Most of this homework (problems 1–5) is about discrete symmetries of Dirac fermions, the charge conjugation **C** and the parity (reflection of space) **P**. But the reading assignment (problem 6, added on Tuesday 10/18 at 10 PM) is on a different subject, classifying the finite multiplets of the Lorentz symmetry.

1. Let's start with the charge conjugation **C** which exchanges particles with antiparticles, for example the electrons e^- with the positrons e^+,

$$\widehat{\mathbf{C}}\left|e^-(\mathbf{p},s)\right\rangle \;=\; \left|e^+(\mathbf{p},s)\right\rangle, \quad \widehat{\mathbf{C}}\left|e^+(\mathbf{p},s)\right\rangle \;=\; \left|e^-(\mathbf{p},s)\right\rangle. \tag{1}$$

Note that the operator $\widehat{\mathbf{C}}$ is unitary and squares to one (repeating the exchange brings us back to the original particles), hence $\widehat{\mathbf{C}}^\dagger = \widehat{\mathbf{C}}^{-1} = \widehat{\mathbf{C}}$.

(a) In the fermionic Fock space, the $\widehat{\mathbf{C}}$ operator act on multi-particle states by turning each particle into an antiparticle and vice verse according to eqs. (1). Show that this action implies

$$\widehat{\mathbf{C}}\,\hat{a}^\dagger_{\mathbf{p},s}\widehat{\mathbf{C}} \;=\; \hat{b}^\dagger_{\mathbf{p},s}\,, \quad \widehat{\mathbf{C}}\,\hat{b}^\dagger_{\mathbf{p},s}\widehat{\mathbf{C}} \;=\; \hat{a}^\dagger_{\mathbf{p},s}\,, \quad \widehat{\mathbf{C}}\,\hat{a}_{\mathbf{p},s}\widehat{\mathbf{C}} \;=\; \hat{b}_{\mathbf{p},s}\,, \quad \widehat{\mathbf{C}}\,\hat{b}_{\mathbf{p},s}\widehat{\mathbf{C}} \;=\; \hat{a}_{\mathbf{p},s}\,. \tag{2}$$

(b) The quantum Dirac fields $\widehat{\Psi}(x)$ and $\widehat{\overline{\Psi}}(x)$ are linear combinations of creation and annihilation operators. Use eqs. (2) and the plane-wave relations $v(p,s) = \gamma^2 u^*(p,s)$ and $u(p,s) = \gamma^2 v^*(p,s)$ from the previous homework to show that

$$\widehat{\mathbf{C}}\widehat{\Psi}(x)\widehat{\mathbf{C}} \;=\; \gamma^2\widehat{\Psi}^*(x) \quad\text{and}\quad \widehat{\mathbf{C}}\widehat{\overline{\Psi}}(x)\widehat{\mathbf{C}} \;=\; \widehat{\overline{\Psi}}^*(x)\gamma^2 \tag{3}$$

where $*$ stands for the hermitian conjugation of the component fields but without transposing a column vector (of 4 Dirac components) into a row vector or vice verse, thus

$$\widehat{\Psi} = \begin{pmatrix} \hat{\psi}_1 \\ \hat{\psi}_2 \\ \hat{\psi}_3 \\ \hat{\psi}_4 \end{pmatrix}, \qquad \widehat{\Psi}^* = \begin{pmatrix} \hat{\psi}_1^\dagger \\ \hat{\psi}_2^\dagger \\ \hat{\psi}_3^\dagger \\ \hat{\psi}_4^\dagger \end{pmatrix}, \qquad \begin{aligned} \widehat{\overline{\Psi}} &= \left(\hat{\psi}_1^\dagger, \hat{\psi}_2^\dagger, \hat{\psi}_3^\dagger, \hat{\psi}_4^\dagger\right) \times \gamma^0, \\ \widehat{\overline{\Psi}}^* &= \left(\hat{\psi}_1, \hat{\psi}_2, \hat{\psi}_3, \hat{\psi}_4\right) \times \gamma^0. \end{aligned} \tag{4}$$

(c) Show that the Dirac equation transforms covariantly under the charge conjugation (3). Hint: prove and use $\gamma^\mu\gamma^2 = -\gamma^2(\gamma^\mu)^*$ for all γ^μ in the Weyl basis.

(d) Show that that the *classical* Dirac Lagrangian is invariant under the charge conjugation (up to a total spacetime derivative). Note that in the classical limit the Dirac fields *anticommute* with each other, $\Psi_\alpha^*\Psi_\beta = -\Psi_\beta\Psi_\alpha^*$. Also, similar to the hermitian conjugation of quantum fields, the complex conjugation of fermionic fields reverses their order: $(F_1F_2)^* = F_2^*F_1^* = -F_1^*F_2^*$.

