

Introduction to Path Integrals

Path Integrals in Quantum Mechanics

Before explaining how the path integrals (or rather, the *functional integrals* work in quantum field theory, let me review the path integrals in the ordinary quantum mechanics of a single particle.

In the coordinate basis, motion of a quantum particle is described by the propagation amplitude

$$U(t_B, \mathbf{x}_B; T_A, \mathbf{x}_A) = \langle \mathbf{x}_B | e^{-i(t_B - t_A)\hat{H}/\hbar} | \mathbf{x}_A \rangle \quad (1)$$

for moving from point \mathbf{x}_A at time t_A to point \mathbf{x}_B at time t_B ; this amplitude is also called the *evolution kernel*. In the semi-classical regime, this kernel is given by the WKB approximation

$$U(B; A) \approx \text{prefactor} \times \exp(iS[\mathbf{x}_{\text{cl}}(t)]/\hbar) \quad (2)$$

where

$$S[\mathbf{x}(t)] = \int_{t_A}^{t_B} L(\mathbf{x}(t), \dot{\mathbf{x}}(t)) dt \quad (3)$$

is the action integral of the *classical mechanics* and $\mathbf{x}_{\text{cl}}(t)$ is the classical path from A to B that obeys the Euler–Lagrange equations of motion. In action terms, this path minimizes the the functional $S[\mathbf{x}(t)]$ under conditions $\mathbf{x}(t_A) = \mathbf{x}_A$ and $\mathbf{x}(t_B) = \mathbf{x}_B$. If there are several classical paths from A to B , then $S[\mathbf{x}]$ has several *local minima*, they all contribute to the evolution kernel with appropriate phases, and we get interference:

$$U(B; A) \approx \sum_{\substack{\text{classical} \\ \text{paths } i}} \text{prefactor}_i \times \exp(iS[\mathbf{x}_i(t)]/\hbar). \quad (4)$$

In the exact quantum mechanics, a sum (4) over classical paths becomes an integral over all possible path from A to B ,

$$U(B; A) = \iiint_{\mathbf{x}(t_A)=\mathbf{x}_A}^{\mathbf{x}(t_B)=\mathbf{x}_B} \mathcal{D}[\mathbf{x}(t)] \exp(iS[\mathbf{x}(t)]/\hbar). \quad (5)$$

However, unlike the sum (4), the integral here is not limited to the classical paths that obey

the Euler–Lagrange equations of motion. Instead, we integrate is over *all* differentiable paths $\mathbf{x}(t)$ from A to B , and they do not obey any equations of motion except by accident. But in the semiclassical $\hbar \rightarrow 0$ limit, the contributions of most paths to the integral is washed out by interference with similar paths whose action differs by only $O(\hbar)$. The only survivors of this wash-out are the stationary “points” of the functional $S[\mathbf{x}(t)]$, which are precisely the classical paths from A to B . This is how the WKB approximation (4) — and eventually the classical mechanics — emerge in the $\hbar \rightarrow 0$ limit.

The problem with the *path integral* (5) is how to define the integration measure $\mathcal{D}[\mathbf{x}(t)]$ for paths. The basic method is to discretize the time: Slice the continuous time interval $t_A \leq t \leq t_B$ into a large but finite set of discrete times

$$(t_0, t_1, t_2, \dots, t_{N-1}, t_N), \quad t_n = t_A + n\Delta t, \quad \Delta t = \frac{t_B - t_A}{N}, \quad t_0 = t_A, \quad t_N = t_B, \quad (6)$$

but eventually take the $N \rightarrow \infty$ limit. This gives us

$$\mathcal{D}[\mathbf{x}(t)] \stackrel{\text{def}}{=} \lim_{N \rightarrow \infty} d^3\mathbf{x}_1 d^3\mathbf{x}_2 \cdots d^3\mathbf{x}_{N-1} \times \text{normalization_factor}, \quad \text{where } \mathbf{x}_n \equiv \mathbf{x}(t_n). \quad (7)$$

Note that we do not integrate over the $\mathbf{x}_0 \equiv \mathbf{x}(t_A)$ and $\mathbf{x}_N \equiv \mathbf{x}(t_B)$ because they are fixed by the boundary conditions in eq. (5).

The non-obvious part of eq. (7) is the `normalization_factor`. We shall see later in these notes that this factor depends on N , on the net time $T = t_B - t_A$, and even on the particle’s mass, and the exact formula for this factor is not easy to guess. Fortunately, there is a different version of path integration that does not suffer from such normalization factors.

