu}f^\mu_{\mathbf{k},\lambda}f^{*\nu}_{\mathbf{k},\lambda'} = -\delta_{\lambda,\lambda'}\,. \tag{2}$$

In this problem, we shall calculate the Feynman propagator for the massive vector field (1).

(a) First, a lemma: Show that any polarization vectors obeying the constraints (2) also obey

$$\sum_\lambda f^\mu_{\mathbf{k},\lambda}f^{*\nu}_{\mathbf{k},\lambda} = -g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2}\,. \tag{3}$$

(b) Next, calculate the "vacuum sandwich" of two vector fields and show that

$$\begin{aligned}\langle 0|\,\hat{A}^\mu(x)\hat{A}^\nu(y)\,|0\rangle &= \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_\mathbf{k}}\left[\left(-g^{\mu\nu}+\frac{k^\mu k^\nu}{m^2}\right)e^{-ik(x-y)}\right]_{k^0=+\omega_\mathbf{k}}\\ &= \left(-g^{\mu\nu}-\frac{\partial^\mu\partial^\nu}{m^2}\right)D(x-y).\end{aligned} \tag{4}$$

(c) Now consider a free scalar field (of the same mass m as the vector field) and its Feynman propagator $G_F^{\text{scalar}}(x-y)$. Show that

$$\left(-g^{\mu\nu}-\frac{\partial^\mu\partial^\nu}{m^2}\right)G_F^{\text{scalar}}(x-y) = \langle 0|\,\mathbf{T}\hat{A}^\mu(x)\hat{A}^\nu(y)\,|0\rangle + \frac{i}{m^2}\,\delta^{\mu 0}\delta^{\nu 0}\delta^{(4)}(x-y). \tag{5}$$

To avoid the δ–function singularity in formulae like (5), the time-ordered product of the vector fields (or rather, just of their $\hat{A}^0$ components) is *modified*⋆ according to

$$\mathbf{T}^* \hat{A}^\mu(x)\hat{A}^\nu(y) \;=\; \mathbf{T}\hat{A}^\mu(x)\hat{A}^\nu(y) \;+\; \frac{i}{m^2}\,\delta^{\mu 0}\delta^{\nu 0}\delta^{(4)}(x-y). \tag{6}$$

Consequently, the Feynman propagator for the massive vector field is defined using the modified time-ordered product of the two fields,

$$G_F^{\mu\nu}(x-y) \;\stackrel{\text{def}}{=}\; \langle 0|\, \mathbf{T}^* \hat{A}^\mu(x)\hat{A}^\nu(y)\, |0\rangle \tag{7}$$

(d) Show that this propagator obtains as

$$G_F^{\mu\nu}(x-y) \;=\; \int\!\frac{d^4\mathbf{k}}{(2\pi)^4}\,\left(-g^{\mu\nu}+\frac{k^\mu k^\nu}{m^2}\right)\times\frac{ie^{-ik(x-y)}}{k^2-m^2+i0}\,. \tag{8}$$

(e) Finally, write the classical action for the free vector field as

$$S \;=\; \tfrac{1}{2}\int\! d^4x\, A_\mu(x)\,\mathcal{D}^{\mu\nu} A_\nu(x) \tag{9}$$

where $\mathcal{D}^{\mu\nu}$ is a differential operator, and show that the Feynman propagator (8) is a Green's function of this operator,

$$\mathcal{D}_x^{\mu\nu} G^F_{\nu\lambda}(x-y) \;=\; +i\delta^\mu_\lambda \delta^{(4)}(x-y). \tag{10}$$

⋆ See *Quantum Field Theory* by Claude Itzykson and Jean–Bernard Zuber.

2. This problem is about the continuous Lorentz group $SO^+(3,1)$ and its generators $\hat{J}^{\mu\nu} = -\hat{J}^{\nu\mu}$. In 3D terms, the six independent $\hat{J}^{\mu\nu}$ generators comprise the 3 components of the angular momentum $\hat{J}^i = \frac{1}{2}\epsilon^{ijk}\hat{J}^{jk}$ — which generate the rotations of space — plus 3 generators $\hat{K}^i = \hat{J}^{0i} = -\hat{J}^{i0}$ of the Lorentz boosts.

