PHY-396 K. Problem set \#7. Due October 27, 2016.

Most of this homework (problems 1-5) is about discrete symmetries of Dirac fermions, the charge conjugation $\mathbf{C}$ and the parity (reflection of space) $\mathbf{P}$. But the reading assignment (problem 6, added on Tuesday $10 / 18$ at 10 PM ) is on a different subject, classifying the finite multiplets of the Lorentz symmetry.

1. Let's start with the charge conjugation $\mathbf{C}$ which exchanges particles with antiparticles, for example the electrons $e^{-}$with the positrons $e^{+}$,

$$
\begin{equation*}
\widehat{\mathbf{C}}\left|e^{-}(\mathbf{p}, s)\right\rangle=\left|e^{+}(\mathbf{p}, s)\right\rangle, \quad \widehat{\mathbf{C}}\left|e^{+}(\mathbf{p}, s)\right\rangle=\left|e^{-}(\mathbf{p}, s)\right\rangle . \tag{1}
\end{equation*}
$$

Note that the operator $\widehat{\mathbf{C}}$ is unitary and squares to one (repeating the exchange brings us back to the original particles), hence $\widehat{\mathbf{C}}^{\dagger}=\widehat{\mathbf{C}}^{-1}=\widehat{\mathbf{C}}$.
(a) In the fermionic Fock space, the $\widehat{\mathbf{C}}$ operator act on multi-particle states by turning each particle into an antiparticle and vice verse according to eqs. (1). Show that this action implies

$$
\begin{equation*}
\widehat{\mathbf{C}} \hat{a}_{\mathbf{p}, s}^{\dagger} \widehat{\mathbf{C}}=\hat{b}_{\mathbf{p}, s}^{\dagger}, \quad \widehat{\mathbf{C}} \hat{b}_{\mathbf{p}, s}^{\dagger} \widehat{\mathbf{C}}=\hat{a}_{\mathbf{p}, s}^{\dagger}, \quad \widehat{\mathbf{C}} \hat{a}_{\mathbf{p}, s} \widehat{\mathbf{C}}=\hat{b}_{\mathbf{p}, s}, \quad \widehat{\mathbf{C}} \hat{b}_{\mathbf{p}, s} \widehat{\mathbf{C}}=\hat{a}_{\mathbf{p}, s} \tag{2}
\end{equation*}
$$

(b) The quantum Dirac fields $\widehat{\Psi}(x)$ and $\widehat{\bar{\Psi}}(x)$ are linear combinations of creation and annihilation operators. Use eqs. (2) and the plane-wave relations $v(p, s)=\gamma^{2} u^{*}(p, s)$ and $u(p, s)=\gamma^{2} v^{*}(p, s)$ from the previous homework to show that

$$
\begin{equation*}
\widehat{\mathbf{C}} \widehat{\Psi}(x) \widehat{\mathbf{C}}=\gamma^{2} \widehat{\Psi}^{*}(x) \quad \text { and } \quad \widehat{\mathbf{C}} \widehat{\bar{\Psi}}(x) \widehat{\mathbf{C}}=\widehat{\bar{\Psi}}^{*}(x) \gamma^{2} \tag{3}
\end{equation*}
$$

where $*$ stands for the hermitian conjugation of the component fields but without transposing a column vector (of 4 Dirac components) into a row vector or vice verse,
thus

$$
\widehat{\Psi}=\left(\begin{array}{c}
\hat{\psi}_{1}  \tag{4}\\
\hat{\psi}_{2} \\
\hat{\psi}_{3} \\
\hat{\psi}_{4}
\end{array}\right), \quad \widehat{\Psi}^{*}=\left(\begin{array}{c}
\hat{\psi}_{1}^{\dagger} \\
\hat{\psi}_{2}^{\dagger} \\
\hat{\psi}_{3}^{\dagger} \\
\hat{\psi}_{4}^{\dagger}
\end{array}\right), \quad \begin{gathered}
\overline{\bar{\Psi}}=\left(\hat{\psi}_{1}^{\dagger}, \hat{\psi}_{2}^{\dagger}, \hat{\psi}_{3}^{\dagger}, \hat{\psi}_{4}^{\dagger}\right) \times \gamma^{0}, \\
\widehat{\bar{\Psi}}^{*}=\left(\hat{\psi}_{1}, \hat{\psi}_{2}, \hat{\psi}_{3}, \hat{\psi}_{4}\right) \times \gamma^{0} .
\end{gathered}
$$