Let’s consider paths in the phase space (\mathbf{x}, \mathbf{p}) rather than just the \mathbf{x} -space. In other words, let’s treat $\mathbf{x}(t)$ and $\mathbf{p}(t)$ as independent variables and write the action integral (3) in the Hamiltonian language

$$S[\mathbf{x}(t), \mathbf{p}(t)] = \int_A^B [\mathbf{p}(t) \cdot d\mathbf{x}(t) - H(\mathbf{x}(t), \mathbf{p}(t)) dt] \quad (8)$$

as a functional of both $\mathbf{x}(t)$ and $\mathbf{p}(t)$. A classical path is a minimax of this functional — a (local) minimum with respect to variations of the $\mathbf{x}(t)$ but a (local) maximum with respect to

variations of the $\mathbf{p}(x)$. Also, the position $\mathbf{x}(t)$ is subject to boundary conditions at the start A and finish B , but there are no boundary conditions for the momentum $\mathbf{p}(t)$. In the quantum mechanics,

$$U(B; A) = \iiint_{\mathbf{x}(t_A)=\mathbf{x}_A}^{\mathbf{x}(t_B)=\mathbf{x}_B} \mathcal{D}'[\mathbf{x}(t)] \iiint \mathcal{D}[\mathbf{p}(t)] \exp(iS[\mathbf{x}(t), \mathbf{p}(t)]/\hbar) \quad (9)$$

where

$$\mathcal{D}'[\mathbf{x}(t)] \times \mathcal{D}[\mathbf{p}(t)] = \lim_{N \rightarrow \infty} \prod_{n=1}^{N-1} d^3\mathbf{x}_n \times \prod_{n=1}^N \frac{d^3\mathbf{p}_n}{(2\pi\hbar)^3}. \quad (10)$$

This time, there are no funny normalization factors: all we have is the $d^3\mathbf{p}/(2\pi\hbar)^3$ for each momentum variable, and that's standard convention in quantum mechanics. Note that for a given N , we integrate over N momenta but only $N - 1$ positions because of the boundary conditions on both ends; to make this difference explicit, I have marked the $\mathcal{D}'[\mathbf{x}(t)]$ with a prime.

Deriving the Phase-Space Path Integral from the Hamiltonian QM

Let's start with a mathematical **lemma**:

$$\lim_{N \rightarrow \infty} \left(e^{\hat{a}/N} \times e^{\hat{b}/N} \right)^N = e^{\hat{a}+\hat{b}} \quad (11)$$

even if the operators \hat{a} and \hat{b} do not commute with each other. **Proof:**

$$e^{\hat{a}/N} \times e^{\hat{b}/N} = 1 + \frac{\hat{a} + \hat{b}}{N} + O(1/N^2), \quad (12)$$

and

$$\lim_{N \rightarrow \infty} \left(1 + \frac{\hat{a} + \hat{b}}{N} + O(1/N^2) \right)^N = e^{\hat{a}+\hat{b}} \quad (13)$$

regardless of the details of the $O(1/N^2)$ terms. ■

Now consider a quantum particle living in three space dimensions with a Hamiltonian operator of the form

$$\hat{H} = K(\hat{\mathbf{p}}) + V(\hat{\mathbf{x}}) \quad (14)$$

where the kinetic energy $\hat{K} \equiv K(\hat{\mathbf{p}})$ does not depend on the position $\hat{\mathbf{x}}$ and the potential energy $\hat{V} = V(\hat{\mathbf{x}})$ does not depend on the momentum $\hat{\mathbf{p}}$. Using the lemma (11), we may write the evolution operator for the particle as

$$\hat{U}(t_B - t_A) \equiv e^{-i\hat{H}(t_B - t_A)/\hbar} = \lim_{N \rightarrow \infty} \left(e^{-i\hat{V}\Delta t/\hbar} \times e^{-i\hat{K}\Delta t/\hbar} \right)^N \quad (15)$$

where $\Delta t = (t_B - t_A)/N$ as in eq. (6). Consequently, in the coordinate basis

