

1. The $f_\pi \approx 93$ MeV is called the *pion decay constant* because it controls the decay rate of the charged pions, mostly into muons and neutrinos, $\pi^+ \rightarrow \mu^+ \nu_\mu$ and $\pi^- \rightarrow \mu^- \bar{\nu}_\mu$. In this problem, we shall learn how this works.

The weak interactions at energies $O(M_\pi) \ll M_W$ are governed by the Fermi's current-current effective Lagrangian

$$\mathcal{L} = -2\sqrt{2}G_F J_L^{+\alpha} J_{L\alpha}^- \quad (1)$$

where $L_L^{\pm\alpha} = \frac{1}{2}(J_V^{\pm\alpha} - J_A^{\pm\alpha})$ are the left-handed charged currents. In terms of the quark and lepton fields,

$$\begin{aligned} J_L^{+\alpha} &= \frac{1}{2}\bar{\Psi}(\nu_\mu)(1 - \gamma^5)\gamma^\alpha\Psi(\mu) + \cos\theta_c \times \frac{1}{2}\bar{\Psi}(u)(1 - \gamma^5)\gamma^\alpha\Psi(d) + \dots, \\ J_L^{-\alpha} &= \frac{1}{2}\bar{\Psi}(\mu)(1 - \gamma^5)\gamma^\alpha\Psi(\nu_\mu) + \cos\theta_c \times \frac{1}{2}\bar{\Psi}(d)(1 - \gamma^5)\gamma^\alpha\Psi(u) + \dots, \end{aligned} \quad (2)$$

where the \dots stand for terms involving other fermions of the Standard Model, and $\theta_c \approx 13^\circ$ is the Cabibbo angle.

For the pion decay process, the axial part one of the currents annihilates the charged pion

$$\langle \text{vacuum} | \bar{\Psi}_d \gamma^5 \gamma^\alpha \Psi_u | \pi^+(p) \rangle = \langle \text{vacuum} | \bar{\Psi}_u \gamma^5 \gamma^\alpha \Psi_d | \pi^-(p) \rangle = f_\pi \times p^\alpha \quad (3)$$

while the other current creates the lepton pair.

- (a) Show that the tree-level pion decay amplitude is

$$\mathcal{M}(\pi^+ \rightarrow \mu^+ \nu_\mu) = G_f f_\pi \cos\theta_c \times p^\alpha(\pi) \times \bar{u}(\nu_\mu)(1 - \gamma^5)\gamma_\alpha v(\mu^+). \quad (4)$$

- (b) Sum over the fermion spins and calculate the decay rate $\Gamma(\pi^+ \rightarrow \mu^+ \nu_\mu)$. FYI, $f_\pi \approx 93$ MeV, $M_\pi \approx 140$ MeV, $M_\mu \approx 106$ MeV, and $G_F \approx 1.17 \cdot 10^{-5}$ GeV⁻².

- (c) The charged pions decay to muons much more often than they decay to electrons,

$$\frac{\Gamma(\pi^+ \rightarrow e^+ \nu_e)}{\Gamma(\pi^+ \rightarrow \mu^+ \nu_\mu)} = \frac{M_e^2 (1 - (M_e/M_\pi)^2)^2}{M_\mu^2 (1 - (M_\mu/M_\pi)^2)^2} \approx 1.2 \cdot 10^{-4}. \quad (5)$$

Derive this formula, then explain this preference for the heavier final-state lepton in terms of mis-matched helicity and chirality of the charged lepton.

The rest of this homework concerns the axial anomaly in gauge theories.

2. Consider the axial anomaly in a non-abelian gauge theory, for example QCD with N_f massless quark flavors,

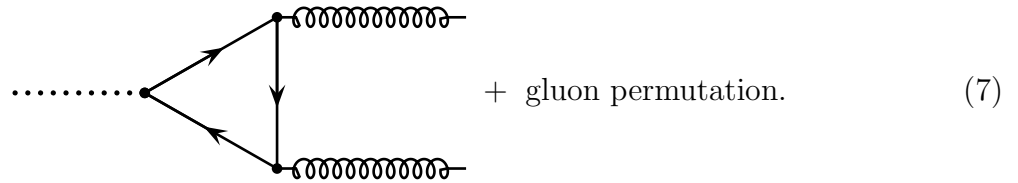
$$J_A^\mu = \sum_{i,f} \bar{\Psi}_{if} \gamma^5 \gamma^\mu \Psi^{if}, \quad \partial_\mu J_A^\mu = -\frac{N_f g^2}{16\pi^2} \epsilon^{\alpha\beta\mu\nu} \text{tr}(F_{\alpha\beta} F_{\mu\nu}) \quad (6)$$

where $F_{\mu\nu}$ is the non-abelian gauge field strength.

