1. The $f_\pi \approx 93$ MeV is called the *pion decay constant* because it controls the decay rate of the charged pions, mostly into muons and neutrinos, $\pi^+ \to \mu^+\nu_\mu$ and $\pi^- \to \mu^-\bar{\nu}_\mu$. In this problem, we shall learn how this works.

   The weak interactions at energies $O(M_\pi) \ll M_W$ are governed by the Fermi's current-current effective Lagrangian

$$\mathcal{L} \;=\; -2\sqrt{2}G_F J_L^{+\alpha} J_{L\alpha}^{-} \tag{1}$$

   where $L_L^{\pm\alpha} = \frac{1}{2}(J_V^{\pm\alpha} - J_A^{\pm\alpha})$ are the left-handed charged currents. In terms of the quark and lepton fields,

$$\begin{aligned} J_L^{+\alpha} \;&=\; \tfrac{1}{2}\overline{\Psi}(\nu_\mu)(1-\gamma^5)\gamma^\alpha\Psi(\mu) \;+\; \cos\theta_c \times \tfrac{1}{2}\overline{\Psi}(u)(1-\gamma^5)\gamma^\alpha\Psi(d) \;+\; \cdots, \\ J_L^{-\alpha} \;&=\; \tfrac{1}{2}\overline{\Psi}(\mu)(1-\gamma^5)\gamma^\alpha\Psi(\nu_\mu) \;+\; \cos\theta_c \times \tfrac{1}{2}\overline{\Psi}(d)(1-\gamma^5)\gamma^\alpha\Psi(u) \;+\; \cdots, \end{aligned} \tag{2}$$

   where the $\cdots$ stand for terms involving other fermions of the Standard Model, and $\theta_c \approx 13^\circ$ is the Cabibbo angle.

   For the pion decay process, the axial part one of the currents annihilates the charged pion

$$\langle\text{vacuum}|\,\overline{\Psi}_d\gamma^5\gamma^\alpha\Psi_u\,\big|\pi^+(p)\big\rangle \;=\; \langle\text{vacuum}|\,\overline{\Psi}_u\gamma^5\gamma^\alpha\Psi_d\,\big|\pi^-(p)\big\rangle \;=\; f_\pi \times p^\alpha \tag{3}$$

   while the other current creates the lepton pair.

   (a) Show that the tree-level pion decay amplitude is

$$\mathcal{M}(\pi^+ \to \mu^+\nu_\mu) \;=\; G_f f_\pi \cos\theta_c \times p^\alpha(\pi) \times \bar{u}(\nu_\mu)(1-\gamma^5)\gamma_\alpha v(\mu^+). \tag{4}$$

   (b) Sum over the fermion spins and calculate the decay rate $\Gamma(\pi^+ \to \mu^+\nu_\mu)$. FYI, $f_\pi \approx 93$ MeV, $M_\pi \approx 140$ MeV, $M_\mu \approx 106$ MeV, and $G_F \approx 1.17 \cdot 10^{-5}\,\text{GeV}^{-2}$.

   (c) The charged pions decay to muons much more often than they decay to electrons,

$$\frac{\Gamma(\pi^+ \to e^+\nu_e)}{\Gamma(\pi^+ \to \mu^+\nu_\mu)} \;=\; \frac{M_e^2}{M_\mu^2}\,\frac{(1-(M_e/M_\pi)^2)^2}{(1-(M_\mu/M_\pi)^2)^2} \;\approx 1.2 \cdot 10^{-4}. \tag{5}$$

   Derive this formula, then explain this preference for the heavier final-state lepton in terms of mis-matched helicity and chirality of the charged lepton.

The rest of this homework concerns the axial anomaly in gauge theories.