$$\langle \mathbf{x}_B | \hat{U}(t_B - t_A) | \mathbf{x}_A \rangle = \lim_{N \rightarrow \infty} \int d^3 \mathbf{x}_1 \cdots \int d^3 \mathbf{x}_{N-1} \prod_{n=1}^N \langle \mathbf{x}_n | e^{-i\hat{V}\Delta t/\hbar} \times e^{-i\hat{K}\Delta t/\hbar} | \mathbf{x}_{n-1} \rangle \quad (16)$$

where we have identified $\mathbf{x}_0 \equiv \mathbf{x}_A$ and $\mathbf{x}_N \equiv \mathbf{x}_B$. Each Dirac bracket in the above product evaluates to

$$\begin{aligned} \langle \mathbf{x}_n | e^{-i\hat{V}\Delta t/\hbar} \times e^{-i\hat{K}\Delta t/\hbar} | \mathbf{x}_{n-1} \rangle &= \\ &= e^{-iV(\mathbf{x}_n)\Delta t/\hbar} \times \langle \mathbf{x}_n | e^{-i\hat{K}\Delta t/\hbar} | \mathbf{x}_{n-1} \rangle \\ &= e^{-iV(\mathbf{x}_n)\Delta t/\hbar} \times \int \frac{d^3 \mathbf{p}_n}{(2\pi\hbar)^3} \langle \mathbf{x}_n | \mathbf{p}_n \rangle e^{-iK(\mathbf{p}_n)\Delta t/\hbar} \langle \mathbf{p}_n | \mathbf{x}_{n-1} \rangle \\ &= \int \frac{d^3 \mathbf{p}_n}{(2\pi\hbar)^3} e^{-iV(\mathbf{x}_n)\Delta t/\hbar} \times e^{i\mathbf{x}_n \cdot \mathbf{p}_n/\hbar} \times e^{-iK(\mathbf{p}_n)\Delta t/\hbar} \times e^{-i\mathbf{x}_{n-1} \cdot \mathbf{p}_n/\hbar} \\ &= \int \frac{d^3 \mathbf{p}_n}{(2\pi\hbar)^3} \exp \left[\frac{i}{\hbar} \left(\mathbf{p}_n \cdot (\mathbf{x}_n - \mathbf{x}_{n-1}) - V(\mathbf{x}_n)\Delta t - K(\mathbf{p}_n)\Delta t \right) \right]. \end{aligned} \quad (17)$$

Plugging this formula back into eq. (16) and combining all the exponentials, we arrive at

$$U(B; A) = \lim_{N \rightarrow \infty} \int d^3 \mathbf{x}_1 \cdots \int d^3 \mathbf{x}_{N-1} \int \frac{d^3 \mathbf{p}_1}{(2\pi\hbar)^3} \cdots \int \frac{d^3 \mathbf{p}_N}{(2\pi\hbar)^3} \exp(iS/\hbar), \quad (18)$$

where

$$S = \sum_{n=1}^N \mathbf{p}_n \cdot (\mathbf{x}_n - \mathbf{x}_{n-1}) - \Delta t \times \sum_{n=1}^N \left(V(\mathbf{x}_n) + K(\mathbf{p}_n) \right) \quad (19)$$

is the discretized action for a discretized path. Indeed, in the large N limit

$$\sum_{n=1}^N \left[\mathbf{p}_n \cdot (\mathbf{x}_n - \mathbf{x}_{n-1}) + (V(\mathbf{x}_n) + K(\mathbf{p}_n)) \times \Delta t \right] \xrightarrow{N \rightarrow \infty} \int_A^B \left(\mathbf{p}(t) \cdot d\mathbf{x}(t) - H(\mathbf{x}(t), \mathbf{p}(t)) dt \right) \equiv S[\mathbf{x}(t), \mathbf{p}(t)]. \quad (20)$$

Consequently, we should interpret the product of coordinate and momentum integrals in eq. (18) as the discretized integral over the paths in the momentum space,

$$\int d^3\mathbf{x}_1 \cdots \int d^3\mathbf{x}_{N-1} \int \frac{d^3\mathbf{p}_1}{(2\pi\hbar)^3} \cdots \int \frac{d^3\mathbf{p}_N}{(2\pi\hbar)^3} \xrightarrow{N \rightarrow \infty} \iiint \mathcal{D}'[\mathbf{x}(t)] \iiint \mathcal{D}[\mathbf{p}(t)] \quad (21)$$

in perfect agreement with eq. (10). And eq. (18) itself is the proof of the path-integral formula

$$U(B; A) = \iiint_{\mathbf{x}(t_A)=\mathbf{x}_A}^{\mathbf{x}(t_B)=\mathbf{x}_B} \mathcal{D}'[\mathbf{x}(t)] \iiint \mathcal{D}[\mathbf{p}(t)] \exp(iS[\mathbf{x}(t), \mathbf{p}(t)]/\hbar). \quad (9)$$

