- In my notations, the A_μ and their components A_μ^a are the canonically normalized vector fields, while the $\mathcal{A}_\mu = gA_\mu$ and the $\mathcal{A}_\mu^a = gA_\mu^a$ are normalized by the symmetry action. Likewise, the tension fields $F_{\mu\nu}$ and their components $F_{\mu\nu}^a$ are canonically normalized while the $\mathcal{F}_{\mu\nu} = gF_{\mu\nu}$ and the $\mathcal{F}_{\mu\nu}^a = gF_{\mu\nu}^a$ are normalized by the symmetry action.

1. In class, I have focused on the *fundamental multiplet* of the local $SU(N)$ symmetry, *i.e.*, a set of N fields (complex scalars or Dirac fermions) which transform as a complex N–vector,

$$\Psi'(x) \;=\; U(x)\Psi(x) \quad i.e. \quad \Psi'^i(x) \;= \sum_j U^i{}_j(x)\Psi^j(x), \quad i,j = 1,2,\ldots,N \tag{1}$$

where $U(x)$ is an x–dependent unitary $N \times N$ matrix, $\det U(x) \equiv 1$. Now consider $N^2 - 1$ real fields $\Phi^a(x)$ forming an *adjoint multiplet*: In matrix form

$$\Phi(x) \;=\; \sum_a \Phi^a(x) \times \frac{\lambda^a}{2} \tag{2}$$

is a traceless hermitian $N \times N$ matrix which transforms under the local $SU(N)$ symmetry according to

$$\Phi'(x) \;=\; U(x)\Phi(x)U^\dagger(x). \tag{3}$$

Note that this transformation law preserves the $\Phi^\dagger = \Phi$ and $\mathrm{tr}(\Phi) = 0$ conditions.

The covariant derivatives D_μ act on an adjoint multiplet of fields as

$$D_\mu\Phi(x) \;=\; \partial_\mu\Phi(x) \;+\; i[\mathcal{A}_\mu(x), \Phi(x)] \;\equiv\; \partial_\mu\Phi(x) \;+\; i\mathcal{A}_\mu(x)\Phi(x) \;-\; i\Phi(x)\mathcal{A}_\mu(x), \tag{4}$$

or in components

$$D_\mu\Phi^a(x) = \;\partial_\mu\Phi^a(x) \;-\; f^{abc}\mathcal{A}_\mu^b(x)\Phi^c(x). \tag{5}$$

(a) Verify that these derivatives are indeed covariant — the $D_\mu\Phi(x)$ transforms under the local $SU(N)$ symmetry exactly like the $\Phi(x)$ itself.

(b) Verify the Leibniz rule for the covariant derivatives of matrix products. Let $\Phi(x)$ and $\Xi(x)$ be two adjoint multiplets while $\Psi(x)$ is a fundamental multiplet and $\Psi^\dagger(x)$ is its hermitian conjugate (a row vector of Ψ^*_i). Show that

$$\begin{aligned} D_\mu(\Phi\Xi) &= (D_\mu\Phi)\Xi + \Phi(D_\mu\Xi), \\ D_\mu(\Phi\Psi) &= (D_\mu\Phi)\Psi + \Phi(D_\mu\Psi), \\ D_\mu(\Psi^\dagger\Xi) &= (D_\mu\Psi^\dagger)\Xi + \Psi^\dagger(D_\mu\Xi). \end{aligned} \tag{6}$$

(c) Show that for an adjoint multiplet $\Phi(x)$,

$$[D_\mu, D_\nu]\Phi(x) = i[\mathcal{F}_{\mu\nu}(x), \Phi(x)] = ig[F_{\mu\nu}(x), \Phi(x)] \tag{7}$$

or in components $[D_\mu, D_\nu]\Phi^a(x) = -gf^{abc}F^b_{\mu\nu}(x)\Phi^c(x)$.

In class, I have argued (using covariant derivatives) that the tension fields $\mathcal{F}_{\mu\nu}(x)$ themselves transform according to eq. (3). In other words, the $\mathcal{F}^a_{\mu\nu}(x)$ form an adjoint multiplet of the $SU(N)$ symmetry group.