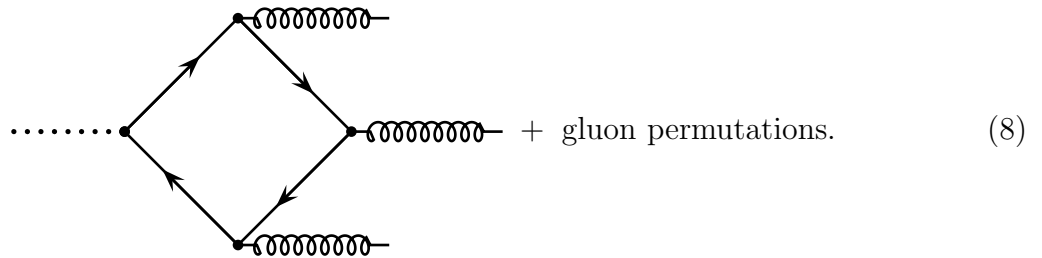
- (a) Expand the right hand side of eq. (6) into 2-gluon, 3-gluon, and 4-gluon terms and show that the 4-gluon term vanishes identically.

Hint: Use the cyclic symmetry of the trace.

The two-gluon anomaly term obtains from the triangle diagrams



This works exactly as discussed in class for the QED, except in QCD we should trace $F_{\alpha\beta} F_{\gamma\delta}$ over the quark colors. But in QCD there is also the three-gluon anomaly (*cf.* part (a)) which obtains from the quadrangle diagrams



- (b) Evaluate the quadrangle diagrams using the Pauli-Villars regularization and derive the three-gluon anomaly in QCD. Note that for the regulators

$$\frac{1}{\not{p} - M} \gamma^5 \not{q} \frac{1}{\not{p} + \not{q} - M} = \gamma^5 \left(\frac{1}{\not{p} + \not{q} - M} - \frac{1}{\not{p} - M} \right) - 2M \gamma^5 \frac{1}{\not{p} - M} \frac{1}{\not{p} + \not{q} - M}. \quad (9)$$

3. Next, a reading assignment: §22.2–3 of *Weinberg* about the chiral anomaly. Pay particular attention to the Jacobian of the fermion path integral and to regularization of the functional trace.
4. In any *even* spacetime dimension $d = 2n$, a massless Dirac fermion has an axial symmetry $\Psi(x) \rightarrow \exp(i\theta\Gamma)\Psi(x)$ where Γ generalizes the γ^5 . Classically, the axial current $J_A^\mu = \bar{\Psi}\Gamma\gamma^\mu\Psi$ is conserved, but when the fermion is coupled to a gauge field — abelian or non-abelian — the axial symmetry is broken by the anomaly and

$$\partial_\mu J_A^\mu = -\frac{2}{n!} \left(\frac{g}{4\pi}\right)^n \epsilon^{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n} \text{tr}\left(F_{\alpha_1\beta_1}F_{\alpha_2\beta_2}\cdots F_{\alpha_n\beta_n}\right). \quad (10)$$

Generalize Weinberg's calculation of the anomaly via Jacobian of the fermionic path integral to any even spacetime dimension $d = 2n$.

For your information, in $2n$ Euclidean dimensions $\{\gamma^\mu, \gamma^\nu\} = +2\delta^{\mu\nu}$, $\Gamma = i^{n-2}\gamma^1\gamma^2\cdots\gamma^{2n}$, $\{\Gamma, \gamma^\mu\} = 0$, $\Gamma^2 = +1$, and for any $2n = d$ matrices $\gamma^\alpha, \dots, \gamma^\omega$, $\text{tr}(\Gamma\gamma^\alpha\gamma^\beta\cdots\gamma^\omega) = 2^n i^{2-n} \epsilon^{\alpha\beta\cdots\omega}$.