

Phase Space Factors

For quantum transitions to un-bound states — for example, an atom emitting a photon, or a radioactive decay, or scattering — which is a kind of unbound \rightarrow unbound transition, — the **transition rate** is given by the **Fermi's golden rule**:

$$\Gamma \stackrel{\text{def}}{=} \frac{d \text{probability}}{d \text{time}} = \frac{2\pi\rho}{\hbar} \times \left| \langle \text{final} | \hat{T} | \text{initial} \rangle \right|^2 \quad (1)$$

where $\hat{T} = \hat{H}_{\text{perturbation}} + \text{higher order corrections}$, and ρ is the *density of final states*,

$$\rho = \frac{dN_{\text{final states}}}{dE_{\text{final}}} . \quad (2)$$

For an example, consider an atom emitting a photon of specific polarization λ . Using the large-box normalization for the photon's states, we have

$$dN_{\text{final}} = \left(\frac{L}{2\pi} \right)^3 d^3\mathbf{k}_\gamma = \frac{L^3}{(2\pi)^3} \times k_\gamma^2 dk_\gamma d^2\Omega_\gamma \quad (3)$$

while $dE_{\text{final}} = dE_\gamma = \hbar c \times dk_\gamma$, hence

$$\rho = L^3 \times \frac{k_\gamma^2}{(2\pi)^3 \hbar c} \times d^2\Omega_\gamma . \quad (4)$$

The L^3 factor here cancels against the $L^{-3/2}$ factor in the matrix element $\langle \text{atom}' + \gamma | \hat{T} | \text{atom} \rangle$ due to the photon's wave function in the large-box normalization. As to the remaining $d^2\Omega_\gamma$ factor, we should integrate over it to get the total decay rate, or divide by it to get the partial emission rate $d\Gamma/d\Omega$ for the photons emitted in a particular direction, thus

$$\begin{aligned} \frac{d\Gamma(\lambda)}{d\Omega} &= \frac{k^2}{(2\pi)^3 \hbar c} \times L^3 \left| \langle \text{atom}' + \gamma(\mathbf{k}, \lambda) | \hat{T} | \text{atom} \rangle \right|^2 , \\ \Gamma_{\text{total}} &= \int d\Omega \sum_\lambda \frac{k^2}{(2\pi)^3 \hbar c} \times L^3 \left| \langle \text{atom}' + \gamma(\mathbf{k}, \lambda) | \hat{T} | \text{atom} \rangle \right|^2 . \end{aligned} \quad (5)$$

In relativistic normalization of quantum states and matrix elements, there are no $L^{-3/2}$ factors but instead there are $\sqrt{2E}$ factors for each final-state or initial state particle, and

they must be compensated by dividing the density of states ρ by the $\prod_i(2E_i)$. Also, we must allow for motion of all the final-state particles (*i.e.*, both the photon and the recoiled atom) but impose the momentum conservation as a constraint. Thus, for a decay of 1 initial particle into n final particles,

$$\Gamma = \frac{1}{2E_{\text{in}}} \int \frac{d^3\mathbf{p}'_1}{(2\pi)^3 2E'_1} \cdots \int \frac{d^3\mathbf{p}'_n}{(2\pi)^3 2E'_n} |\langle p'_1, \dots, p'_n | \mathcal{M} | p_{\text{in}} \rangle|^2 \times (2\pi^4) \delta^{(4)}(p'_1 + \cdots + p'_n - p_{\text{in}}), \quad (6)$$

where the δ function takes care of both momentum conservation and of the denominator dE_f in the density-of-states factor (2). Likewise, the transition rate for a generic $2 \rightarrow n$ scattering process is given by

$$\Gamma = \frac{1}{2E_1 \times 2E_2} \int \frac{d^3\mathbf{p}'_1}{(2\pi)^3 2E'_1} \cdots \int \frac{d^3\mathbf{p}'_n}{(2\pi)^3 2E'_n} |\langle p'_1, \dots, p'_n | \mathcal{M} | p_1, p_2 \rangle|^2 \times (2\pi^4) \delta^{(4)}(p'_1 + \cdots + p'_n - p_1 - p_2). \quad (7)$$

