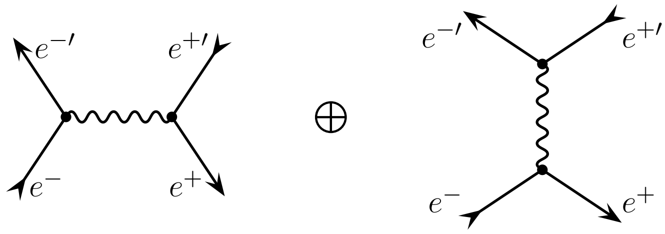
1. Consider the elastic scattering $e^-e^+ \to e^-e^+$ of ultra-relativistic electrons and positrons. This process is called the *Bhabha scattering* after Homi Bhabha who has calculated the cross-section back in 1935. His calculation was the leading order in perturbation theory; in modern terms, it corresponds to the three-level of QED. Today, the Bhabha cross-section is known to very high precision, so the observed rate of Bhabha scatterings at electron-positron colliders is used to monitor the collider's luminosity.

   At the tree level of QED, there are two diagrams contributing to the Bhabha scattering, namely



(1)

   (a) Evaluate the two diagrams and write down the amplitude $\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$. Mind the sign rules for the fermions.

   Now comes the real work: calculating the un-polarized partial cross-section

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{c.m.}} = \frac{\overline{|\mathcal{M}|^2}}{64\pi^2 s} \tag{2}$$

   where $\overline{|\mathcal{M}|^2}$ stands for $|\mathcal{M}|^2$ summed over final particle spins and averaged over the spins of the initial particles. Note the two diagrams (1) must be added together before squaring the amplitude, because

$$|\mathcal{M}_1 + \mathcal{M}_2|^2 = |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 + 2\,\mathrm{Re}\big(\mathcal{M}_1^*\mathcal{M}_2\big) \neq |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2. \tag{3}$$

   For simplicity, assume $E \gg m_e$ and neglect the electron's mass throughout your calculation. You may find it convenient to express products of momenta in terms of Mandelstam's variables s, t, and u. In the $m_e \approx 0$ approximation, $p_1^2 = p_2^2 = p_1'^2 = p_2'^2 = m_e^2 \approx 0$ while

$$(p_1p_2) = (p_1'p_2') \approx \tfrac{1}{2}s, \quad (p_1p_1') = (p_2p_2') \approx -\tfrac{1}{2}t, \quad (p_1p_2') = (p_2p_1') \approx -\tfrac{1}{2}u. \tag{4}$$

(b) Let's start with the second diagram's amplitude $\mathcal{M}_2$. Sum / average the $|\mathcal{M}_2|^2$ over all spins and show that

$$\frac{1}{4}\sum_{\text{all spins}} |\mathcal{M}_2|^2 \;=\; 2e^4 \times \frac{t^2+u^2}{s^2}\,. \tag{5}$$

(c) Similarly, show that for the first diagram

$$\frac{1}{4}\sum_{\text{all spins}} |\mathcal{M}_1|^2 \;=\; 2e^4 \times \frac{s^2+u^2}{t^2}\,. \tag{6}$$

(d) Now consider the interference $\mathcal{M}_1^* \times \mathcal{M}_2$ between the two diagrams. Show that

$$\frac{1}{4}\sum_{\text{all spins}} \mathcal{M}_1^* \times \mathcal{M}_2 \;=\; 2e^4 \times \frac{u^2}{st}\,. \tag{7}$$

(e) Finally assemble all the terms together and show that for the Bhabha scattering

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{c.m.}} \;=\; \frac{\alpha^2}{2s} \times \frac{s^4+t^4+u^4}{s^2\times t^2} \;=\; \frac{\alpha^2}{4s} \times \left(\frac{3+\cos^2\theta}{1-\cos\theta}\right)^2. \tag{8}$$

2. Next, read *carefully* my notes on annihilation and Compton scattering and pay attention to the algebra. Make sure you understand and can follow all the calculations.

3. Now consider a QFT where heavy (*i.e.*, $M_s \gg m_e$) neutral scalar particles have Yukawa–like coupling to electrons, which in turn couple to photons according to the usual QED rules, thus

$$\mathcal{L} \;=\; -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \;+\; \overline{\Psi}(i\not{D}-m_e)\Psi \,+\, \left[\tfrac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi \;-\; \tfrac{1}{2}M_s^2\varphi^2\right] \;+\; g\varphi\times\overline{\Psi}\Psi. \tag{9}$$

In this theory, an electron and a positron colliding with energy $E_{\text{c.m.}} > M_s$ may annihilate into one photon and one scalar particle, $e^- + e^+ \to \gamma + S$.

(a) Draw tree diagrams for the $e^- + e^+ \to \gamma + S$ process and write down the tree-level matrix element $\langle\gamma S|\,\mathcal{M}\,\big|e^- e^+\big\rangle$.

(b) Verify the Ward identity for the photon. Note: the Ward identity does not have to work for individual diagrams, but it must work for the net amplitude.

(c) Sum $|\mathcal{M}|^2$ over the photon's polarizations and average over the fermion's spins. Show that

$$\overline{|\mathcal{M}|^2} \;\equiv\; \frac{1}{4}\sum_{s_-,s_+}\sum_{\lambda}|\mathcal{M}|^2 \;=\; e^2g^2\left(\frac{A_{11}}{(t-m_e^2)^2} \;+\; \frac{A_{22}}{(u-m_e^2)^2} \;+\; \frac{2\Re A_{12}}{(t-m_e^2)(u-m_e^2)}\right) \tag{10}$$

where

$$\begin{aligned} A_{11} &= -\tfrac{1}{4}\,\mathrm{Tr}\Big((\not{p}_+ - m_e)(\not{q} + m_e)\gamma^\mu(\not{p}_- + m_e)\gamma_\mu(\not{q} + m_e)\Big),\\ A_{22} &= -\tfrac{1}{4}\,\mathrm{Tr}\Big((\not{p}_+ - m_e)\gamma^\mu(\tilde{\not{q}} + m_e)(\not{p}_- + m_e)(\tilde{\not{q}} + m_e)\gamma_\mu\Big),\\ A_{12} &= -\tfrac{1}{4}\,\mathrm{Tr}\Big((\not{p}_+ - m_e)(\not{q} + m_e)\gamma^\mu(\not{p}_- + m_e)(\tilde{\not{q}} + m_e)\gamma_\mu\Big). \end{aligned} \tag{11}$$

Since $M_s \gg m_e$, the initial electron and positron must be ultra-relativistic. So let's simplify our calculation by neglecting the electron's mass both in the traces (11) and in the denominators in eq. (10).

(d) Evaluate the Dirac traces (11) in the $m_e \approx 0$ approximation and express them in terms of the Mandelstam variables s, t, u. Show that

$$\text{for } m_e \approx 0, \quad A_{11} \;\approx\; A_{22} \;\approx\; tu, \quad A_{12} \;\approx\; (t - M_S^2)(u - M_s^2). \tag{12}$$

Note: because of the scalar's mass, the kinematic relations between various momentum products such as $(k_\gamma p_\mp)$ and between the Mandelstam's s, t, and u are different from the $e^+e^- \to \gamma\gamma$ annihilation.

(e) Finally, assemble the net $|\mathcal{M}|^2$ (in the $m_e \approx 0$ approximation), work out the kinematics in the CM frame, and calculate the partial cross-section

$$\frac{d\sigma(e^-e^+ \to \gamma S)}{d\Omega_{\text{c.m.}}}\,.$$