(c) Show that the Dirac equation transforms covariantly under the charge conjugation (3). Hint: prove and use $\gamma^{\mu} \gamma^{2}=-\gamma^{2}\left(\gamma^{\mu}\right)^{*}$ for all $\gamma^{\mu}$ in the Weyl basis.
(d) Show that that the classical Dirac Lagrangian is invariant under the charge conjugation (up to a total spacetime derivative). Note that in the classical limit the Dirac fields anticommute with each other, $\Psi_{\alpha}^{*} \Psi_{\beta}=-\Psi_{\beta} \Psi_{\alpha}^{*}$. Also, similar to the hermitian conjugation of quantum fields, the complex conjugation of fermionic fields reverses their order: $\left(F_{1} F_{2}\right)^{*}=F_{2}^{*} F_{1}^{*}=-F_{1}^{*} F_{2}^{*}$.
2. Now consider the parity $\mathbf{P}$, the im-proper Lorentz symmetry that reflects the space but not the time, $(\mathbf{x}, t) \rightarrow(-\mathbf{x},+t)$. This symmetry acts on the Dirac spinor fields according to

$$
\begin{equation*}
\widehat{\Psi}^{\prime}(-\mathbf{x},+t)= \pm \gamma^{0} \widehat{\Psi}(+\mathbf{x},+t) \tag{5}
\end{equation*}
$$

where the overall $\pm$ sign is the intrinsic parity of the fermion species described by the $\widehat{\Psi}$ field.
(a) Verify that the Dirac equation transforms covariantly under (5) and that the Dirac Lagrangian is invariant (apart from $\mathcal{L}(\mathbf{x}, t) \rightarrow \mathcal{L}(-\mathbf{x}, t)$ ).

In the Fock space, eq. (5) becomes

$$
\begin{equation*}
\widehat{\mathbf{P}} \widehat{\Psi}(\mathbf{x}, t) \widehat{\mathbf{P}}= \pm \gamma^{0} \widehat{\Psi}(-\mathbf{x}, t) \tag{6}
\end{equation*}
$$

for some unitary operator $\widehat{\mathbf{P}}$ that squares to one. Let's find how this operator acts on the particles and their states.
(b) First, look up the plane-wave solutions $u(\mathbf{p}, s)$ and $v(\mathbf{p}, s)$ in the previos homework and show that $u(-\mathbf{p}, s)=+\gamma^{0} u(\mathbf{p}, s)$ while $v(-\mathbf{p}, s)=-\gamma^{0} v(\mathbf{p}, s)$.
(c) Now show that eq. (6) implies

$$
\begin{align*}
& \widehat{\mathbf{P}} \hat{a}_{\mathbf{p}, s} \widehat{\mathbf{P}}= \pm \hat{a}_{-\mathbf{p},+s}, \quad \widehat{\mathbf{P}} \hat{a}_{\mathbf{p}, s}^{\dagger} \widehat{\mathbf{P}}= \pm \hat{a}_{-\mathbf{p},+s}^{\dagger}, \\
& \widehat{\mathbf{P}} \hat{b}_{\mathbf{p}, s} \widehat{\mathbf{P}}=\mp \hat{b}_{-\mathbf{p},+s}, \quad \widehat{\mathbf{P}} \hat{b}_{\mathbf{p}, s}^{\dagger} \widehat{\mathbf{P}}=\mp \hat{b}_{-\mathbf{p},+s}^{\dagger}, \tag{7}
\end{align*}
$$

and hence

$$
\begin{equation*}
\widehat{\mathbf{P}}|F(\mathbf{p}, s)\rangle= \pm|F(-\mathbf{p},+s)\rangle \quad \text { and } \quad \widehat{\mathbf{P}}|\bar{F}(\mathbf{p}, s)\rangle=\mp|\bar{F}(-\mathbf{p},+s)\rangle . \tag{8}
\end{equation*}
$$

Note that the fermion $F$ and the antifermion $\bar{F}$ have opposite intrinsic parities!
3. Some electrically neutral particles carry other kinds of changes (forex, the baryon number) that distinguish them from their antiparticles. But other particles - such as the photon or the $\pi^{0}$ meson - have no charges at all and act as their own antiparticles. The charge conjugation symmetry turns such particles $n$ into themselves,

$$
\begin{equation*}
\widehat{\mathbf{C}}|n(\mathbf{p}, s)\rangle= \pm|n(\mathbf{p}, s)\rangle, \tag{9}
\end{equation*}
$$

where the overall $\pm$ sign is called the C-parity or charge-parity of the particle in question. This C-parity - as well as the P-parity under space reflections - limit the allowed decay channels of unstable particles via strong and EM interactions which respect both $\widehat{\mathbf{C}}$ and $\widehat{\mathbf{P}}$ symmetries.

Consider a bound state of a charged Dirac fermion $F$ and the corresponding antifermion, for example a $q \bar{q}$ meson or a positronium "atom" (a hydrogen-atom-like bound state of $e^{-}$and $e^{+}$). For simplicity, let this bound state have zero net momentum. In the Fock space of fermions and antifermions, such a bound state appears as

$$
\begin{equation*}
\left|B\left(\mathbf{p}_{\text {tot }}=0\right)\right\rangle=\int \frac{d^{3} \mathbf{p}_{\text {red }}}{(2 \pi)^{3}} \sum_{s_{1}, s_{2}} \psi\left(\mathbf{p}_{\text {red }}, s_{1}, s_{2}\right) \times \hat{a}^{\dagger}\left(+\mathbf{p}_{\text {red }}, s_{1}\right) \hat{b}^{\dagger}\left(-\mathbf{p}_{\text {red }}, s_{2}\right)|0\rangle \tag{10}
\end{equation*}
$$

for some wave-function $\psi$ of the reduced momentum and of the two spins.