2. Now consider the *parity* $\mathbf{P}$, the im-proper Lorentz symmetry that reflects the space but not the time, $(\mathbf{x},t) \to (-\mathbf{x},+t)$. This symmetry acts on the Dirac spinor fields according to

$$\widehat{\Psi}'(-\mathbf{x},+t) \;=\; \pm\gamma^0\widehat{\Psi}(+\mathbf{x},+t) \tag{5}$$

where the overall $\pm$ sign is the *intrinsic parity* of the fermion species described by the $\widehat{\Psi}$ field.

(a) Verify that the Dirac equation transforms covariantly under (5) and that the Dirac Lagrangian is invariant (apart from $\mathcal{L}(\mathbf{x},t) \to \mathcal{L}(-\mathbf{x},t)$).

In the Fock space, eq. (5) becomes

$$\widehat{\mathbf{P}}\widehat{\Psi}(\mathbf{x},t)\widehat{\mathbf{P}} \;=\; \pm\gamma^0\widehat{\Psi}(-\mathbf{x},t) \tag{6}$$

for some unitary operator $\widehat{\mathbf{P}}$ that squares to one. Let's find how this operator acts on the particles and their states.

(b) First, look up the plane-wave solutions $u(\mathbf{p},s)$ and $v(\mathbf{p},s)$ in the previos homework and show that $u(-\mathbf{p},s)=+\gamma^0 u(\mathbf{p},s)$ while $v(-\mathbf{p},s)=-\gamma^0 v(\mathbf{p},s)$.

(c) Now show that eq. (6) implies

$$\begin{aligned} \widehat{\mathbf{P}}\,\hat{a}_{\mathbf{p},s}\,\widehat{\mathbf{P}} &= \pm\hat{a}_{-\mathbf{p},+s}\,, & \widehat{\mathbf{P}}\,\hat{a}^\dagger_{\mathbf{p},s}\,\widehat{\mathbf{P}} &= \pm\hat{a}^\dagger_{-\mathbf{p},+s}\,, \\ \widehat{\mathbf{P}}\,\hat{b}_{\mathbf{p},s}\,\widehat{\mathbf{P}} &= \mp\hat{b}_{-\mathbf{p},+s}\,, & \widehat{\mathbf{P}}\,\hat{b}^\dagger_{\mathbf{p},s}\,\widehat{\mathbf{P}} &= \mp\hat{b}^\dagger_{-\mathbf{p},+s}\,, \end{aligned} \tag{7}$$

and hence

$$\widehat{\mathbf{P}}\,|F(\mathbf{p},s)\rangle \;=\; \pm\,|F(-\mathbf{p},+s)\rangle \quad\text{and}\quad \widehat{\mathbf{P}}\,\big|\overline{F}(\mathbf{p},s)\big\rangle \;=\; \mp\,\big|\overline{F}(-\mathbf{p},+s)\big\rangle\,. \tag{8}$$

Note that the fermion F and the antifermion $\overline{F}$ have opposite intrinsic parities!

3. Some electrically neutral particles carry other kinds of changes (forex, the baryon number) that distinguish them from their antiparticles. But other particles — such as the photon or the π^0 meson — have no charges at all and act as their own antiparticles. The charge conjugation symmetry turns such particles n into themselves,

$$\widehat{\mathbf{C}}\,|n(\mathbf{p},s)\rangle \;=\; \pm\,|n(\mathbf{p},s)\rangle\,, \tag{9}$$

where the overall $\pm$ sign is called the *C-parity* or *charge-parity* of the particle in question. This C-parity — as well as the P-parity under space reflections — limit the allowed decay channels of unstable particles via strong and EM interactions which respect both $\widehat{\mathbf{C}}$ and $\widehat{\mathbf{P}}$ symmetries.

Consider a bound state of a charged Dirac fermion F and the corresponding antifermion, for example a $q\bar{q}$ meson or a positronium "atom" (a hydrogen-atom-like bound state of e^- and e^+). For simplicity, let this bound state have zero net momentum. In the Fock space of fermions and antifermions, such a bound state appears as

$$|B(\mathbf{p}_{\rm tot}=0)\rangle \;=\; \int\!\frac{d^3\mathbf{p}_{\rm red}}{(2\pi)^3}\sum_{s_1,s_2}\psi(\mathbf{p}_{\rm red},s_1,s_2)\times\hat{a}^\dagger(+\mathbf{p}_{\rm red},s_1)\,\hat{b}^\dagger(-\mathbf{p}_{\rm red},s_2)\,|0\rangle \tag{10}$$

for some wave-function ψ of the reduced momentum and of the two spins.