A note on discretization. Interpreting the sum $\sum_n \mathbf{p}_n \cdot (\mathbf{x}_n - \mathbf{x}_{n-1})$ as the discretized integral $\int \mathbf{p} \cdot d\mathbf{x}$ calls for assigning the momenta \mathbf{p}_n to mid-point discrete times with respect to the coordinates \mathbf{x}_n :

$$\mathbf{x}_n \equiv \mathbf{x}(t = t_A + n\Delta t) \quad \text{but} \quad \mathbf{p}_n \equiv \mathbf{p}(t = t_A + (n - \frac{1}{2})\Delta t). \quad (22)$$

As long as the Hamiltonian can be split into separate kinetic and potential energies according to eq. (14), such different discrete times for the \mathbf{x}_n and \mathbf{p}_n are OK because

$$\int H(\mathbf{x}, \mathbf{p}) dt = \int V(\mathbf{x}) dt + \int K(\mathbf{p}) dt \rightarrow \Delta t \sum_{n=1}^N V(\mathbf{x}_n) + \Delta t \sum_{n=1}^N K(\mathbf{p}_n) \quad (23)$$

and the details of the discretization do not matter in the large N limit. However, *when the classical Hamiltonian is more complicated than a sum of kinetic and potential energies, the path*

integral formalism suffers from the discretization ambiguity. For example, for

$$H(\mathbf{x}, \mathbf{p}) = \frac{\mathbf{p}^2}{2M(\mathbf{x})} \quad (24)$$

we could discretize the action as

$$\begin{aligned} S &\rightarrow \sum_n \mathbf{p}_n \cdot (\mathbf{x}_n - \mathbf{x}_{n-1}) - \Delta t \sum_n \frac{\mathbf{p}_n^2}{2M(\mathbf{x}_n)}, \\ \text{or} &\rightarrow \sum_n \mathbf{p}_n \cdot (\mathbf{x}_n - \mathbf{x}_{n-1}) - \Delta t \sum_n \frac{\mathbf{p}_n^2}{2M(\mathbf{x}_{n-1})}, \\ \text{or} &\rightarrow \sum_n \mathbf{p}_n \cdot (\mathbf{x}_n - \mathbf{x}_{n-1}) - \Delta t \sum_n \frac{\mathbf{p}_n^2}{M(\mathbf{x}_n) + M(\mathbf{x}_{n-1})}, \\ \text{or} &\rightarrow \text{something else,} \end{aligned} \quad (25)$$

all these options lead to different evolution kernels, and there are no general rules how to resolve such ambiguities. Instead, *the discretization ambiguities of the path-integral formalism correspond to the operator-ordering ambiguities of the Hilbert-space formalism of quantum mechanics.* For example, given the classical Hamiltonian of the form (24), we can take the quantum Hamiltonian operators to be

$$\begin{aligned} \hat{H} &= \frac{1}{2M(\hat{\mathbf{x}})} \hat{\mathbf{p}}^2, \quad \text{or } \hat{H} = \hat{\mathbf{p}}^2 \frac{1}{2M(\hat{\mathbf{x}})}, \quad \text{or } \hat{H} = \hat{\mathbf{p}} \frac{1}{2M(\hat{\mathbf{x}})} \hat{\mathbf{p}}, \quad \text{or} \\ \hat{H} &= \frac{1}{2M(\hat{\mathbf{x}})} \hat{\mathbf{p}} M(\hat{\mathbf{x}}) \hat{\mathbf{p}} \frac{1}{M(\hat{\mathbf{x}})}, \quad \text{or something else.} \end{aligned} \quad (26)$$

The Lagrangian Path Integral

In this section, I shall reduce the Hamiltonian path integrals over both $\mathbf{x}(t)$ and $\mathbf{p}(t)$ to the Lagrangian path integrals over the $\mathbf{x}(t)$ alone by integrating over the paths in momentum space. *This works only when the kinetic energy is quadratic in the momentum,*

$$H(\mathbf{p}, \mathbf{x}) = \frac{\mathbf{p}^2}{2M} + V(\mathbf{x}) \implies \hat{H} = \frac{\hat{\mathbf{p}}^2}{2M} + V(\hat{\mathbf{x}}). \quad (27)$$

