1. First, an exercise in Dirac matrices γ^μ. In this problem, you should not assume any explicit matrices for the γ^μ but simply use the anticommutation relations

$$\gamma^\mu\gamma^\nu \;+\; \gamma^\nu\gamma^\mu \;=\; 2g^{\mu\nu}. \tag{1}$$

When necessary, you may also assume that the Dirac matrices are 4×4, and the γ^0 matrix is hermitian while the $\gamma^1,\gamma^2,\gamma^3$ matrices are antihermitian, $(\gamma^0)^\dagger = +\gamma^0$ while $(\gamma^i)^\dagger = -\gamma^i$ for $i=1,2,3$.

(a) Show that $\gamma^\alpha\gamma_\alpha = 4$, $\gamma^\alpha\gamma^\nu\gamma_\alpha = -2\gamma^\nu$, $\gamma^\alpha\gamma^\mu\gamma^\nu\gamma_\alpha = 4g^{\mu\nu}$, and $\gamma^\alpha\gamma^\lambda\gamma^\mu\gamma^\nu\gamma_\alpha = -2\gamma^\nu\gamma^\mu\gamma^\lambda$. Hint: use $\gamma^\alpha\gamma^\nu = 2g^{\nu\alpha} - \gamma^\nu\gamma^\alpha$ repeatedly.

(b) The electron field in the EM background obeys the *covariant Dirac equation* $\bigl(i\gamma^\mu D_\mu \,-\, m\bigr)\Psi(x) \,=\, 0$ where $D_\mu\Psi = \partial_\mu\Psi - ieA_\mu\Psi$. Show that this equation implies

$$\bigl(D_\mu D^\mu \,+\, m^2 \,-\, eF_{\mu\nu}S^{\mu\nu}\bigr)\,\Psi(x) \;=\; 0. \tag{2}$$

Besides the 4 Dirac matrices γ^μ, there is another useful matrix $\gamma^5 \stackrel{\rm def}{=} i\gamma^0\gamma^1\gamma^2\gamma^3$.

(c) Show that the γ^5 anticommutes with each of the γ^μ matrices — $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$ — and commutes with all the spin matrices, $\gamma^5 S^{\mu\nu} = +S^{\mu\nu}\gamma^5$.

(d) Show that the γ^5 is hermitian and that $(\gamma^5)^2 = 1$.

(e) Show that $\gamma^5 = (i/24)\epsilon_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu$ and that $\gamma^{[\kappa}\gamma^\lambda\gamma^\mu\gamma^{\nu]} = +24i\epsilon^{\kappa\lambda\mu\nu}\,\gamma^5$.

(f) Show that $\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} = +6i\epsilon^{\kappa\lambda\mu\nu}\,\gamma_\kappa\gamma^5$.

(g) Show that any 4×4 matrix Γ is a unique linear combination of the following 16 matrices: 1, γ^μ, $\frac{1}{2}\gamma^{[\mu}\gamma^{\nu]} = -2iS^{\mu\nu}$, $\gamma^5\gamma^\mu$, and γ^5.

$*$ My conventions here are: $\epsilon^{0123} = -1$, $\epsilon_{0123} = +1$, $\gamma^{[\mu}\gamma^{\nu]} = \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu$, $\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} = \gamma^\lambda\gamma^\mu\gamma^\nu - \gamma^\lambda\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu\gamma^\lambda - \gamma^\mu\gamma^\lambda\gamma^\nu + \gamma^\nu\gamma^\lambda\gamma^\mu - \gamma^\nu\gamma^\mu\gamma^\lambda$, *etc.*

1*. For extra challenge, let's generalize the Dirac matrices to spacetime dimensions $d \neq 4$. Such matrices always satisfy the Clifford algebra (1), but their sizes depend on d.

Let $\Gamma = i^n \gamma^0 \gamma^1 \cdots \gamma^{d-1}$ be the generalization of the γ^5 to d dimensions; the pre-factor $i^n = \pm i$ or ± 1 is chosen such that $\Gamma = \Gamma^\dagger$ and $\Gamma^2 = +1$.

(a) *For even* d, Γ anticommutes with all the γ^μ. Prove this, then use this fact to show that there are 2^d independent products of the γ^μ matrices, and consequently the matrices should be $2^{d/2} \times 2^{d/2}$.

