

1. First, an exercise in Dirac matrices  $\gamma^\mu$ . In this problem, you should not assume any explicit matrices for the  $\gamma^\mu$  but simply use the anticommutation relations

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}. \quad (1)$$

When necessary, you may also assume that the Dirac matrices are  $4 \times 4$ , and the  $\gamma^0$  matrix is hermitian while the  $\gamma^1, \gamma^2, \gamma^3$  matrices are antihermitian,  $(\gamma^0)^\dagger = +\gamma^0$  while  $(\gamma^i)^\dagger = -\gamma^i$  for  $i = 1, 2, 3$ .

- (a) Show that  $\gamma^\alpha\gamma_\alpha = 4$ ,  $\gamma^\alpha\gamma^\nu\gamma_\alpha = -2\gamma^\nu$ ,  $\gamma^\alpha\gamma^\mu\gamma^\nu\gamma_\alpha = 4g^{\mu\nu}$ , and  $\gamma^\alpha\gamma^\lambda\gamma^\mu\gamma^\nu\gamma_\alpha = -2\gamma^\nu\gamma^\mu\gamma^\lambda$ .  
Hint: use  $\gamma^\alpha\gamma^\nu = 2g^{\nu\alpha} - \gamma^\nu\gamma^\alpha$  repeatedly.
- (b) The electron field in the EM background obeys the *covariant Dirac equation*  
 $(i\gamma^\mu D_\mu - m)\Psi(x) = 0$  where  $D_\mu\Psi = \partial_\mu\Psi - ieA_\mu\Psi$ . Show that this equation implies

$$(D_\mu D^\mu + m^2 - eF_{\mu\nu}S^{\mu\nu})\Psi(x) = 0. \quad (2)$$

Besides the 4 Dirac matrices  $\gamma^\mu$ , there is another useful matrix  $\gamma^5 \stackrel{\text{def}}{=} i\gamma^0\gamma^1\gamma^2\gamma^3$ .

- (c) Show that the  $\gamma^5$  anticommutes with each of the  $\gamma^\mu$  matrices —  $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$  — and commutes with all the spin matrices,  $\gamma^5 S^{\mu\nu} = +S^{\mu\nu}\gamma^5$ .
- (d) Show that the  $\gamma^5$  is hermitian and that  $(\gamma^5)^2 = 1$ .
- (e) Show that  $\gamma^5 = (i/24)\epsilon_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu$  and that  $\gamma^{[\kappa\gamma^\lambda\gamma^\mu\gamma^\nu]} = +24i\epsilon^{\kappa\lambda\mu\nu}\gamma^5$ .
- (f) Show that  $\gamma^{[\lambda\gamma^\mu\gamma^\nu]} = +6i\epsilon^{\kappa\lambda\mu\nu}\gamma_\kappa\gamma^5$ .
- (g) Show that any  $4 \times 4$  matrix  $\Gamma$  is a unique linear combination of the following 16 matrices:  
 $1, \gamma^\mu, \frac{1}{2}\gamma^{[\mu\gamma^\nu]} = -2iS^{\mu\nu}, \gamma^5\gamma^\mu,$  and  $\gamma^5$ .
- \* My conventions here are:  $\epsilon^{0123} = -1, \epsilon_{0123} = +1, \gamma^{[\mu\gamma^\nu]} = \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu,$   
 $\gamma^{[\lambda\gamma^\mu\gamma^\nu]} = \gamma^\lambda\gamma^\mu\gamma^\nu - \gamma^\lambda\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu\gamma^\lambda - \gamma^\mu\gamma^\lambda\gamma^\nu + \gamma^\nu\gamma^\lambda\gamma^\mu - \gamma^\nu\gamma^\mu\gamma^\lambda,$  etc.

1\*. For extra challenge, let's generalize the Dirac matrices to spacetime dimensions  $d \neq 4$ . Such matrices always satisfy the Clifford algebra (1), but their sizes depend on  $d$ .

Let  $\Gamma = i^n \gamma^0 \gamma^1 \dots \gamma^{d-1}$  be the generalization of the  $\gamma^5$  to  $d$  dimensions; the pre-factor  $i^n = \pm i$  or  $\pm 1$  is chosen such that  $\Gamma = \Gamma^\dagger$  and  $\Gamma^2 = +1$ .

