1. First, a reading assignment on renormalization. In class I have explained the MS renormalization scheme and gave you formulae for the β-functions in terms of the residues of simple $1/\epsilon$ poles of the counterterms, see pages 1–7 of my notes. Please read the rest of these notes (pages 8–14) where I prove those formulae and also derive a recursive relation for the higher-order poles $1/\epsilon^2$, $1/\epsilon^3$, *etc.*, in terms of the simmple poles.

2. Second, another reading assignment: my notes about properly discretized path integral for the harmonic oscillator.

3. Third, a simple exercise in using path integrals. Consider a 1D particle living on a circle of radius R, or equivalently a 1D particle in a box of length $L = 2\pi R$ with periodic boundary conditions where moving past the $x = L$ point brings you back to $x = 0$. In other words, the particle's position $x(t)$ is defined modulo L.

   The particle has no potential energy, only the non-relativistic kinetic energy $p^2/2M$.

   (a) As a particle moves from some point $x_1 \pmod L$ at time $t_1 = 0$ to some other point $x_2 \pmod L$ at time $t_2 = T$, it may travel directly from x_1 to x_2, or it may take a few turns around the circle before ending at the x_2. Show that the space of all such paths on a circle is isomorphic to the space of all paths on an infinite line which begin at fixed x_1 at time t_1 and end at time t_2 at any one of the points $x_2' = x_2 + nL$ where $n = 0, \pm1, \pm2, \ldots$ is any whole number.

   Then use path integrals to relate the evolution kernels for the circle and for the infinite line (over the same time interval $t_2 - t_1 = T$) as

$$U_{\text{circle}}(x_2; x_1) \;=\; \sum_{n=-\infty}^{+\infty} U_{\text{line}}(x_2 + nL; x_1). \tag{1}$$

The next question uses Poisson's resummation formula: If a function $F(n)$ of integer n can be analytically continued to a function $F(\nu)$ of arbitrary real ν, then

$$\sum_{n=-\infty}^{+\infty} F(n) \;=\; \int d\nu\, F(\nu) \times \sum_{n=-\infty}^{+\infty} \delta(\nu - n) \;=\; \sum_{\ell=-\infty}^{+\infty} \int d\nu\, F(\nu) \times e^{2\pi i \ell \nu}. \tag{2}$$

(b) The free particle living on an infinite 1D line has evolution kernel

$$U_{\text{line}}(x_2; x_1) \;=\; \sqrt{\frac{M}{2\pi i \hbar T}} \times \exp\left(+\frac{iM(x_2 - x_1)^2}{2\hbar T}\right). \tag{3}$$

Plug this free kernel into eq. (1) and use Poisson's formula to sum over n.

(c) Verify that the resulting evolution kernel for the particle one the circle agrees with the usual QM formula

$$U_{\text{box}}(x_2; x_1) \;=\; \sum_p L^{-1/2} e^{ipx_2/\hbar} \times e^{-iT(p^2/2M)/\hbar} \times L^{-1/2} e^{-ipx_1/\hbar} \tag{4}$$

where p takes circle-quantized values

$$p \;=\; \frac{2\pi\hbar}{L} \times \text{integer}. \tag{5}$$

4. Finally, let's prove the *Mermin–Wagner–Coleman theorem*, which forbids spontaneous breaking of a *continuous* symmetry in $D \leq 2$ dimensions (Minkowski or Euclidean).

Consider a QFT (in any dimenssion D) with a global $U(1)$ phase symmetry, a comples field $\Phi(x)$ which transforms into $e^{i\theta}\Phi(x)$, and the effective scalar potential which has a degenerate ring of minima at $\Phi = Ae^{i\phi}$ (fixed radius $A > 0$, any phase ϕ). In polar coordinates (ρ, ϕ) for the Φ field, the radial field $\delta\rho(x) = \rho(x) - A$ is massive while the angular direction $\phi(x)$ is massless, so the effective low-energy theory has only the massive $\phi(x)$ field.

(a) In terms of the angular $\phi(x)$ field, the $U(1)$ phase symmetry acts as a *shift symmetry:* $\phi'(x) = \phi(x) + \theta$. Show that the only relevant or marginal operator which respects this shift symmetry (as well as the Lorentz or Euclidean symmetry in D dimensions) is the kinetic energy operator $(\partial_\mu \phi)^2$. Thus, at low energies the $\phi(x)$ field is free and massless, and its Lagrangian is

$$\mathcal{L} \;=\; \frac{B}{2} \times (\partial_\mu \phi)^2 \;+\; \text{nothing else} \tag{6}$$

for some constant $B > 0$. (Classically $B = 2A^2$, but it's subject to quantum corrections.)

(b) Next, use path integrals to show that for a free massless scalar field $\phi(x)$ with Lagrangian (6),

$$\langle\Omega|\,\mathbf{T}\exp(+i\phi(x))\exp(-i\phi(y))\,|\Omega\rangle \;=\; \exp\left(\frac{G_0(x-y) \;-\; G_0(0)}{B}\right) \tag{7}$$

where $G_0(x-y)$ is the Feynman propagator of a massless scalar.

(c) Now use parts (a) and (b) to show that at long distances $x - y \to \infty$,

$$\langle\omega|\,\mathbf{T}\Phi(x)\Phi^*(y)\,|\omega\rangle \;=\; C^2 \times \exp\left(\frac{G_0(x-y)}{B}\right) \tag{8}$$

for some positive constant $C^2 > 0$.

For a massless free scalar field, the coordinate-space formula for the propagator is fairly simple: In D Euclidean dimensions,

$$G_0(x-y) \;\equiv\; \int\frac{d^d p_E}{(2\pi)^d}\,\frac{e^{ip(x-y)}}{p_E^2} \;=\; \begin{cases} \dfrac{\Gamma\left(\frac{D}{2}-1\right)}{4\pi^{D/2}} \times |x-y|^{2-D} & \text{for } D \neq 2, \\[2ex] \text{const} \;-\; \dfrac{1}{2\pi} \times \log|x-y| & \text{for } D = 2. \end{cases} \tag{9}$$

(d) Finally, use eqs. (8), (9), and the cluster expansion to argue that in $D > 2$ dimensions $\langle\Phi\rangle \neq 0$ and the $U(1)$ symmetry is spontaneously broken but in $D \leq 2$ dimensions $\langle\Phi\rangle = 0$ and the $U(1)$ symmetry remains unbroken.