(b) *For odd* d, Γ commutes with all the Γ^μ — prove this. Consequently, one can set $\Gamma = +1$ or $\Gamma = -1$; the two choices lead to in-equivalent sets of the γ^μ.

Classify the independent products of the γ^μ for odd d and show that their net number is 2^{d-1}; consequently, the matrices should be $2^{(d-1)/2} \times 2^{(d-1)/2}$.

2. Now let's go back to $d = 3 + 1$ and learn about the Weyl spinors and Weyl spinor fields. Since all the spin matrices $S^{\mu\nu}$ commute with the γ^5, for all *continuous* Lorentz symmetries $L^\mu_{\ \nu}$ their Dirac-spinor representations $M_D(L) = \exp\bigl(-\frac{i}{2}\Theta_{\alpha\beta}S^{\alpha\beta}\bigr)$ are block-diagonal in the eigenbasis of the γ^5. This makes the Dirac spinor Ψ a *reducible* multiplet of the continuous Lorentz group $SO^+(3,1)$ — it comprises two different irreducible 2-component spinor multiplets, called the left-handed Weyl spinor ψ_L and the right-handed Weyl spinor ψ_R.

This decomposition becomes clear in the Weyl convention for the Dirac matrices where

$$\gamma^\mu = \begin{pmatrix} 0 & \bar\sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix} \quad \text{where} \quad \begin{aligned} \sigma^\mu &\stackrel{\text{def}}{=} \bigl(\mathbf{1}_{2\times 2}, -\boldsymbol{\sigma}\bigr), \\ \bar\sigma^\mu &\stackrel{\text{def}}{=} \bigl(\mathbf{1}_{2\times 2}, +\boldsymbol{\sigma}\bigr), \end{aligned} \tag{3}$$

and consequently

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} \quad \Longrightarrow \quad M_D(L) = \begin{pmatrix} M_L(L) & 0 \\ 0 & M_R(L) \end{pmatrix}. \tag{4}$$

(a) Check that the γ^5 matrix indeed has this form and write down explicit matrices for the $S^{\mu\nu}$ in the Weyl convention.

(b) Show that for a space rotation R through angle θ around axis $\mathbf{n}$,

$$M_L(R) \;=\; M_R(R) \;=\; \exp\bigl(-\tfrac{i}{2}\,\theta\mathbf{n}\cdot\boldsymbol{\sigma}\bigr). \tag{5}$$

Likewise, show that for a Lorentz boost B of speed v in the direction $\mathbf{n}$,

$$M_L(B) \;=\; \exp(-\tfrac{1}{2}\,r\,\mathbf{n}\cdot\boldsymbol{\sigma}\bigr) \quad\text{while}\quad M_R(B) \;=\; \exp(+\tfrac{1}{2}\,r\,\mathbf{n}\cdot\boldsymbol{\sigma}\bigr) \tag{6}$$

where $r = \operatorname{artanh}(v)$ is the *rapidity* of the boost. For successive boosts in the same direction, the rapidities ass up, $r_{1+2} = r_1 + r_2$. Consequently, a finite Lorentz boost of rapidity r in the direction $\mathbf{n}$ is $B = \exp(r\mathbf{n}\cdot\hat{\mathbf{K}})$.

(c) The more familiar β and γ parameters of a Lorentz boost are related to the rapidity as

$$\beta \;=\; \tanh(r), \quad \gamma \;=\; \cosh(r), \quad \beta\gamma \;=\; \sinh(r). \tag{7}$$

Show that in terms of these parameters, eqs. (6) translate to

$$M_L(B) \;=\; \sqrt{\gamma}\times\sqrt{1\,-\,\beta\,\mathbf{n}\cdot\boldsymbol{\sigma}}\,, \qquad M_R(B) \;=\; \sqrt{\gamma}\times\sqrt{1\,+\,\beta\,\mathbf{n}\cdot\boldsymbol{\sigma}}\,. \tag{8}$$

(d) Show that for any continuous Lorentz symmetry L, the $M_L(L)$ and the $M_R(L)$ matrices are related to each other according to

$$M_R(L) \;=\; \sigma_2\times M_L^*(L)\times\sigma_2\,, \quad M_L(L) \;=\; \sigma_2\times M_R^*(L)\times\sigma_2\,. \tag{9}$$