Suppose this bound state has a definite orbital angular momentum L — which controls the symmetry of the wave function ψ with respect to $\mathbf{p}_{\rm red} \to -\mathbf{p}_{\rm red}$ — and definite net spin S — which controls the symmetry of ψ under $s_1 \leftrightarrow s_2$. Turns out that the L and the S of the bound state also determine its C-parity and P-parity.

(a) Show that $C = (-1)^{L+S}$.

(b) Show that $P = (-1)^{L+1}$.

Now let's apply these results to the positronium — a hydrogen-atom-like bound state of a positron e^+ and an electron e^-. The ground state of positronium is hydrogen-like 1S ($n = 1$, $L = 0$), with the net spin which could be either $S = 0$ or $S = 1$.

(c) Explain why the $S = 0$ state annihilates into photons much faster than the $S = 1$ state.
Hint#1: The annihilation rate of positronium into n photons happens in the $n^{\rm th}$ order of QED perturbation theory, so the rate $\propto \alpha^n$ (for $\alpha \approx 1/137$).
Hint#2: Since the EM fields couple linearly to the electric charges and currents (which are reversed by $\widehat{\mathbf{C}}$), each photon has $C = -1$.

4. A Dirac spinor field $\Psi(x)$ comprises two 2-component Weyl spinor fields,

$$\widehat{\Psi}(x) \;=\; \begin{pmatrix} \hat{\psi}_L(x) \\ \hat{\psi}_R(x) \end{pmatrix}. \tag{11}$$

Spell out the actions of the C, P, and the combined CP symmetry on the Weyl spinors. In particular, show that C and P interchange the two spinors, while the combined CP symmetry acts on the ψ_L and the ψ_R independently from each other.

5. Now consider the bilinear products of a Dirac field $\Psi(x)$ and its conjugate $\overline{\Psi}(x)$. Generally, such products have form $\overline{\Psi}\Gamma\Psi$ where Γ is one of 16 matrices discussed in the previous homework (problem 1.g); altogether, we have

$$S = \overline{\Psi}\Psi, \quad V^\mu = \overline{\Psi}\gamma^\mu\Psi, \quad T^{\mu\nu} = \overline{\Psi}\tfrac{i}{2}\gamma^{[\mu}\gamma^{\nu]}\Psi, \quad A^\mu = \overline{\Psi}\gamma^5\gamma^\mu\Psi, \quad P = \overline{\Psi}i\gamma^5\Psi. \tag{12}$$

(a) Show that all the bilinears (12) are Hermitian.
Hint: First, show that $\left(\overline{\Psi}\Gamma\Psi\right)^\dagger = \overline{\Psi}\overline{\Gamma}\Psi$.
Note: despite the Fermi statistics, $\left(\Psi^\dagger_\alpha\Psi_\beta\right)^\dagger = +\Psi^\dagger_\beta\Psi_\alpha$.

(b) Show that under *continuous* Lorentz symmetries, the S and the P transform as scalars, the V^μ and the A^μ as vectors, and the $T^{\mu\nu}$ as an antisymmetric tensor.

(c) Find the transformation rules of the bilinears (12) under parity and show that while S is a true scalar and V is a true (polar) vector, P is a pseudoscalar and A is an axial vector.

Next, consider the charge-conjugation properties of the Dirac bilinears. To avoid the operator-ordering problems, take the classical limit where $\Psi(x)$ and $\Psi^\dagger(x)$ *anticommute* with each other, $\Psi_\alpha\Psi^\dagger_\beta = -\Psi^\dagger_\beta\Psi_\alpha$.

(d) Show that **C** turns $\overline{\Psi}\Gamma\Psi$ into $\overline{\Psi}\Gamma^c\Psi$ where $\Gamma^c = \gamma^0\gamma^2\Gamma^\top\gamma^0\gamma^2$.

(e) Calculate Γ^c for all 16 independent matrices Γ and find out which Dirac bilinears are C–even and which are C–odd.

6. Finally, a reading assignment: my notes on finite multiplets of the $\mathrm{Spin}(3,1) \cong SL(2\mathbf{C})$ group, the double cover of the continuous Lorentz group $SO^+(3,1)$.

⋆ For extra challenge, solve the exercises interspersed with the notes.