1. First, a modified textbook problem 9.2(c–e) about the Euclidean path integrals at finite temperature. Questions (a–c) below concern a free scalar field, questions (d–e) concern free fermionic fields, and question (f) is about the free electromagnetic field.

   (a) Consider a free scalar field in $3+1$ dimensions at finite temperature T. Use Euclidean path integral to calculate the partition function and hence the Helmholz free energy. Show that formally

$$\mathcal{F}(T) \;=\; \frac{T}{2}\times\operatorname{Tr}\log\bigl(-\partial_E^2+m^2\bigr) \tag{1}$$

   where the ∂_E^2 operator acts on functions$(x_1,x_2,x_3,x_4)_E$ which are periodic in the Euclidean time x_4 with period $\beta=1/T$. Then take the trace in momentum space and the Poisson resummation formula (see homework set#19, eq. (2)) to show that

$$\begin{aligned}\mathcal{F}(T) \;&=\; \text{const} \;+\; \frac{1}{2}\sum_{\ell=-\infty}^{+\infty}\int\!\frac{d^4p_E}{(2\pi)^4}\,\exp(i\ell\beta p_4)\times\log(p_E^2+m^2) &(2)\\ &=\; \mathcal{F}(0) \;+\; \sum_{\ell=1}^{\infty}\int\!\frac{d^4p_E}{(2\pi)^4}\,\exp(i\ell\beta p_4)\times\log(p_E^2+m^2). &(3)\end{aligned}$$

   (b) Take the $\int dp_4$ integral in eq. (3). Deform the integration contour so it runs on two 'banks' of a branch cut to show that

$$\int\limits_{-\infty}^{+\infty}\frac{dp_4}{2\pi}\,\exp(i\ell\beta p_4)\times\log(p_4^2+E^2) \;=\; -\frac{\exp(-\ell\beta E)}{\ell\beta}\,. \tag{4}$$

   (c) Now use eqs. (3) and (4) to show that the free energy of a free scalar field above the zero-point energy is

$$\mathcal{F}(T)\,-\,\mathcal{F}(0) \;=\int\!\frac{d^3\mathbf{p}}{(2\pi)^3}\,T\,\log\Bigl(1-e^{-\beta E_p}\Bigr) \;=\int\!\frac{d^3\mathbf{p}}{(2\pi)^3}\,\Bigl(\mathcal{F}_{\substack{\text{harmonic}\\ \text{oscillator}}}(T,E_p)\,-\,\tfrac{1}{2}E_p\Bigr)\,. \tag{5}$$

Next, consider a free fermion $0+1$ dimensions, basically a two-level system in Quantum Mechanics. In the Hamiltonian formulation this means

$$\hat{H} = \omega\hat{\psi}^\dagger\hat{\psi} \quad \text{where} \quad \{\hat{\psi},\hat{\psi}^\dagger\} = 1 \quad \text{and} \quad \omega = \text{constant} > 0, \tag{6}$$

while in the Lagrangian formulation, $\psi(t)$ and $\psi^*(t)$ are Grassmann-number-valued functions of the time and

$$L_E = \psi^* \times \frac{d\psi}{dt_E} + \omega \times \psi^*\psi. \tag{7}$$

(d) Use the path integral to calculate the partition function for both periodic and antiperiodic boundary conditions for the fermionic variables in the Euclidean time, $\psi(t_E+\beta) = \pm\psi(t_E)$. SHow that the periodic conditions lead to an unphysical partiction function, while the antiperiodic conditions lead to the correct partiction function of a two-level system.

(e) Now apply the lesson of part (d) to a Dirac fermionic field in $3+1$ dimensions. Calculate the partition function and hence the free energy using the Euclidean path integral over Dirac fielda which are antiperiodic in the Euclidean time, $\Psi(\mathbf{x},x_4+\beta) = \Psi(\mathbf{x},x_4)$.

The last question (f) below involves the path integral for the electromagnetic field. You should answer it only when I have finished explaining how to take such integrals, including the explanation of the Faddeev–Popov determinant and of the gauge-averaging leading to the Feynman gauge condition. If I do not finish this subject the end of the Tuesday 4/11 class, then question (f) is posponed to the next homework.

(f) At finite temperature, the electromagnetic field $A^\mu(x)$ — just like any other bosonic field — is periodic in the Euclidean time, $A^\mu(\mathbf{x},x_4+\beta) = A^\mu(\mathbf{x},x_4)$. Use the path integral to show that formally, the EM free energy is

$$\mathcal{F}(T) = T \times \operatorname{Tr}\log\bigl(-\partial_E^2\bigr). \tag{8}$$

Then recycle calculations from parts (a–c) to shwo that

$$\mathcal{F}(T) - \mathcal{F}(0) = \int\frac{d^3\mathbf{p}}{(2\pi)^3}\, 2T \times \left(1 - e^{-\beta|p|}\right). \tag{9}$$

2. The second problem is a big reading assignment: §12.1 of the *Peskin & Schroeder* textbook about integrating out of the short-distance modes and the Wilsonian renormalization. Also, read §12.4 of the Weinberg's book (vol. 1) about the same subjects.

   These issues are important, and I wish I could spend a lecture or two explaining them. Alas, the class time is too short, hence this assignment.