   (a) In 4D terms, the commutation relations of the Lorentz generators are

$$\left[\hat{J}^{\alpha\beta}, \hat{J}^{\mu\nu}\right] = ig^{\beta\mu}\hat{J}^{\alpha\nu} - ig^{\alpha\mu}\hat{J}^{\beta\nu} - ig^{\beta\nu}\hat{J}^{\alpha\mu} + ig^{\alpha\nu}\hat{J}^{\beta\mu}. \tag{11}$$

   Show that in 3D terms, these relations become

$$\left[\hat{J}^i, \hat{J}^j\right] = i\epsilon^{ijk}\hat{J}^k, \quad \left[\hat{J}^i, \hat{K}^j\right] = i\epsilon^{ijk}\hat{K}^k, \quad \left[\hat{K}^i, \hat{K}^j\right] = -i\epsilon^{ijk}\hat{J}^k. \tag{12}$$

   The Lorentz symmetry dictates the commutation relations of the $\hat{J}^{\mu\nu}$ with any operators comprising a Lorentz multiplet. In particular, for any Lorentz vector $\hat{V}^\mu$

$$\left[\hat{V}^\lambda, \hat{J}^{\mu\nu}\right] = ig^{\lambda\mu}\hat{V}^\nu - ig^{\lambda\nu}\hat{V}^\mu. \tag{13}$$

   (b) Spell out these commutation relations in 3D terms, then use them to show that the Lorentz boost generators $\hat{\mathbf{K}}$ do not commute with the Hamiltonian $\hat{H}$.

   (c) Show that even in the non-relativistic limit, the Galilean boosts $t' = t,\ \mathbf{x}' = \mathbf{x} + \mathbf{v}t$ and their generators $\hat{\mathbf{K}}_G$ do not commute with the Hamiltonian.

   Note: Only the *time-independent* symmetries commute with the Hamiltonian. But when the action of a symmetry is manifestly time dependent — like a Galilean boost $\mathbf{x}' = \mathbf{x} + \mathbf{v}t$ or a Lorentz boost — the symmetry operators do not commute with the time evolution and hence with the Hamiltonian.

   Next, consider the little group $G(p)$ of Lorentz symmetries preserving some momentum 4–vector p^μ. For the moment, allow the p^μ to be time-like, light-like, or even space-like — anything goes as long as $p \neq 0$.

   (d) Show that the little group $G(p)$ is generated by the 3 components of the vector

$$\hat{\mathbf{R}} = p^0\hat{\mathbf{J}} + \mathbf{p}\times\hat{\mathbf{K}} \tag{14}$$

   after a suitable component-by-component rescaling.

Suppose the momentum p^μ belonds to a massive particle, thus $p^\mu p_\mu = m^2 > 0$. For simplicity, assume the particle moves in z direction with velocity β, thus $p^\mu = (E,0,0,p)$ for $E = \gamma m$ and $p = \beta\gamma m$. In this case, the properly normalized generators of the little group $G(p)$ are the

$$\begin{aligned}
\tilde{J}^x &= \frac{1}{m}\,\hat{R}^x = \gamma\hat{J}^x - \beta\gamma\hat{K}^y,\\
\tilde{J}^y &= \frac{1}{m}\,\hat{R}^y = \gamma\hat{J}^y + \beta\gamma\hat{K}^x,\\
\tilde{J}^z &= \frac{1}{\gamma m}\,\hat{R}^z = \hat{J}^z, \quad \text{the helicity.}
\end{aligned} \tag{15}$$

(e) Show that these generators have angular-momentum-like commutators with each other, $[\tilde{J}^i,\tilde{J}^j] = i\epsilon^{ijk}\tilde{J}^k$. Consequently, the little group $G(p)$ is isomorphic to the rotation group $SO(3)$.