For such Hamiltonians,

$$\begin{aligned} \mathbf{p} \cdot \dot{\mathbf{x}} - H(\mathbf{p}, \mathbf{x}) &= \mathbf{p} \cdot \dot{\mathbf{x}} - \frac{\mathbf{p}^2}{2M} - V(\mathbf{x}) = -\frac{(\mathbf{p} - M\dot{\mathbf{x}})^2}{2M} + \frac{M\dot{\mathbf{x}}^2}{2} - V(\mathbf{x}) \\ &= L(\dot{\mathbf{x}}, \mathbf{x}) - \frac{(\mathbf{p} - M\dot{\mathbf{x}})^2}{2M} \end{aligned} \quad (28)$$

and consequently

$$S^{\text{Ham}}[\mathbf{x}(t), \mathbf{p}(t)] = S^{\text{Lagr}}[\mathbf{x}(t)] - \frac{1}{2M} \int dt (\mathbf{p} - M\dot{\mathbf{x}})^2. \quad (29)$$

Therefore, in the path integral formalism,

$$\begin{aligned} U(B; A) &= \int_A^B \mathcal{D}'[\mathbf{x}(t)] \int \mathcal{D}[\mathbf{p}(t)] \exp\left(\frac{i}{\hbar} S^{\text{Ham}}[\mathbf{x}(t), \mathbf{p}(t)]\right) \\ &= \int_A^B \mathcal{D}'[\mathbf{x}(t)] \exp\left(\frac{i}{\hbar} S^{\text{Lagr}}[\mathbf{x}(t)]\right) \times \int \mathcal{D}[\mathbf{p}(t)] \exp\left(\frac{-i}{2M\hbar} \int dt (\mathbf{p} - M\dot{\mathbf{x}})^2\right). \end{aligned} \quad (30)$$

On the second line here, we integrate over the coordinate-space paths $\mathbf{x}(t)$ after integrating over the momentum-space paths $\mathbf{p}(t)$, so as far as $\int \mathcal{D}[\mathbf{p}(t)]$ is concerned, we can treat the coordinate-space path $\mathbf{x}(t)$ as a constant. Also, the path-integral measure is linear so we may shift the integration variable by a constant, thus

$$\begin{aligned} \int \mathcal{D}[\mathbf{p}(t)] \exp\left(\frac{-i}{2M\hbar} \int dt (\mathbf{p} - M\dot{\mathbf{x}})^2\right) &= \int \mathcal{D}[\mathbf{p}(t) - M\dot{\mathbf{x}}(t)] \exp\left(\frac{-i}{2M\hbar} \int dt (\mathbf{p} - M\dot{\mathbf{x}})^2\right) \\ &= \int \mathcal{D}[\mathbf{p}'(t)] \exp\left(\frac{-i}{2M\hbar} \int dt \mathbf{p}'^2(t)\right) \\ &= \text{const.} \end{aligned} \quad (31)$$

Plugging this formula back into eq. (30) gives us the Lagrangian path integral

$$U(B; A) = \text{const} \times \int_{\mathbf{x}(t_A)=\mathbf{x}_A}^{\mathbf{x}(t_B)=\mathbf{x}_B} \mathcal{D}'[\mathbf{x}(t)] \exp\left(\frac{i}{\hbar} S^{\text{Lagr}}[\mathbf{x}(t)]\right). \quad (32)$$

In this formalism there is no independent momentum-space path $\mathbf{p}(t)$, we integrate only over the coordinate-space path $\mathbf{x}(t)$, and the action is given by the Lagrangian formula (3). However, the price of this simplification is the un-known overall constant multiplying the path integral (32).