In terms of the scattering cross-section σ , the rate (7) is $\Gamma = \sigma \times \text{flux}$ of initial particles. In the large-box normalization the flux is $L^{-3}|\mathbf{v}_1 - \mathbf{v}_2|$, so in the continuum normalization it's simply the relative speed $|\mathbf{v}_1 - \mathbf{v}_2|$. Consequently, the total scattering cross-section is given by

$$\sigma_{\text{tot}} = \frac{1}{4E_1 E_2 |\mathbf{v}_1 - \mathbf{v}_2|} \int \frac{d^3\mathbf{p}'_1}{(2\pi)^3 2E'_1} \cdots \int \frac{d^3\mathbf{p}'_n}{(2\pi)^3 2E'_n} |\langle p'_1, \dots, p'_n | \mathcal{M} | p_1, p_2 \rangle|^2 \times (2\pi^4) \delta^{(4)}(p'_1 + \cdots + p'_n - p_1 - p_2). \quad (8)$$

In particle physics, all the factors in eqs (6) or (8) besides the matrix elements — as well as the integrals over such factors — are collectively called the *phase space* factors.

A note on Lorentz invariance of decay rates or cross-sections. The matrix elements $\langle \text{final} | \mathcal{M} | \text{initial} \rangle$ are Lorentz invariant, and so are all the integrals over the final-particles' momenta and the δ -functions. The only non-invariant factor in the decay-rate formula (6) is the pre-integral $1/E_{\text{init}}$, hence the decay rate of a moving particle is

$$\Gamma(\text{moving}) = \Gamma(\text{rest frame}) \times \frac{M}{E} \quad (9)$$

where M/E is precisely the time dilation factor in the moving frame.

As to the scattering cross-section, it should be invariant under Lorentz boosts along the initial axis of scattering, thus the same cross-section in any frame where $\mathbf{p}_1 \parallel \mathbf{p}_2$. This includes the *lab frame* where one of the two particles is initially at rest, the *center-of-mass frame* where $\mathbf{p}_1 + \mathbf{p}_2 = 0$, and any other frame where the two particles collide head-on. And indeed, the pre-integral factor in eq. (8) for the cross-section

$$\frac{1}{4E_1 E_2 |\mathbf{v}_1 - \mathbf{v}_2|} = \frac{1}{4|E_1 \mathbf{p}_2 - E_2 \mathbf{p}_1|} \quad (10)$$

is invariant under Lorentz boosts along the scattering axis.

Let's simplify eq. (8) for a 2 particle \rightarrow 2 particle scattering process in the center-of-mass frame where $\mathbf{p}_1 + \mathbf{p}_2 = 0$. In this frame, the pre-exponential factor (10) becomes

$$\frac{1}{4|\mathbf{p}| \times (E_1 + E_2)} \quad (11)$$

while the remaining phase space factors amount to

$$\begin{aligned} \mathcal{P}_{\text{int}} &= \int \frac{d^3 \mathbf{p}'_1}{(2\pi)^3 2E'_1} \int \frac{d^3 \mathbf{p}'_2}{(2\pi)^3 2E'_2} (2\pi)^4 \delta^{(3)}(\mathbf{p}'_1 + \mathbf{p}'_2) \delta(E'_1 + E'_2 - E_{\text{net}}) \\ &= \int \frac{d^3 \mathbf{p}'_1}{(2\pi)^3 \times 2E'_1 \times 2E'_2} (2\pi) \delta(E'_1(\mathbf{p}'_1) + E'_2(-\mathbf{p}'_1) - E_{\text{net}}) \\ &= \int d^2 \Omega_{\mathbf{p}'} \times \int_0^\infty dp' \frac{p'^2}{16\pi^2 E'_1 E'_2} \times \delta(E'_1 + E'_2 - E_{\text{tot}}) \\ &= \int d^2 \Omega_{\mathbf{p}'} \left[\frac{p'^2}{16\pi^2 E'_1 E'_2} \Big/ \frac{d(E'_1 + E'_2)}{dp'} \right]_{E'_1 + E'_2 = E_{\text{tot}}}^{\text{when}}. \end{aligned} \quad (12)$$