Hint: all 3 Pauli matrices σ_i, are related to their comlplex congugates σ_i^* according to $\sigma_2\sigma_i^*\sigma_2 = -\sigma_i$,

In the Weyl convention for the Dirac matrices, the Dirac spinor field $\Psi(x)$ splits into the left-handed Weyl spinor field $\psi_L(x)$ and the right-handed Weyl spinor field $\psi_R(x)$ according to

$$\Psi_{\text{Dirac}}(x) \;=\; \begin{pmatrix}\psi_L(x),\\ \psi_R(x)\end{pmatrix} \quad\text{where}\quad \begin{aligned}\psi'_L(x') \;&=\; M_L(L)\psi_L(x),\\ \psi'_R(x') \;&=\; M_R(L)\psi_R(x).\end{aligned} \tag{10}$$

(e) Show that the hermitian conjugate of each Weyl spinor transforms equivalently to the other spinor. Specifically, the $\sigma_2 \times \psi_L^*(x)$ transforms under continuous Lorentz symmetries like the $\psi_R(x)$, while the $\sigma_2 \times \psi_R^*(x)$ transforms like the $\psi_L(x)$.

Note: the * superscript on a multi-component quantum field means hermitian conjugation of each component field but without transposing the compoents, thus

$$\psi_L = \begin{pmatrix} \psi_{L1} \\ \psi_{L2} \end{pmatrix}, \quad \psi_L^* = \begin{pmatrix} \psi_{L1}^\dagger \\ \psi_{L2}^\dagger \end{pmatrix}, \quad \text{while} \quad \psi_L^\dagger = (\, \psi_{L1}^\dagger \quad \psi_{L2}^\dagger \,), \tag{11}$$

and likewise for the ψ_R and its conjugates.

Finally, consider the Dirac Lagrangian $\overline{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi$.

(f) Express this Lagrangian in terms of the Weyl spinor fields $\psi_L(x)$ and $\psi_R(x)$ (and their conjugates $\psi_L^\dagger(x)$ and $\psi_R^\dagger(x)$).

(g) Show that for $m = 0$ — and only for $m = 0$ — the two Weyl spinor fields become independent from each other.

3. The third problem is about the plane-wave solutions of the Dirac equation, $e^{-ipx}u_\alpha$ and $e^{+ipx}v_\alpha$ for some x-independent Dirac spinors $u_\alpha(p,s)$ and $v_\alpha(p,s)$.

(a) Check that these waves indeed solve the Dirac equation provided $p^2 = m^2$ while

$$(\not{p} - m)u(p,s) = 0, \quad (\not{p} + m)v(p,s) = 0. \tag{12}$$

By convention, we always take $E = p^0 = +\sqrt{\mathbf{p}^2 + m^2}$ — that's why we have both $e^{-ipx}u_\alpha$ and $e^{+ipx}v_\alpha$ types of wave — while the spinor coefficients $u(p,s)$ and $v(p,s)$ are normalized to

$$u^\dagger(p,s)u(p,s') = v^\dagger(p,s)v(p,s') = 2E\delta_{s,s'}\,. \tag{13}$$

In this problem we shall write down explicit formulae for these spinors in the Weyl basis for the γ^μ matrices.

(b) Show that for $\mathbf{p}=0$,

$$u(\mathbf{p}=\mathbf{0},s)\ =\ \begin{pmatrix}\sqrt{m}\,\xi_s\\ \sqrt{m}\,\xi_s\end{pmatrix} \tag{14}$$

where ξ_s is a two-component $SO(3)$ spinor encoding the electron's spin state. The ξ_s are normalized to $\xi_s^\dagger\xi_{s'}=\delta_{s,s'}$.