To calculate this constant we should first discretize the time and only then integrate out the discrete momenta \mathbf{p}_n . For finite N , the discretized Hamiltonian-formalism action (19) can be written as

$$\begin{aligned}
S_{\text{discr}}^{\text{Ham}}(\mathbf{x}_0, \dots, \mathbf{x}_N; \mathbf{p}_1, \dots, \mathbf{p}_N) &= \sum_n \mathbf{p}_n \cdot (\mathbf{x}_n - \mathbf{x}_{n-1}) - \frac{\Delta t}{2M} \sum_n \mathbf{p}_n^2 - \Delta t \sum_n V(\mathbf{x}_n) \\
&= -\frac{\Delta t}{2M} \sum_n \left(\mathbf{p}_n - M \frac{\mathbf{x}_n - \mathbf{x}_{n-1}}{\Delta t} \right)^2 \\
&\quad + \frac{M}{2\Delta t} \sum_n (\mathbf{x}_n - \mathbf{x}_{n-1})^2 - \Delta t \sum_n V(\mathbf{x}_n) \\
&= -\frac{\Delta t}{2M} \sum_n \left(\mathbf{p}_n - M \frac{\mathbf{x}_n - \mathbf{x}_{n-1}}{\Delta t} \right)^2 + S_{\text{discr}}^{\text{Lagr}}(\mathbf{x}_0, \dots, \mathbf{x}_N)
\end{aligned} \tag{33}$$

where

$$\begin{aligned}
S_{\text{discr}}^{\text{Lagr}}(\mathbf{x}_0, \dots, \mathbf{x}_N) &= \Delta t \sum_n \left[\frac{M}{2} \left(\frac{\mathbf{x}_n - \mathbf{x}_{n-1}}{\Delta t} \right)^2 - V(\mathbf{x}_n) \right] \\
&\xrightarrow{N \rightarrow \infty} \int dt \left[\frac{M}{2} \left(\frac{d\mathbf{x}}{dt} \right)^2 - V(\mathbf{x}) \right] = S^{\text{Lagr}}[\mathbf{x}(t)]
\end{aligned} \tag{34}$$

is the discretized action for of the Lagrangian formalism. In light of eq. (33) we may write the discretized path integral (18) as

$$\begin{aligned}
&\int d^3\mathbf{x}_1 \cdots \int d^3\mathbf{x}_{N-1} \int \frac{d^3\mathbf{p}_1}{(2\pi\hbar)^3} \cdots \int \frac{d^3\mathbf{p}_N}{(2\pi\hbar)^3} \exp \left(\frac{i}{\hbar} S_{\text{discr}}^{\text{Ham}}(\mathbf{x}_0, \dots, \mathbf{x}_N; \mathbf{p}_1, \dots, \mathbf{p}_N) \right) = \\
&= \int d^3\mathbf{x}_1 \cdots \int d^3\mathbf{x}_{N-1} \exp \left(\frac{i}{\hbar} S_{\text{discr}}^{\text{Lagr}}(\mathbf{x}_0, \dots, \mathbf{x}_N) \right) \times \\
&\quad \times \prod_{n=1}^N \int \frac{d^3\mathbf{p}_n}{(2\pi\hbar)^3} \exp \left(\frac{-i\Delta t}{2M\hbar} \left(\mathbf{p}_n - M \frac{\mathbf{x}_n - \mathbf{x}_{n-1}}{\Delta t} \right)^2 \right)
\end{aligned} \tag{35}$$

where we integrate over all the momenta \mathbf{p}_n before we integrate over the coordinates. Consequently, in each integral on the last line of eq. (35) we may shift the integration variable from \mathbf{p}_n to $\mathbf{p}'_n = \mathbf{p}_n - M\Delta\mathbf{x}_n/\Delta t$, thus

$$\begin{aligned}
\int \frac{d^3\mathbf{p}_n}{(2\pi\hbar)^3} \exp \left(\frac{-i\Delta t}{2M\hbar} \left(\mathbf{p}_n - M \frac{\mathbf{x}_n - \mathbf{x}_{n-1}}{\Delta t} \right)^2 \right) &= \int \frac{d^3\mathbf{p}'_n}{(2\pi\hbar)^3} \exp \left(\frac{-i\Delta t}{2M\hbar} \mathbf{p}'_n{}^2 \right) \\
&= \left(\frac{M}{2\pi i \hbar \Delta t} \right)^{3/2}.
\end{aligned} \tag{36}$$

Plugging this formula back into eq. (35), we arrive at the Lagrangian path integral

$$\begin{aligned}
U(B; A) &= \lim_{N \rightarrow \infty} \left(\frac{MN}{2\pi i \hbar (t_B - t_A)} \right)^{3N/2} \times \int d^3 \mathbf{x}_1 \cdots \int d^3 \mathbf{x}_{N-1} \exp \left(\frac{i}{\hbar} S_{\text{discr}}^{\text{Lagr}}(\mathbf{x}_0, \dots, \mathbf{x}_N) \right) \\
&\equiv \iiint_{\mathbf{x}(t_A)=\mathbf{x}_A}^{\mathbf{x}(t_B)=\mathbf{x}_B} \mathcal{D}'[\mathbf{x}(t)] \exp \left(\frac{i}{\hbar} S^{\text{Lagr}}[\mathbf{x}(t)] \right).
\end{aligned} \tag{37}$$

