1. As a warm-up exercise, consider N scalar fields ϕ_i of the same mass m with $O(N)$ symmetric quartic couplings to each other,

$$\mathcal{L} \;=\; \frac{1}{2}\sum_i(\partial_\mu\phi_i)^2 \;-\; \frac{m^2}{2}\sum_i\phi_i^2 \;-\; \frac{\lambda}{8}\left(\sum_i\phi_i^2\right)^2 . \tag{1}$$

   (a) Write down the Feynman propagators and vertices for this theory.

   (b) Calculate the tree-level scattering amplitudes $\mathcal{M}$, the partial cross-sections $d\sigma/d\Omega$ (in the center-of-mass frame), and the total cross-sections for the following 3 processes:

      (i) $\phi_1+\phi_2\to\phi_1+\phi_2$.

      (ii) $\phi_1+\phi_1\to\phi_2+\phi_2$.

      (iii) $\phi_1+\phi_1\to\phi_1+\phi_1$.

2. Next, consider the so-called *linear sigma model* comprising N massless scalar or pseudoscalar fields π_i and a massive scalar field σ with both quartic and cubic couplings to the pions, specifically

$$\begin{aligned}\mathcal{L} \;&=\; \frac{1}{2}\sum\nolimits_i(\partial_\mu\pi_i)^2 \;+\; \frac{1}{2}(\partial_\mu\sigma)^2 \;-\; \frac{\lambda}{8}\Big(\sum\nolimits_i\pi_i^2 \;+\; \sigma^2 \;+\; 2f\sigma\Big)^2\\ &=\; \frac{1}{2}\sum_i(\partial_\mu\pi_i)^2 \;+\; \frac{1}{2}(\partial_\mu\sigma)^2 \;-\; \frac{M_\sigma^2}{2}\times\sigma^2\\ &\qquad -\; \frac{\kappa}{2}\times\Big(\sigma^3 + \sigma\sum\nolimits_i\pi_i^2\Big) \;-\; \frac{\lambda}{8}\Big(\sum\nolimits_i\pi_i^2 \;+\; \sigma^2\Big)^2\end{aligned} \tag{2}$$

where $M_\sigma^2=\lambda f^2$ and $\kappa=\lambda f$. $\quad\Longrightarrow\quad \kappa^2 \;=\; \lambda\times M_\sigma^2\,.$ (3)

Both the masslessness of the π_i fields and the relation (3) between the couplings κ and λ and the sigma's mass^2 peculiar to this model stem from the *spontaneous breaking down* of the $O(N+1)$ symmetry, which I shall explain in class later this semester. I shall also explain the relation of this model to the approximate chiral symmetry of QCD and hence to the real-life pi-mesons and their low-energy scattering amplitudes.

But in this homework, you should simply take the Lagrangian (2) as it is, and explore its implications for the scattering of π particles.

(a) Write down all the vertices and all the propagators for the Feynman rules for this theory.

(b) Draw *all* the tree diagrams and calculate the tree-level scattering amplitudes of two pions to two pions, $\mathcal{M}_{\text{tree}}(\pi^j + \pi^k \to \pi^\ell + \pi^m)$.

(c) Show that thanks to the relation (3) between the cubic and the quartic couplings, in the low-energy limit $E_{\text{tot}} \ll M_\sigma$, all the amplitudes $\mathcal{M}_{\text{tree}}(\pi^j + \pi^k \to \pi^\ell + \pi^m)$ become small as $O(E_{\text{tot}}^2/M_\sigma^2)$ or smaller.

Then use Mandelstam's variables s, t, u to show that when any of the incoming or outgoing pions' energy becomes small (while the other pions' energies are $O(M_\sigma)$), the scattering amplitudes become small as $O(E_{\text{small}}/M_\sigma)$ or smaller.

Later in class, we shall learn that this behavior stems from the *Goldstone–Nambu theorem.*

(d) Write down specific tree-level amplitudes, partial cross-sections (in the CM frame), and total cross-sections for the processes

(i) $\pi^1 + \pi^2 \to \pi^1 + \pi^2$

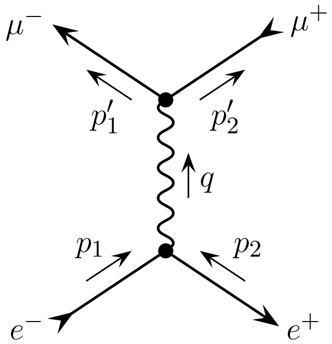
(ii) $\pi^1 + \pi^1 \to \pi^2 + \pi^2$

(iii) $\pi^1 + \pi^1 \to \pi^1 + \pi^1$

in the low-energy limit $E_{\text{cm}} \ll M_\sigma$.

3. Now, a reading assignment: §4.7 of the *Peskin&Schroeder* textbook about the Feynman rules of the Yukawa theory. Find out where the sign rules for the fermionic lines come from. Also find out the origin of the Yukawa potential $V(r) \propto e^{-mr}/r$. (You can find a shorter explanation of the Yukawa potential on the last two pages of my notes on QED Feynman rules.