Suppose this bound state has a definite orbital angular momentum $L$ - which controls the symmetry of the wave function $\psi$ with respect to $\mathbf{p}_{\text {red }} \rightarrow-\mathbf{p}_{\text {red }}$ - and definite net spin $S$ — which controls the symmetry of $\psi$ under $s_{1} \leftrightarrow s_{2}$. Turns out that the $L$ and the $S$ of the bound state also determine its C-parity and P-parity.
(a) Show that $C=(-1)^{L+S}$.
(b) Show that $P=(-1)^{L+1}$.

Now let's apply these results to the positronium - a hydrogen-atom-like bound state of a positron $e^{+}$and an electron $e^{-}$. The ground state of positronium is hydrogen-like 1 S ( $n=1, L=0$ ), with the net spin which could be either $S=0$ or $S=1$.
(c) Explain why the $S=0$ state annihilates into photons much faster than the $S=1$ state.

Hint\#1: The annihilation rate of positronium into $n$ photons happens in the $n^{\text {th }}$ order of QED perturbation theory, so the rate $\propto \alpha^{n}$ (for $\alpha \approx 1 / 137$ ).
Hint\#2: Since the EM fields couple linearly to the electric charges and currents (which are reversed by $\widehat{\mathbf{C}}$ ), each photon has $C=-1$.
4. A Dirac spinor field $\Psi(x)$ comprises two 2-component Weyl spinor fields,

$$
\begin{equation*}
\widehat{\Psi}(x)=\binom{\hat{\psi}_{L}(x)}{\hat{\psi}_{R}(x)} . \tag{11}
\end{equation*}
$$

Spell out the actions of the C, P, and the combined CP symmetry on the Weyl spinors. In particular, show that C and P interchange the two spinors, while the combined CP symmetry acts on the $\psi_{L}$ and the $\psi_{R}$ independently from each other.
5. Now consider the bilinear products of a Dirac field $\Psi(x)$ and its conjugate $\bar{\Psi}(x)$. Generally, such products have form $\bar{\Psi} \Gamma \Psi$ where $\Gamma$ is one of 16 matrices discussed in the previous homework (problem 1.g); altogether, we have

$$
\begin{equation*}
S=\bar{\Psi} \Psi, \quad V^{\mu}=\bar{\Psi} \gamma^{\mu} \Psi, \quad T^{\mu \nu}=\bar{\Psi} \frac{i}{2} \gamma^{[\mu} \gamma^{\nu]} \Psi, \quad A^{\mu}=\bar{\Psi} \gamma^{5} \gamma^{\mu} \Psi, \quad P=\bar{\Psi} i \gamma^{5} \Psi \tag{12}
\end{equation*}
$$

(a) Show that all the bilinears (12) are Hermitian.

Hint: First, show that $(\bar{\Psi} \Gamma \Psi)^{\dagger}=\overline{\Psi \Gamma} \Psi$.
Note: despite the Fermi statistics, $\left(\Psi_{\alpha}^{\dagger} \Psi_{\beta}\right)^{\dagger}=+\Psi_{\beta}^{\dagger} \Psi_{\alpha}$.
(b) Show that under continuous Lorentz symmetries, the $S$ and the $P$ transform as scalars, the $V^{\mu}$ and the $A^{\mu}$ as vectors, and the $T^{\mu \nu}$ as an antisymmetric tensor.
(c) Find the transformation rules of the bilinears (12) under parity and show that while $S$ is a true scalar and $V$ is a true (polar) vector, $P$ is a pseudoscalar and $A$ is an axial vector.

Next, consider the charge-conjugation properties of the Dirac bilinears. To avoid the operator-ordering problems, take the classical limit where $\Psi(x)$ and $\Psi^{\dagger}(x)$ anticommute with each other, $\Psi_{\alpha} \Psi_{\beta}^{\dagger}=-\Psi_{\beta}^{\dagger} \Psi_{\alpha}$.
(d) Show that $\mathbf{C}$ turns $\bar{\Psi} \Gamma \Psi$ into $\bar{\Psi} \Gamma^{c} \Psi$ where $\Gamma^{c}=\gamma^{0} \gamma^{2} \Gamma^{\top} \gamma^{0} \gamma^{2}$.
(e) Calculate $\Gamma^{c}$ for all 16 independent matrices $\Gamma$ and find out which Dirac bilinears are C -even and which are C -odd.
6. Finally, a reading assignment: my notes on finite multiplets of the Spin(3,1) $\cong S L(2 \mathbf{C})$ group, the double cover of the continuous Lorentz group $\mathrm{SO}^{+}(3,1)$.

* For extra challenge, solve the exercises interspersed with the notes.

