1. In class, I have calculated the (infinite parts of the) $\delta_2^{(q)}$ and $\delta_1^{(q)}$ counterterms for the quarks. Your task is to calculate the analogous $\delta_2^{(\mathrm{gh})}$ and $\delta_1^{(\mathrm{gh})}$ counterterms for the *ghosts fields.*

   (a) Draw one-loop diagrams whose divergences are cancelled by the $\delta_2^{(\mathrm{gh})}$ and $\delta_1^{(\mathrm{gh})}$, and calculate the group factors for each diagrams.

   (b) Calculate the momentum integrals for the diagrams. Focus on the UV divergences and ignore the finite parts of the integrals.

   (c) Assemble your results and show that the *difference* $\delta_1^{(\mathrm{gh})} - \delta_2^{(\mathrm{gh})}$ for the ghosts is exactly the same as the $\delta_1^{(q)} - \delta_2^{(q)}$ difference for the quarks.

2. Now consider the three gauge couplings of the $SU(3) \times SU(2) \times U(1)$ Standard Model and their one-loop beta-functions

$$\beta_1^{\text{1 loop}} = \frac{b_1 g_1^3}{16\pi^2}\,, \quad \beta_2^{\text{1 loop}} = \frac{b_2 g_2^3}{16\pi^2}\,, \quad \beta_3^{\text{1 loop}} = \frac{b_3 g_3^3}{16\pi^2}\,. \tag{1}$$

   In this exercise, you do not need to calculate these beta-function from scratch by evaluating the UV divergences of a bunch of loop diagrams. Instead, use eqs. (84–85) from the last page of my notes on QCD beta-function.

   (a) Calculate the b_1, b_2, b_3 coefficients it for the minimal version of the Standard Model: the $SU(3) \times SU(2) \times U(1)$ gauge fields, one Higgs doublet, three families of quarks and leptons, and nothing else.

   ⋆ FYI, each family comprises 8 left-handed Weyl fields in the $(\mathbf{3}, \mathbf{2}, y = +\frac{1}{6})$ and $(\mathbf{1}, \mathbf{2}, y = -\frac{1}{2})$ multiplets of the gauge symmetry and 7 right-handed Weyl fermions in the $(\mathbf{3}, \mathbf{1}, y = +\frac{2}{3})$, $(\mathbf{3}, \mathbf{1}, y = -\frac{1}{3})$, and $(\mathbf{1}, \mathbf{1}, y = -1)$ multiplets.

   (b) Re-calculate the b_1, b_2, b_3 for the MSSM — the Minimal Supersymmetric Standard Model. FYI, here is complete list of the MSSM fields:

   ∘ The $SU(3) \times SU(2) \times U(1)$ gauge fields, same as the non-SUSY SM.

   • For each vector field there is a Majorana fermion (gaugino) with similar $SU(3) \times SU(2) \times U(1)$ quantum numbers. Altogether, there is an adjoint multiplet of gauginos for each factor of the gauge symmetry.

- ◦ 3 families of quarks and leptons, same as the non-SUSY SM.
- For each Weyl fermion — left-handed or right-handed — in these three families, the MSSM also have a complex scalar field (squark or slepton) with similar $SU(3) \times SU(2) \times U(1)$ quantum numbers. Altogether, this makes 45 complex scalar fields in similar multiplets to the quarks and leptons.
- The Higgs sector of the MSSM comprises *two* $SU(2)$ doublets of complex scalars accompanied by one $SU(2)$ doublet of Dirac fermions (the higgsinos); all these doublets have $y = \frac{1}{2}$.

— There are all kinds of Yukawa and ϕ^4 interactions between the MSSM fields, but you do not need them for the one-loop calculation of the gauge couplings' beta-functions.

In Grand Unified Theories

$$\alpha_3 \;=\; \alpha_2 \;=\; \tfrac{5}{3}\alpha_1 \;=\; \alpha_{\text{GUT}} \quad \text{at the GUT scale.} \tag{2}$$

At lower energy scales $E \ll M_{\text{GUT}}$ the SM couplings are given (lo the leading one-loop order) by the Georgi–Quinn–Weinberg equations

$$\begin{aligned} \frac{1}{\alpha_3(E)} &= \frac{1}{\alpha_{\text{GUT}}} + b_3 \times \frac{1}{2\pi} \log \frac{M_{\text{GUT}}}{E}\,, \\ \frac{1}{\alpha_2(E)} &= \frac{1}{\alpha_{\text{GUT}}} + b_2 \times \frac{1}{2\pi} \log \frac{M_{\text{GUT}}}{E}\,, \\ \frac{1}{\alpha_1(E)} &= \frac{5/3}{\alpha_{\text{GUT}}} + b_1 \times \frac{1}{2\pi} \log \frac{M_{\text{GUT}}}{E}\,. \end{aligned} \tag{3}$$

(c) Derive these equations from eqs. (1).

(d) Experimentally, at $E = M_Z \approx 91$ GeV

$$\frac{1}{\alpha_3(M_Z)} \approx 8.5, \qquad \frac{1}{\alpha_2(M_Z)} \approx 29.5, \qquad \frac{1}{\alpha_1(M_Z)} \approx 98.5. \tag{4}$$

Check that these couplings are consistent with eqs. (3) for the MSSM but not for the non-SUSY minimal Standard Model. For the MSSM, calculate the Grand Unification scale M_{GUT} and the unified gauge coupling α_{GUT}.

Although the top quark, the Higgs, and all the additional particles of the MSSM are heavier than Z, for this exercise you should ignore the thresholds due to these masses. Instead, use the b_3, b_2, b_1 coefficients of the massless theory — the minimal SM or the MSSM — for all energies between the M_Z and the $M_{\rm GUT}$.

3. Next, a reading assignment: $16.7 of *Peskin & Schroeder* about the “magnetic anti-screening” explanation of the asymptotic freedom in QCD.

4. Finally, another reading assignment: §19.3 of *Peskin & Schroeder* about the chiral symmetry of QCD and the pions.

   Chapter 19 of *Weinberg* has a deeper discussion of pions (and Goldstone bosons in general); you are advised to read it, but not necessarily this week.