(a) For even  $d$ ,  $\Gamma$  anticommutes with all the  $\gamma^\mu$ . Prove this, then use this fact to show that there are  $2^d$  independent products of the  $\gamma^\mu$  matrices, and consequently the matrices should be  $2^{d/2} \times 2^{d/2}$ .

(b) For odd  $d$ ,  $\Gamma$  commutes with all the  $\gamma^\mu$  — prove this. Consequently, one can set  $\Gamma = +1$  or  $\Gamma = -1$ ; the two choices lead to in-equivalent sets of the  $\gamma^\mu$ .

Classify the independent products of the  $\gamma^\mu$  for odd  $d$  and show that their net number is  $2^{d-1}$ ; consequently, the matrices should be  $2^{(d-1)/2} \times 2^{(d-1)/2}$ .

2. Now let's go back to  $d = 3 + 1$  and learn about the Weyl spinors and Weyl spinor fields. Since all the spin matrices  $S^{\mu\nu}$  commute with the  $\gamma^5$ , for all *continuous* Lorentz symmetries  $L^\mu_\nu$  their Dirac-spinor representations  $M_D(L) = \exp(-\frac{i}{2} \Theta_{\alpha\beta} S^{\alpha\beta})$  are block-diagonal in the eigenbasis of the  $\gamma^5$ . This makes the Dirac spinor  $\Psi$  a *reducible* multiplet of the continuous Lorentz group  $SO^+(3,1)$  — it comprises two different irreducible 2-component spinor multiplets, called the left-handed Weyl spinor  $\psi_L$  and the right-handed Weyl spinor  $\psi_R$ .

This decomposition becomes clear in the Weyl convention for the Dirac matrices where

$$\gamma^\mu = \begin{pmatrix} 0 & \bar{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix} \quad \text{where} \quad \begin{aligned} \sigma^\mu &\stackrel{\text{def}}{=} (\mathbf{1}_{2 \times 2}, -\boldsymbol{\sigma}), \\ \bar{\sigma}^\mu &\stackrel{\text{def}}{=} (\mathbf{1}_{2 \times 2}, +\boldsymbol{\sigma}), \end{aligned} \quad (3)$$

and consequently

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} \implies M_D(L) = \begin{pmatrix} M_L(L) & 0 \\ 0 & M_R(L) \end{pmatrix}. \quad (4)$$

(a) Check that the  $\gamma^5$  matrix indeed has this form and write down explicit matrices for the  $S^{\mu\nu}$  in the Weyl convention.

(b) Show that for a space rotation  $R$  through angle  $\theta$  around axis  $\mathbf{n}$ ,

$$M_L(R) = M_R(R) = \exp\left(-\frac{i}{2}\theta\mathbf{n}\cdot\boldsymbol{\sigma}\right). \quad (5)$$

Likewise, show that for a Lorentz boost  $B$  of speed  $v$  in the direction  $\mathbf{n}$ ,

$$M_L(B) = \exp\left(-\frac{1}{2}r\mathbf{n}\cdot\boldsymbol{\sigma}\right) \quad \text{while} \quad M_R(B) = \exp\left(+\frac{1}{2}r\mathbf{n}\cdot\boldsymbol{\sigma}\right) \quad (6)$$

where  $r = \text{artanh}(v)$  is the *rapidity* of the boost. For successive boosts in the same direction, the rapidities add up,  $r_{1+2} = r_1 + r_2$ . Consequently, a finite Lorentz boost of rapidity  $r$  in the direction  $\mathbf{n}$  is  $B = \exp(r\mathbf{n}\cdot\hat{\mathbf{K}})$ .

(c) The more familiar  $\beta$  and  $\gamma$  parameters of a Lorentz boost are related to the rapidity as

$$\beta = \tanh(r), \quad \gamma = \cosh(r), \quad \beta\gamma = \sinh(r). \quad (7)$$

Show that in terms of these parameters, eqs. (6) translate to

$$M_L(B) = \sqrt{\gamma} \times \sqrt{1 - \beta\mathbf{n}\cdot\boldsymbol{\sigma}}, \quad M_R(B) = \sqrt{\gamma} \times \sqrt{1 + \beta\mathbf{n}\cdot\boldsymbol{\sigma}}. \quad (8)$$

(d) Show that for any continuous Lorentz symmetry  $L$ , the  $M_L(L)$  and the  $M_R(L)$  matrices are related to each other according to

$$M_R(L) = \sigma_2 \times M_L^*(L) \times \sigma_2, \quad M_L(L) = \sigma_2 \times M_R^*(L) \times \sigma_2. \quad (9)$$