(d) Verify the $\mathcal{F}'_{\mu\nu}(x) = U(x)\mathcal{F}_{\mu\nu}(x)U^\dagger(x)$ transformation law directly from the definition $\mathcal{F}_{\mu\nu} \stackrel{\text{def}}{=} \partial_\mu\mathcal{A}_\nu - \partial_\mu\mathcal{A}_\nu + i[\mathcal{A}_\mu, \mathcal{A}_\nu]$ and from the non-abelian gauge transform formula for the $\mathcal{A}_\mu$ fields.

(e) Verify the Bianchi identity for the non-abelian tension fields $\mathcal{F}_{\mu\nu}(x)$:

$$D_\lambda\mathcal{F}_{\mu\nu} + D_\mu\mathcal{F}_{\nu\lambda} + D_\nu\mathcal{F}_{\lambda\mu} = 0. \tag{8}$$

Note the covariant derivatives in this equation.

Finally, consider the $SU(N)$ Yang–Mills theory — the non-abelian gauge theory that does not have any fields except $\mathcal{A}^a(x)$ and $\mathcal{F}^a(x)$; its Lagrangian is

$$\mathcal{L}_{\text{YM}} = -\frac{1}{2g^2}\,\text{tr}\big(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\big) = \sum_a \tfrac{-1}{4}F^a_{\mu\nu}F^{a\mu\nu}. \tag{9}$$

(f) Show that the Euler–Lagrange field equations for the Yang–Mills theory can be written in covariant form as $D_\mu\mathcal{F}^{\mu\nu} = 0$.
Hint: first show that for an infinitesimal variation $\delta\mathcal{A}_\mu(x)$ of the non-abelian gauge fields, the tension fields vary according to $\delta\mathcal{F}_{\mu\nu}(x) = D_\mu\delta\mathcal{A}_\nu(x) - D_\nu\delta\mathcal{A}_\mu(x)$.

2. Continuing the previous problem, consider an $SU(N)$ gauge theory in which N^2-1 vector fields $A^a_\mu(x)$ interact with some "matter" fields $\phi_\alpha(x)$,

$$\mathcal{L} \;=\; -\frac{1}{2g^2}\,\mathrm{tr}\big(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\big) \;+\; \mathcal{L}_{\mathrm{mat}}(\phi, D_\mu\phi). \tag{10}$$

For the moment, let me keep the matter fields completely generic — they can be scalars, or vectors, or spinors, or whatever, and form any kind of a multiplet of the local $SU(N)$ symmetry as long as such multiplet is complete and non-trivial. All we need to know right now is that there are well-defined covariant derivatives $D_\mu\phi$ that depend on the gauge fields A^a_μ, which give rise to the currents

$$J^{a\mu} \;=\; -\frac{\partial\mathcal{L}_{\mathrm{mat}}}{\partial A^a_\mu}\,. \tag{11}$$

Collectively, these N^2-1 currents should form an adjoint multiplet $J^\mu = \sum_a(\frac{1}{2}\lambda^a)J^{a\mu}$ of the $SU(N)$ symmetry.

(a) Show that in this theory the equation of motion for the A^a_μ fields are $D_\mu F^{a\mu\nu} = J^{a\nu}$ and that consistency of these equations requires the currents to be *covariantly conserved,*

$$D_\mu J^\mu \;=\; \partial_\mu J^\mu \;+\; ig[A_\mu, J^\mu] \;=\; 0, \tag{12}$$

or in components,

$$D_\mu J^{\mu a} \;=\; \partial_\mu J^{a\mu} \;-\; gf^{abc}A^b_\mu J^{c\mu} \;=\; 0. \tag{13}$$

Note: a covariantly conserved current does *not* lead to a conserved charge, $(d/dt)\int d^3\mathbf{x}\,J^{a0}(\mathbf{x},t) \neq 0!$

Now consider a simple example of matter fields — a fundamental multiplet $\Psi(x)$ of N Dirac fermions $\Psi^i(x)$, with a Lagrangian

$$\mathcal{L}_{\mathrm{mat}} \;=\; \overline{\Psi}\big(i\gamma^\mu D_\mu \;-\; m\big)\Psi, \qquad \mathcal{L}_{\mathrm{net}} \;=\; \mathcal{L}_{\mathrm{mat}} \;-\; \frac{1}{2g^2}\,\mathrm{tr}\big(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\big). \tag{14}$$

(b) Derive the $SU(N)$ currents $J^{a\mu}$ for these fermions and verify that under the $SU(N)$ symmetries they transform covariantly into each other as members of an adjoint multiplet. That is, the $N\times N$ matrix $J^\mu = \sum_a(\frac{1}{2}\lambda^a)J^{a\mu}$ transforms according to eq. (3).