(c) For other momenta, $u(p,s)=M_D(\text{boost})\times u(\mathbf{p}=0,s)$ for the boost that turns $(m,\vec{0})$ into p^μ. Use eqs. (8) to show that

$$u(p,s)\ =\ \begin{pmatrix}\sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}}\,\xi_s\\ \sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}}\,\xi_s\end{pmatrix}\ =\ \begin{pmatrix}\sqrt{p_\mu\bar{\sigma}^\mu}\,\xi_s\\ \sqrt{p_\mu\sigma^\mu}\,\xi_s\end{pmatrix}. \tag{15}$$

(d) Use similar arguments to show that

$$v(p,s)\ =\ \begin{pmatrix}+\sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}}\,\eta_s\\ -\sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}}\,\eta_s\end{pmatrix}\ =\ \begin{pmatrix}+\sqrt{p_\mu\bar{\sigma}^\mu}\,\eta_s\\ -\sqrt{p_\mu\sigma^\mu}\,\eta_s\end{pmatrix} \tag{16}$$

where η_s are two-component $SO(3)$ spinors normalized to $\eta_s^\dagger\eta_{s'}=\delta_{s,s'}$.

Physically, the η_s should have opposite spins from ξ_s — the holes in the Dirac sea have opposite spins (as well as p^μ) from the missing negative-energy particles. Mathematically, this requires $\eta_s^\dagger\mathbf{S}\eta_s=-\xi_s^\dagger\mathbf{S}\xi_s$; we may solve this condition by letting $\eta_s=\sigma_2\xi_s^*=\pm i\xi_{-s}^*$.

(e) Check that $\eta_s=\sigma_2\xi_s^*=\pm i\xi_{-s}^*$ indeed provides for the $\eta_s^\dagger\mathbf{S}\eta_s=-\xi_s^\dagger\mathbf{S}\xi_s$, then show that this leads to $v(p,s)=\gamma^2u^*(p,s)$.

(f) Show that for the ultra-relativistic electrons or positrons of definite helicity $\lambda=\pm\frac{1}{2}$, the Dirac plane waves become *chiral* — *i.e.*, dominated by one of the two irreducible Weyl spinor components $\psi_L(x)$ or $\psi_R(x)$ of the Dirac spinor $\Psi(x)$, while the other component becomes negligible. Specifically,

$$\begin{aligned}
u(p,-\tfrac{1}{2})\ &\approx\ \sqrt{2E}\begin{pmatrix}\xi_L\\ 0\end{pmatrix}, \qquad & u(p,+\tfrac{1}{2})\ &\approx\ \sqrt{2E}\begin{pmatrix}0\\ \xi_R\end{pmatrix},\\
v(p,-\tfrac{1}{2})\ &\approx\ -\sqrt{2E}\begin{pmatrix}0\\ \eta_L\end{pmatrix}, \qquad & v(p,+\tfrac{1}{2})\ &\approx\ \sqrt{2E}\begin{pmatrix}\eta_R\\ 0\end{pmatrix}.
\end{aligned} \tag{17}$$

Note that for the electron waves the helicity agrees with the chirality — they are both left or both right, — but for the positron waves the chirality is opposite from the helicity.

Back in problem 2(g) we saw that for $m = 0$ the LH and the RH Weyl spinor fields decouple from each other. Now this exercise show us which particle modes comprise each Weyl spinor: *The $\psi_L(x)$ and its hermitian conjugate $\psi_L^\dagger(x)$ contain the left-handed fermions and the right-handed antifermions, while the $\psi_R(x)$ and the $\psi_R^\dagger(x)$ contain the right-handed fermions and the left-handed antifermions.*

4. Finally, let's establish some basis-independent properties of the Dirac spinors $u(p,s)$ and $v(p,s)$ — although you may use the Weyl basis to verify them.

   (a) Show that

$$\bar{u}(p,s)u(p,s') \;=\; +2m\delta_{s,s'}\,, \quad \bar{v}(p,s)v(p,s') \;=\; -2m\delta_{s,s'}\,; \tag{18}$$

   note the $\pm 2m$ mormalization factors here, unlike the $+2E$ factors in eq. (13) for the $u^\dagger u$ and the $v^\dagger v$.

   (b) There are only two independent $SO(3)$ spinors, hence $\sum_s \xi_s \xi_s^\dagger = \sum_s \eta_s^\dagger \eta_s = \mathbf{1}_{2\times 2}$. Use this fact to show that

$$\sum_{s=1,2} u_\alpha(p,s)\bar{u}_\beta(p,s) \;=\; (\not{p}+m)_{\alpha\beta} \quad \text{and} \quad \sum_{s=1,2} v_\alpha(p,s)\bar{v}_\beta(p,s) \;=\; (\not{p}-m)_{\alpha\beta}\,. \tag{19}$$