Now suppose the momentum p^μ belongs to a massless particle, $p^\mu p_\mu = 0$. Again, assume for simplicity that the particle moves in the z direction, thus $p^\mu = (E,0,0,E)$. In this case, we cannot normalize the generators of the little group as in eq. (15); instead, let's normalize them according to

$$\hat{\mathbf{I}} = \frac{1}{E}\hat{\mathbf{R}} = \hat{\mathbf{J}} + \vec{\beta}\times\hat{\mathbf{K}}, \tag{16}$$

or in components,

$$\hat{I}^x = \hat{J}^x - \hat{K}^y, \quad \hat{I}^y = \hat{J}^y + \hat{K}^x, \quad \hat{I}^z = \hat{J}^z. \tag{17}$$

(f) Show that these generators obey similar commutation relations to the $\hat{p}^x$, $\hat{p}^y$, and $\hat{J}^z$ operators, namely

$$[\hat{J}^z,\hat{I}^x] = +i\hat{I}^y, \quad [\hat{J}^z,\hat{I}^y] = -i\hat{I}^x, \quad [\hat{I}^x,\hat{I}^y] = 0. \tag{18}$$

Consequently, the little group $G(p)$ is isomorphic to the ISO(2) group of *rotations and translations* in the xy plane.

(g) Finally, show that for a tachyonic momentum with $p^\mu p_\mu < 0$, the properly normalized generators of the little group have similar commutation relations to the $\hat{K}^x$, $\hat{K}^y$, and $\hat{J}^z$ operators. Consequently, the little group $G(p)$ is isomorphic to the $SO^+(2,1)$, the continuous Lorentz group in $2+1$ spacetime dimensions.

3. Now let's focus on the massless particles. As explained in class, the finite unitary multiplets of the $G(p) \cong \mathrm{ISO}(2)$ group generated by the (17) operators are singlets $|\lambda\rangle$, althouth they are non-trivial singlets for $\lambda \neq 0$. Specifically, the state $|\lambda\rangle$ is an eigenstate of the helicity operator $\hat{J}^z$ (for the momentum in the z direction) and are annihilated by the $\hat{I}^{x,y}$ operators,

$$\hat{J}^z |\lambda\rangle = \lambda |\lambda\rangle, \quad \hat{I}^x |\lambda\rangle = 0, \quad \hat{I}^y |\lambda\rangle = 0. \tag{19}$$

(a) Show that in 4D terms the state $|p,\lambda\rangle$ of a *massless* particle satisfies

$$\epsilon_{\alpha\beta\gamma\delta} \hat{J}^{\beta\gamma} \hat{P}^\delta |p,\lambda\rangle = 2\lambda \hat{P}_\alpha |p,\lambda\rangle . \tag{20}$$

(b) Use eq. (20) to show that *continuous* Lorentz transforms do not change helicities of *massless* particles,

$$\forall L \in \mathrm{SO}^+(3,1), \quad \widehat{\mathcal{D}}(L) |p,\lambda\rangle = |Lp, \text{same}\,\lambda\rangle \times e^{i\,\text{phase}}. \tag{21}$$

4. Finally, a reading assignment. To help you understand the relations between the continuous symmetries, their generators, the multiplets, and the representations of the generators and of the finite symmetries, read about the rotational symmetry and its generators in chapter 3 of the J. J. Sakurai's book *Modern Quantum Mechanics.** Please focus on sections 1, 2, 3, second half of section 5 (representations of the rotation operators), and section 10; the other sections 4, 6, 7, 8, and 9 are not relevant to the present class material.

PS: If you have already read the Sakurai's book before but it has been a while, please read it again.

$\star$ The UT Math–Physics–Astronomy library has several hard copies but no electronic copies of the book. However, you can find several pirate scans of the book (in PDF format) all over the web; Google them up if you cannot find a legitimate copy.