Hint: for any complex N–vectors ξ^i and η^j,

$$\sum_a \left(\eta^\dagger \lambda^a \xi\right) \times \left(\lambda^a\right)^i_{\ j} \;=\; 2\,\eta^*_j \xi^i \;-\; \frac{2}{N}\left(\eta^\dagger \xi\right) \times \delta^i_j \,.. \tag{15}$$

(c) Finally, verify the covariant conservation $D_\mu J^{a\mu} = 0$ of these currents when the fermionic fields $\Psi^i(x)$ and $\overline{\Psi}_i(x)$ obey their equations of motion.

3. This problem is about general multiplets of general gauge groups. Consider a Lie group G with generators $\hat{T}^a$ obeying commutation relations $[\hat{T}^a, \hat{T}^b] = if^{abc}\hat{T}^c$. Under an infinitesimal local symmetry

$$\mathcal{G}(x) \;=\; 1 \;+\; i\Lambda^a(x)\hat{T}^a \;+\; \cdots, \quad \text{infinitesimal } \Lambda^a(x), \tag{16}$$

the gauge fields $\mathcal{A}^a_\mu(x)$ transform as

$$\mathcal{A}^a_\mu(x) \;\to\; \mathcal{A}^a_\mu(x) \;-\; D_\mu \Lambda^a(x) \;=\; \mathcal{A}^a_\mu(x) \;-\; \partial_\mu \Lambda^a(x) \;-\; f^{abc}\Lambda^b(x)\mathcal{A}^c_\mu(x). \tag{17}$$

Other fields of the gauge theory (scalar, spinor, or whatever) must form complete multiplets of the gauge group G. In any such multiplet (m), the generators $\hat{T}^a$ are represented by the $\mathrm{size}(m) \times \mathrm{size}(m)$ matrices $(T^a_{(m)})^\alpha_{\ \beta}$ satisfying similar commutation relations, $[T^a_{(m)}, T^b_{(m)}] = if^{abc}T^c_{(m)}$. The fields $\Psi^\alpha(x)$ belonging to such a multiplet transform under infinitesimal gauge transforms (16) as

$$\Psi^\alpha(x) \;\to\; \Psi^\alpha(x) \;+\; i\Lambda^a(x)(T^a_{(m)})^\alpha_{\ \beta}\Psi^\beta(x), \tag{18}$$

and the covariant derivatives D_μ act on these fields as

$$D_\mu \Psi^\alpha(x) \;=\; \partial_\mu \Psi^\alpha(x) \;+\; i\mathcal{A}^a_\mu(x)(T^a_{(m)})^\alpha_{\ \beta}\Psi^\beta(x). \tag{19}$$

Note different matrices $T^a_{(m)}$ in the covariant derivatives of fields belonging to different multiplet (m). But the gauge fields $\mathcal{A}^a_\mu$ are the same for all the matter fields of the same gauge theory!

- Verify covariance of the derivatives (19) under infinitesimal gauge transforms (16).

  ○ The derivatives (19) are covariant under any gauge transformations, infinitesional of finite. But proving the covariance under the finite gauge transforms is much harder, so your homework is limited to the infinitesimal case.

4. In the previous homework (set#10, problem#2), we had continuous global symmetry $G = SU(N)_L \times SU(N)_R \times U(1)$ spontaneously broken down to $H = SU(N)_V$. Now let's gauge the entire $SU(N)_L \times SU(N)_R \times U(1)$ symmetry and work out the Higgs mechanism.