On the last 3 lines here $E'_1 = E'_1(\mathbf{p}'_1) = \sqrt{p'^2 + m_1^2}$ while $E'_2 = E'_2(\mathbf{p}'_2 = -\mathbf{p}'_1) = \sqrt{p'^2 + m_2^2}$. Consequently,

$$\frac{dE'_1}{dp'} = \frac{p'}{E'_1}, \quad \frac{dE'_2}{dp'} = \frac{p'}{E'_2}, \quad (13)$$

hence

$$\frac{d(E'_1 + E'_2)}{dp'} = \frac{p'}{E'_1} + \frac{p'}{E'_2} = \frac{p'}{E'_1 E'_2} \times (E'_2 + E'_1 = E_{\text{tot}}), \quad (14)$$

and therefore

$$\mathcal{P}_{\text{int}} = \frac{1}{16\pi^2} \times \frac{p'}{E_{\text{tot}}} \times \int d^2\Omega_{\mathbf{p}'}. \quad (15)$$

Including the pre-integral factor (11), we arrive at the net phase space factor

$$\mathcal{P} = \frac{p'}{p} \times \frac{1}{64\pi^2 E_{\text{tot}}^2} \times \int d^2\Omega_{\mathbf{p}'}. \quad (16)$$

The matrix element \mathcal{M} for the scattering should be put inside the direction-angle integral in this phase-space formula. Thus, the total scattering cross-section is

$$\sigma_{\text{tot}}(1 + 2 \rightarrow 1' + 2') = \frac{p'}{p} \times \frac{1}{64\pi^2 E_{\text{cm}}^2} \times \int d^2\Omega \left| \langle p'_1 + p'_2 | \mathcal{M} | p_1 + p_2 \rangle \right|^2, \quad (17)$$

while the partial cross-section for scattering in a particular direction is

$$\frac{d\sigma(1 + 2 \rightarrow 1' + 2')}{d\Omega_{\text{cm}}} = \frac{p'}{p} \times \frac{1}{64\pi^2 E_{\text{cm}}^2} \times \left| \langle p'_1 + p'_2 | \mathcal{M} | p_1 + p_2 \rangle \right|^2. \quad (18)$$

Note: the total cross-section is the same in frames where the initial momenta are collinear, but in the partial cross-section, $d\Omega$ depends on the frame of reference, so eq. (18) applies only in the center-of mass frame. Also, the E_{cm} factor in denominators of both formulae stands for the net energy in the center-of-mass frame. In frame-independent terms,

$$E_{\text{cm}}^2 = (p_1 + p_2)^2 = (p'_1 + p'_2)^2 = s. \quad (19)$$

Finally, let me write down the phase-space factor for a 2-body decay (1 particle \rightarrow 2 particles) in the rest frame of the initial particle. The under-the-integral factors for such a decay are the same as in eq. (15) for a $2 \rightarrow 2$ scattering, but the pre-integral factor is $1/2M_{\text{init}}$ instead of the (11), thus

$$\mathcal{P} = \frac{p'}{32\pi^2 M^2}, \quad (20)$$

meaning

$$\frac{d\Gamma(0 \rightarrow 1' + 2')}{d\Omega} = \frac{p'}{32\pi^2 M^2} \times \left| \langle p'_1 + p'_2 | \mathcal{M} | p_0 \rangle \right|^2, \quad (21)$$

$$\Gamma(0 \rightarrow 1' + 2') = \frac{p'}{32\pi^2 M^2} \times \int d^2\Omega \left| \langle p'_1 + p'_2 | \mathcal{M} | p_0 \rangle \right|^2. \quad (22)$$