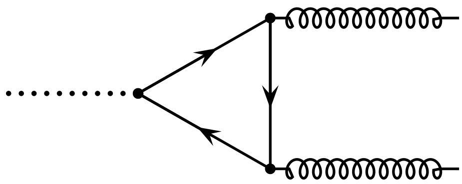
2. Consider the axial anomaly in a non-abelian gauge theory, for example QCD with N_f massless quark flavors,

$$J^\mu_A \;=\; \sum_{i,f} \overline{\Psi}_{if}\gamma^5\gamma^\mu\Psi^{if}, \qquad \partial_\mu J^\mu_A \;=\; -\frac{N_f g^2}{16\pi^2}\,\epsilon^{\alpha\beta\mu\nu}\,\mathrm{tr}\Big(F_{\alpha\beta}F_{\mu\nu}\Big) \tag{6}$$

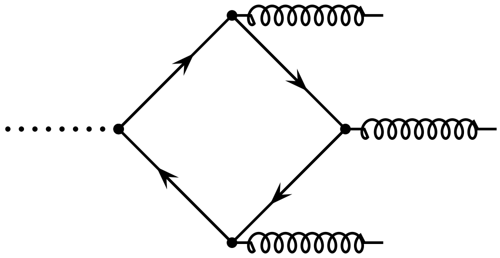
where $F_{\mu\nu}$ is the non-abelian gauge field strength.

(a) Expand the right hand side of eq. (6) into 2–gluon, 3–gluon, and 4–gluon terms and show that the 4–gluon term vanishes identically.
Hint: Use the cyclic symmetry of the trace.

The two-gluon anomaly term obtains from the triangle diagrams

 + gluon permutation. (7)

This works exactly as discussed in class for the QED, except in QCD we should trace $F_{\alpha\beta}F_{\gamma\delta}$ over the quark colors. But in QCD there is also the three-gluon anomaly (*cf.* part (a)) which obtains from the quadrangle diagrams

 + gluon permutations. (8)

(b) Evaluate the quadrangle diagrams using the Pauli–Villars regularization and derive the three-gluon anomaly in QCD. Note that for the regulators

$$\frac{1}{\not{p}-M}\,\gamma^5\,\not{q}\,\frac{1}{\not{p}+\not{q}-M} \;=\; \gamma^5\left(\frac{1}{\not{p}+\not{q}-M}-\frac{1}{\not{p}-M}\right) - 2M\gamma^5\,\frac{1}{\not{p}-M}\,\frac{1}{\not{p}+\not{q}-M}\,. \tag{9}$$

3. Next, a reading assignment: §22.2–3 of *Weinberg* about the chiral anomaly. Pay particular attention to the Jacobian of the fermion path integral and to regularization of the functional trace.

4. In any *even* spacetime dimension $d = 2n$, a massless Dirac fermion has an axial symmetry $\Psi(x) \to \exp(i\theta\Gamma)\Psi(x)$ where Γ generalizes the γ^5. Classically, the axial current $J_A^\mu = \overline{\Psi}\Gamma\gamma^\mu\Psi$ is conserved, but when the fermion is coupled to a gauge field — abelian or non-abelian — the axial symmetry is broken by the anomaly and

$$\partial_\mu J_A^\mu \;=\; -\frac{2}{n!}\left(\frac{g}{4\pi}\right)^n \epsilon^{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n}\,\mathrm{tr}\Big(F_{\alpha_1\beta_1}F_{\alpha_2\beta_2}\cdots F_{\alpha_n\beta_n}\Big). \tag{10}$$

   Generalize Weinberg's calculation of the anomaly via Jacobian of the fermionic path integral to any even spacetime dimension $d = 2n$.

   For your information, in $2n$ Euclidean dimensions $\{\gamma^\mu,\gamma^\nu\} = +2\delta^{\mu\nu}$, $\Gamma = i^{n-2}\gamma^1\gamma^2\cdots\gamma^{2n}$, $\{\Gamma,\gamma^\mu\} = 0$, $\Gamma^2 = +1$, and for any $2n = d$ matrices $\gamma^\alpha,\ldots,\gamma^\omega$, $\mathrm{tr}(\Gamma\gamma^\alpha\gamma^\beta\cdots\gamma^\omega) = 2^n i^{2-n}\epsilon^{\alpha\beta\cdots\omega}$.