Note however that in the Lagrangian formalism, the $\mathcal{D}'[\mathbf{x}(t)]$ is not just the limit of $d^{3(N-1)} \mathbf{x} \equiv d^3 \mathbf{x}_1 \cdots d^3 \mathbf{x}_{N-1}$ but also includes the normalisation factor

$$C(N, M, t_B - t_A) = \left(\frac{MN}{2\pi i \hbar (t_B - t_A)} \right)^{3N/2}. \tag{38}$$

This normalization factor depends on N , on the net time $T = t_B - t_A$, and on the particle's mass M , but it does not depend on the potential $V(x)$ or the initial and final points x_A and x_B . Consequently, *without discretizing time, a Lagrangian path integral calculation yields the amplitude $U(B; A)$ up to an unknown overall factor $F(M, T)$* . However, we may obtain this factor by comparing with a similar path integral for a free particle: the overall $F(M, T)$ factor is the same in both cases, and the free amplitude is known to be

$$U_{\text{free}}(B; A) = \left(\frac{M}{2\pi i \hbar T} \right)^{3/2} \times \exp \left(\frac{iM(x_B - x_A)^2}{2\hbar T} \right). \tag{39}$$

Alternatively, all kind of quantities can be obtained from the ratios of path integrals, and such ratios do not depend on the overall normalization of the $\mathcal{D}[x(t)]$; this is the method most commonly used in the quantum field theory.

The Partition Function

The partition function of a quantum system with a Hamiltonian \hat{H} is the trace

$$Z(t) \stackrel{\text{def}}{=} \text{Tr} \hat{U}(t; 0) \equiv \text{Tr} \exp(-it\hat{H}/\hbar) = \sum_{\text{eigenvalues } E_n} \exp(-itE_n/\hbar). \tag{40}$$

This time-dependent partition function is related to the temperature-dependent partition func-

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$$Z(\beta) = \text{Tr} \exp(-\beta \hat{H}) \quad (41)$$

via analytical continuation of time t to imaginary values

$$t \rightarrow i\hbar\beta = \frac{-i\hbar}{k_B \times \text{Temperature}}. \quad (42)$$

In the path integral formalism, the partition function is given by

$$Z(T) = \int dx U(t, x; 0, x) = \iiint_{x(T)=x(0)} \mathcal{D}[x(t)] e^{iS[x(t)]/\hbar}. \quad (43)$$

Note no prime over \mathcal{D} because the paths $x(t)$ are subject to only one boundary condition — periodicity in time, $x(T) = x(0)$. Without discretizing time, the path integral (43) can be calculated up to an overall normalization constant. Consequently, when we extract the Hamiltonian's spectrum $\{E_n\}$ from the partition function $Z(T)$, the multiplicity of all the eigenvalues can be determined only up to some unknown overall factor.

For example, consider a harmonic oscillator with action

$$S[x(t)] = \frac{M}{2} \int dt (\dot{x}^2(t) - \omega^2 x^2(t)). \quad (44)$$

This action is a quadratic functional of the $x(t)$, and it can be diagonalized via Fourier transform,

$$x(t) = \sum_{n=-\infty}^{+\infty} y_n \times e^{2\pi i n t/T}, \quad y_n^* = y_{-n}, \quad (45)$$

$$S[x(t)] = \sum_{n=-\infty}^{+\infty} C_n y_n^* y_n, \quad (46)$$

$$C_n = C_{-n} = \frac{MT}{2} \times \left(\left(\frac{2\pi n}{T} \right)^2 - \omega^2 \right). \quad (47)$$

Note that the discrete frequencies $2\pi n/T$ of the Fourier transform (45) are completely determined by the boundary conditions $x(T) = x(0)$ and have nothing to do with the oscillator's

frequency ω . By linearity of the transform (45),

$$\begin{aligned} \iint_{\text{periodic}} \mathcal{D}[x(t)] &= \prod_{n=-\infty}^{+\infty} \int dy_n \times \text{a constant Jacobian} \\ &= J \times \int dy_0 \prod_{n=1}^{\infty} \int d \operatorname{Re} y_n \int d \operatorname{Im} y_n. \end{aligned} \quad (48)$$

To be precise, the Jacobian J here depends on T and on the mass M via the normalization of the Lagrangian path integral, but it does not depend on any of the y_n variables, and it does not depend on the oscillator's frequency ω .