4. Finally, an exercise in QED (Quantum Electro Dynamics). Consider the muon pair production in electron-positron collisions, $e^-e^+ \to \mu^-\mu^+$. As I explained in class, at the tree level there is only one diagram contributing to this process,



which yields the amplitude

$$\left\langle \mu^-, \mu^+ \middle| \mathcal{M} \middle| e^-, e^+ \right\rangle \;=\; \frac{e^2}{s} \times \bar{u}(\mu^-)\gamma^\nu v(\mu^+) \times \bar{v}(e^+)\gamma_\nu u(e^-). \tag{4}$$

In class I have focused on the un-polarized cross-section for this process, but in this exercise you should focus on the polarized amplitudes for definite helicities of all 4 particles involved.

For simplicity, let us assume that all the particles are ultra-relativistic so that their Dirac spinors $u(e^-)$, $v(e^+)$, $u(\mu^-)$, $v(\mu^+)$ all have definite chiralities,

$$\begin{aligned} u_L \;&\approx\; \sqrt{2E}\begin{pmatrix}\xi_L\\ 0\end{pmatrix}, \qquad & u_R \;&\approx\; \sqrt{2E}\begin{pmatrix}0\\ \xi_R\end{pmatrix},\\ v_L \;&\approx\; -\sqrt{2E}\begin{pmatrix}0\\ \eta_L\end{pmatrix}, \qquad & v_R \;&\approx\; \sqrt{2E}\begin{pmatrix}\eta_R\\ 0\end{pmatrix}. \end{aligned} \tag{5}$$

cf. homework set#6, eq. (17).

(a) Show that in the approximation (5),

$$\bar{v}(e^+_L)\gamma_\nu u(e^-_L) \;=\; \bar{v}(e^+_R)\gamma_\nu u(e^-_R) \;=\; 0, \tag{6}$$

which means there is no muon pairs production unless the initial electron and positron have *opposite helicities*.

(b) Show that the μ^- and the μ^+ must also have *opposite helicities* because

$$\bar{u}(\mu_L^-)\gamma^\nu v(\mu_L^+) \;=\; \bar{u}(\mu_R^-)\gamma^\nu v(\mu_R^+) \;=\; 0. \tag{7}$$

(c) Let's work in the center-of-mass frame where the initial e^- and e^+ collide along the z axis, $p_1^\nu = (E, 0, 0, +E)$, $p_2^\nu = (E, 0, 0, -E)$. Calculate the 4–vector $\bar{v}(e^+)\gamma^\nu u(e^-)$ in this frame and show that

$$\bar{v}(e_L^+)\gamma_\nu u(e_R^-) \;=\; 2E \times (0, -i, +1, 0)_\nu\,, \qquad \bar{v}(e_R^+)\gamma_\nu u(e_L^-) \;=\; 2E \times (0, +i, +1, 0)_\nu\,. \tag{8}$$

(d) In the CM frame the muons fly away in opposite directions at some angle θ to the electron / positron directions. Without loss of generality we may assume the muons' momenta being in the xz plane, thus

$$p_1'^\nu \;=\; (E, +E\sin\theta, 0, +E\cos\theta), \qquad p_1'^\nu \;=\; (E, -E\sin\theta, 0, -E\cos\theta) \tag{9}$$

Calculate the 4–vector $\bar{u}(\mu^-)\gamma_\nu v(\mu^+)$ for the muons and show that

$$\begin{aligned} \bar{u}(\mu_R^-)\gamma^\nu v(\mu_L^+) \;&=\; 2E \times (0, -i\cos\theta, -1, +i\sin\theta), \\ \bar{u}(\mu_L^-)\gamma^\nu v(\mu_R^+) \;&=\; 2E \times (0, +i\cos\theta, -1, -i\sin\theta). \end{aligned} \tag{10}$$

(e) Now calculate the amplitudes (4) for all possible combinations of particles' helicities, calculate the partial cross-sections, and show that

$$\begin{aligned} \frac{d\sigma(e_L^- + e_R^+ \to \mu_L^- + \mu_R^+)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e_R^- + e_L^+ \to \mu_R^- + \mu_L^+)}{d\Omega_{\text{c.m.}}} = \frac{\alpha^2}{4s} \times (1+\cos\theta)^2, \\ \frac{d\sigma(e_L^- + e_R^+ \to \mu_R^- + \mu_L^+)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e_R^- + e_L^+ \to \mu_L^- + \mu_R^+)}{d\Omega_{\text{c.m.}}} = \frac{\alpha^2}{4s} \times (1-\cos\theta)^2, \\ \frac{d\sigma(e_L^- + e_L^+ \to \mu_{\text{any}}^- + \mu_{\text{any}}^+)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e_R^- + e_R^+ \to \mu_{\text{any}}^- + \mu_{\text{any}}^+)}{d\Omega_{\text{c.m.}}} = 0, \\ \frac{d\sigma(e_{\text{any}}^- + e_{\text{any}}^+ \to \mu_L^- + \mu_L^+)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e_{\text{any}}^- + e_{\text{any}}^+ \to \mu_R^- + \mu_R^+)}{d\Omega_{\text{c.m.}}} = 0. \end{aligned} \tag{11}$$

(f) Finally, sum / average over the helicities and calculate the un-polarized cross-section for the muon pair production.