Hint: all 3 Pauli matrices  $\sigma_i$ , are related to their complex conjugates  $\sigma_i^*$  according to  $\sigma_2\sigma_i^*\sigma_2 = -\sigma_i$ ,

In the Weyl convention for the Dirac matrices, the Dirac spinor field  $\Psi(x)$  splits into the left-handed Weyl spinor field  $\psi_L(x)$  and the right-handed Weyl spinor field  $\psi_R(x)$  according to

$$\Psi_{\text{Dirac}}(x) = \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix} \quad \text{where} \quad \begin{aligned} \psi'_L(x') &= M_L(L)\psi_L(x), \\ \psi'_R(x') &= M_R(L)\psi_R(x). \end{aligned} \quad (10)$$

- (e) Show that the hermitian conjugate of each Weyl spinor transforms equivalently to the other spinor. Specifically, the  $\sigma_2 \times \psi_L^*(x)$  transforms under continuous Lorentz symmetries like the  $\psi_R(x)$ , while the  $\sigma_2 \times \psi_R^*(x)$  transforms like the  $\psi_L(x)$ .

Note: the  $*$  superscript on a multi-component quantum field means hermitian conjugation of each component field but without transposing the components, thus

$$\psi_L = \begin{pmatrix} \psi_{L1} \\ \psi_{L2} \end{pmatrix}, \quad \psi_L^* = \begin{pmatrix} \psi_{L1}^\dagger \\ \psi_{L2}^\dagger \end{pmatrix}, \quad \text{while} \quad \psi_L^\dagger = (\psi_{L1}^\dagger \quad \psi_{L2}^\dagger), \quad (11)$$

and likewise for the  $\psi_R$  and its conjugates.

Finally, consider the Dirac Lagrangian  $\bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi$ .

- (f) Express this Lagrangian in terms of the Weyl spinor fields  $\psi_L(x)$  and  $\psi_R(x)$  (and their conjugates  $\psi_L^\dagger(x)$  and  $\psi_R^\dagger(x)$ ).
- (g) Show that for  $m = 0$  — and only for  $m = 0$  — the two Weyl spinor fields become independent from each other.

3. The third problem is about the plane-wave solutions of the Dirac equation,  $e^{-ipx}u_\alpha$  and  $e^{+ipx}v_\alpha$  for some  $x$ -independent Dirac spinors  $u_\alpha(p, s)$  and  $v_\alpha(p, s)$ .

- (a) Check that these waves indeed solve the Dirac equation provided  $p^2 = m^2$  while

$$(\not{p} - m)u(p, s) = 0, \quad (\not{p} + m)v(p, s) = 0. \quad (12)$$

By convention, we always take  $E = p^0 = +\sqrt{\mathbf{p}^2 + m^2}$  — that's why we have both  $e^{-ipx}u_\alpha$  and  $e^{+ipx}v_\alpha$  types of wave — while the spinor coefficients  $u(p, s)$  and  $v(p, s)$  are normalized to

$$u^\dagger(p, s)u(p, s') = v^\dagger(p, s)v(p, s') = 2E\delta_{s,s'}. \quad (13)$$

In this problem we shall write down explicit formulae for these spinors in the Weyl basis for the  $\gamma^\mu$  matrices.

(b) Show that for  $\mathbf{p} = 0$ ,

$$u(\mathbf{p} = \mathbf{0}, s) = \begin{pmatrix} \sqrt{m} \xi_s \\ \sqrt{m} \xi_s \end{pmatrix} \quad (14)$$

where  $\xi_s$  is a two-component  $SO(3)$  spinor encoding the electron's spin state. The  $\xi_s$  are normalized to  $\xi_s^\dagger \xi_{s'} = \delta_{s,s'}$ .

(c) For other momenta,  $u(p, s) = M_D(\text{boost}) \times u(\mathbf{p} = 0, s)$  for the boost that turns  $(m, \vec{0})$  into  $p^\mu$ . Use eqs. (8) to show that

$$u(p, s) = \begin{pmatrix} \sqrt{E - \mathbf{p} \cdot \boldsymbol{\sigma}} \xi_s \\ \sqrt{E + \mathbf{p} \cdot \boldsymbol{\sigma}} \xi_s \end{pmatrix} = \begin{pmatrix} \sqrt{p_\mu \bar{\sigma}^\mu} \xi_s \\ \sqrt{p_\mu \sigma^\mu} \xi_s \end{pmatrix}. \quad (15)$$