In terms of the Fourier variables y_n , the path integral (43) becomes

$$\begin{aligned} Z &= J \times \int dy_0 \prod_{n=0}^{\infty} \int d \operatorname{Re} y_n \int d \operatorname{Im} y_n \exp \left(\frac{i}{\hbar} S = \frac{iC_0}{\hbar} y_0^2 + \sum_{n=1}^{\infty} \frac{2iC_n}{\hbar} |y_n|^2 \right) \\ &= J \times \sqrt{\frac{\pi i \hbar}{C_0}} \times \prod_{n=1}^{\infty} \frac{\pi i \hbar}{2C_n}. \end{aligned} \quad (49)$$

The coefficients C_n are spelled out in eq. (47), but it's convenient to rewrite them as

$$C_0 = -\frac{M}{2T} \times (\omega T)^2, \quad C_{n>0} = \frac{2\pi^2 M n^2}{T} \times \left(1 - \left(\frac{\omega T}{2\pi n} \right)^2 \right). \quad (50)$$

Consequently, the partition function (49) becomes

$$Z(T) = J \times \frac{\sqrt{-2\pi i \hbar T / M}}{\omega T} \times \prod_{n=1}^{\infty} \frac{(i \hbar T) / (4\pi n^2 M)}{1 - \left(\frac{\omega T}{2\pi n} \right)^2} = \frac{-iF}{(\omega T) \prod_{n=1}^{\infty} \left(1 - \left(\frac{\omega T}{2\pi n} \right)^2 \right)} \quad (51)$$

where

$$F = J \times \sqrt{\frac{-2\pi i \hbar T}{M}} \times \prod_{n=1}^{\infty} \frac{i \hbar T}{4\pi M n^2} \quad (52)$$

combines all the factors that do not depend on the oscillator's frequency ω . *A priori*, F could be a function of M or T , but by the non-relativistic dimensional analysis, a dimensionless function $F(M, T, \hbar)$ which does not depend on anything else must be a constant. It is not clear whether

this constant is finite or infinite: it contains an infinite product over n that is badly divergent, and the Jacobian J is also badly divergent. To resolve this issue, we need to discretize time and then go through a calculation similar to the above but more complicated; I have written it down in a separate [supplementary note](#), and you should read as a part of your next homework. For now, just take it without proof that all the divergences cancel out and F is finite.

The remaining infinite product in the denominator of eq. (51) is absolutely convergent, and it may be evaluated just by looking at its poles and zeros. The analytic function

$$s(x) = \frac{1}{x} \times \prod_{n=1}^{\infty} \left(\frac{1}{1 - (x/n)^2} = \frac{n}{n-x} \times \frac{n}{n+x} \right) \quad (53)$$

has no zeroes, it has simple poles at all integers (positive, negative, and zero), it does not have any worse-than-pole singularities in the complex x plane, and it does not grow when $\text{Im } x \rightarrow \pm\infty$. These facts completely determine this function to be

$$\frac{1}{x} \times \prod_{n=1}^{\infty} \frac{1}{1 - (x/n)^2} = \frac{\pi}{\sin(\pi x)} \quad (54)$$

where the normalization comes from the residue of the pole at $x = 0$. In eq. (51) we have a similar product for $x = \omega T/2\pi$, hence

$$Z(T) = \frac{-iF/2}{\sin(\omega T/2)}. \quad (55)$$

To extract the oscillator's eigenvalues from this partition function, we expand it as

$$Z(T) = \frac{F}{2i \sin(\omega T/2)} = \frac{F}{e^{i\omega T/2} - e^{-i\omega T/2}} = F \times \sum_{n=0}^{\infty} e^{-i\omega T(n+\frac{1}{2})}. \quad (56)$$

Comparing this series to eq. (40), we immediately see that the eigenvalues are $E_n = \hbar\omega(n + \frac{1}{2})$ and they all have the same multiplicity F . Of course, we all new those facts back in the undergraduate school (if not earlier), but now we know how to derive them in the path-integral formalism.