   The present theory comprises N^2 complex scalar fields $\Phi_i^j(x)$ organized into an $N \times N$ matrix, and $2N^2 - 1$ real vector fields $B_\mu(x)$, $L_\mu^a(x)$, and $R_\mu^a(x)$, the latter organized into traceless hermitian matrices $L_\mu(x) = \sum_a L_\mu^a(x) \times \frac{1}{2}\lambda^a$ and $R_\mu(x) = \sum_a R_\mu^a(x) \times \frac{1}{2}\lambda^a$, where $a = 1, \ldots, (N^2 - 1)$ and λ^a are the Gell-Mann matrices of $SU(N)$. The Lagrangian is

$$\mathcal{L} = -\tfrac{1}{4}B_{\mu\nu}B^{\mu\nu} - \tfrac{1}{2}\operatorname{tr}\left(L_{\mu\nu}L^{\mu\nu}\right) - \tfrac{1}{2}\operatorname{tr}\left(R_{\mu\nu}R^{\mu\nu}\right) + \operatorname{tr}\left(D^\mu\Phi^\dagger\, D_\mu\Phi\right) - V(\Phi^\dagger\Phi), \quad (20)$$

   where

$$\begin{aligned}
B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu\,, \\
L_{\mu\nu} &= \partial_\mu L_\nu - \partial_\nu L_\mu + ig[L_\mu, L_\nu], \\
R_{\mu\nu} &= \partial_\mu R_\nu - \partial_\nu R_\mu + ig[R_\mu, R_\nu], \\
D_\mu\Phi &= \partial_\mu\Phi + ig'B_\mu\Phi + igL_\mu\Phi - ig\Phi R_\mu\,, \\
D_\mu\Phi^\dagger &= (D_\mu\Phi)^\dagger = \partial_\mu\Phi^\dagger - ig'B_\mu\Phi^\dagger + igR_\mu\Phi^\dagger - ig\Phi^\dagger L_\mu\,.
\end{aligned} \quad (21)$$

   For simplicity, I assume equal gauge couplings $g_L = g_R = g$ for the two $SU(N)$ factors of the gauge group, but the abelian coupling g' is different.

   The scalar potential V is precisely as in the previous homework,

$$V = \frac{\alpha}{2}\operatorname{tr}\bigl(\Phi^\dagger\Phi\Phi^\dagger\Phi\bigr) + \frac{\beta}{2}\operatorname{tr}^2\bigl(\Phi^\dagger\Phi\bigr) + m^2\operatorname{tr}\bigl(\Phi^\dagger\Phi\bigr), \qquad \alpha, \beta > 0, \quad m^2 < 0, \quad (22)$$

   hence similar VEVs of the scalar fields: up to a gauge symmetry,

$$\langle\Phi\rangle = C \times \mathbf{1}_{N\times N} \quad \text{where} \quad C = \sqrt{\frac{-m^2}{\alpha + N\beta}}\,, \quad (23)$$

which breaks the $G = SU(N)_L \times SU(N)_R \times U(1)$ symmetry down to the $SU(N)_V$ subgroup.

(a) The Higgs mechanism makes N^2 out of $2N^2 - 1$ vector fields massive. Calculate their masses by plugging $\langle\Phi\rangle$ for the $\Phi(x)$ into the $\mathrm{tr}(D_\mu \Phi^\dagger D^\mu \Phi)$ term of the Lagrangian. In particular, show that the abelian gauge field B_μ and the $X_\mu^a = \frac{1}{\sqrt{2}}(L_\mu^a - R_\mu^a)$ combinations of the $SU(N)$ gauge fields become massive, while the $V_\mu^a = \frac{1}{\sqrt{2}}(L_\mu^a + R_\mu^a)$ combinations remain massless.

(b) Find the effective Lagrangian for the massless vector fields $V_\mu^a(x)$ by freezing all the other fields, *i.e.* setting $B_\mu(x) \equiv 0$, $X_\mu^a(x) \equiv 0$, and $\Phi(x) \equiv \langle\Phi\rangle$. Show that this Lagrangian describes a Yang–Mills theory with gauge group $SU(N)_V$ and gauge coupling $g_V = g/\sqrt{2}$.

$\star$ For extra challenge, allow for un-equal gauge coulings $g_L \neq g_R$. Find which combinations of the $L_\mu^a(x)$ and $R_\mu^a(x)$ fields remain massless in this case, then derive the effective Lagrangian for these massless fields by freezing everything else. As in part (b), you should get an $SU(N)$ Yang–Mills theory, but this time the gauge coupling is

$$g_v = \frac{g_L g_R}{\sqrt{g_L^2 + g_R^2}} \,. \tag{24}$$