(d) Use similar arguments to show that

$$v(p, s) = \begin{pmatrix} +\sqrt{E - \mathbf{p} \cdot \boldsymbol{\sigma}} \eta_s \\ -\sqrt{E + \mathbf{p} \cdot \boldsymbol{\sigma}} \eta_s \end{pmatrix} = \begin{pmatrix} +\sqrt{p_\mu \bar{\sigma}^\mu} \eta_s \\ -\sqrt{p_\mu \sigma^\mu} \eta_s \end{pmatrix} \quad (16)$$

where  $\eta_s$  are two-component  $SO(3)$  spinors normalized to  $\eta_s^\dagger \eta_{s'} = \delta_{s,s'}$ .

Physically, the  $\eta_s$  should have opposite spins from  $\xi_s$  — the holes in the Dirac sea have opposite spins (as well as  $p^\mu$ ) from the missing negative-energy particles. Mathematically, this requires  $\eta_s^\dagger \mathbf{S} \eta_s = -\xi_s^\dagger \mathbf{S} \xi_s$ ; we may solve this condition by letting  $\eta_s = \sigma_2 \xi_s^* = \pm i \xi_{-s}^*$ .

(e) Check that  $\eta_s = \sigma_2 \xi_s^* = \pm i \xi_{-s}^*$  indeed provides for the  $\eta_s^\dagger \mathbf{S} \eta_s = -\xi_s^\dagger \mathbf{S} \xi_s$ , then show that this leads to  $v(p, s) = \gamma^2 u^*(p, s)$ .

(f) Show that for the ultra-relativistic electrons or positrons of definite helicity  $\lambda = \pm \frac{1}{2}$ , the Dirac plane waves become *chiral* — *i.e.*, dominated by one of the two irreducible Weyl spinor components  $\psi_L(x)$  or  $\psi_R(x)$  of the Dirac spinor  $\Psi(x)$ , while the other component becomes negligible. Specifically,

$$\begin{aligned} u(p, -\tfrac{1}{2}) &\approx \sqrt{2E} \begin{pmatrix} \xi_L \\ 0 \end{pmatrix}, & u(p, +\tfrac{1}{2}) &\approx \sqrt{2E} \begin{pmatrix} 0 \\ \xi_R \end{pmatrix}, \\ v(p, -\tfrac{1}{2}) &\approx -\sqrt{2E} \begin{pmatrix} 0 \\ \eta_L \end{pmatrix}, & v(p, +\tfrac{1}{2}) &\approx \sqrt{2E} \begin{pmatrix} \eta_R \\ 0 \end{pmatrix}. \end{aligned} \quad (17)$$

Note that for the electron waves the helicity agrees with the chirality — they are both left or both right, — but for the positron waves the chirality is opposite from the helicity.

Back in problem 2(g) we saw that for  $m = 0$  the LH and the RH Weyl spinor fields decouple from each other. Now this exercise show us which particle modes comprise each Weyl spinor: *The  $\psi_L(x)$  and its hermitian conjugate  $\psi_L^\dagger(x)$  contain the left-handed fermions and the right-handed antifermions, while the  $\psi_R(x)$  and the  $\psi_R^\dagger(x)$  contain the right-handed fermions and the left-handed antifermions.*

4. Finally, let's establish some basis-independent properties of the Dirac spinors  $u(p, s)$  and  $v(p, s)$  — although you may use the Weyl basis to verify them.

(a) Show that

$$\bar{u}(p, s)u(p, s') = +2m\delta_{s,s'}, \quad \bar{v}(p, s)v(p, s') = -2m\delta_{s,s'}; \quad (18)$$

note the  $\pm 2m$  normalization factors here, unlike the  $+2E$  factors in eq. (13) for the  $u^\dagger u$  and the  $v^\dagger v$ .

(b) There are only two independent  $SO(3)$  spinors, hence  $\sum_s \xi_s \xi_s^\dagger = \sum_s \eta_s^\dagger \eta_s = \mathbf{1}_{2 \times 2}$ . Use this fact to show that

$$\sum_{s=1,2} u_\alpha(p, s) \bar{u}_\beta(p, s) = (\not{p} + m)_{\alpha\beta} \quad \text{and} \quad \sum_{s=1,2} v_\alpha(p, s) \bar{v}_\beta(p, s) = (\not{p} - m)_{\alpha\beta}. \quad (19)$$