Feynman Propagator of a Scalar Field

Earlier in class, I have defined the Feynman propagator of a free real scalar field as a time-ordered correlation function of two scalar fields in the vacuum state,

$$G_F(x - y) \overset{\text{def}}{=} \langle 0 | T\hat{\Phi}(x)\hat{\Phi}(y) | 0 \rangle. \quad (1)$$

We saw that

$$G_F(x - y) = \theta(x^0 > y^0)\times D(x - y) + \theta(x^0 < y^0)\times D(y - x) = \begin{cases} D(x - y) & \text{when } x^0 > y^0, \\ D(y - x) & \text{when } x^0 < y^0, \end{cases} \quad (2)$$

where

$$D(x - y) \overset{\text{def}}{=} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \times \exp(-ik(x - y))^{k^0 = +\omega_k}. \quad (3)$$

A complex scalar field has a similar propagator, but the correlation function involves one $\hat{\Phi}$ field and one $\hat{\Phi}^\dagger$ field,

$$\langle 0 | T\hat{\Phi}^\dagger(x)\hat{\Phi}(y) | 0 \rangle = \langle 0 | T\hat{\Phi}(x)\hat{\Phi}^\dagger(y) | 0 \rangle = G_F(x - y). \quad (4)$$

In these notes, I shall show that the propagator (1) is a Green’s function of the Klein–Gordon equation, and then I shall explain why there are many different Green’s functions and which particular Green’s function happens to be the Feynman propagator.

**The Feynman propagator is a Green’s function**

A free scalar field obeys the Klein–Gordon equation $(\partial^2 + m^2)\hat{\Phi}(x) = 0$. Consequently, the Feynman propagator (1) for the $\hat{\Phi}$ is a Green’s function of that equation,

$$(\partial^2 + m^2)G_F(x - y) = -i\delta^{(4)}(x - y). \quad (5)$$

Note the delta-function on the RHS is in all four dimensions of the spacetime.
To prove eq. (5), we start with a Lemma: the time derivative of a time-ordered product of two operators $\hat{A}(t)$ and $\hat{B}(t_0)$ obtains as

$$\frac{\partial}{\partial t}(T\hat{A}(t)\hat{B}(t_0)) = T\left(\frac{\partial \hat{A}(t)}{\partial t}\right)\hat{B}(t_0) + \delta(t - t_0) \times [\hat{A}(t), \hat{B}(t_0)].$$  \hfill (6)

Proof (of the lemma):

$T\hat{A}(t)\hat{B}(t_0) \overset{\text{def}}{=} \theta(t > t_0) \times \hat{A}(t)\hat{B}(t_0) + \theta(t < t_0) \times \hat{B}(t_0)\hat{A}(t),$  \hfill (7)

$$\frac{\partial}{\partial t} \theta(t > t_0) = +\delta(t - t_0), \quad \frac{\partial}{\partial t} \theta(t < t_0) = -\delta(t - t_0),$$  \hfill (8)

therefore

$$\frac{\partial}{\partial t}(T\hat{A}(t)\hat{B}(t_0)) = \frac{\partial}{\partial t} \left(\theta(t > t_0) \times \hat{A}(t)\hat{B}(t_0)\right) + \frac{\partial}{\partial t} \left(\theta(t < t_0) \times \hat{B}(t_0)\hat{A}(t)\right)$$

$$= \delta(t - t_0) \times \hat{A}(t)\hat{B}(t_0) + \theta(t > t_0) \times \frac{\partial \hat{A}(t)}{\partial t} \times \hat{B}(t_0)$$

$$- \delta(t - t_0) \times \hat{B}(t_0) \times \hat{A}(t) + \theta(t < t_0) \times \hat{B}(t) \times \frac{\partial \hat{A}(t)}{\partial t},$$

$\langle \langle \text{reorganizing terms} \rangle \rangle$

$$= \delta(t - t_0) \times \left(\hat{A}(t)\hat{B}(t_0) - \hat{B}(t_0)\hat{A}(t)\right)$$

$$+ \left(\theta(t > t_0)\frac{\partial \hat{A}(t)}{\partial t} \hat{B}(t_0) + \theta(t < t_0)\hat{B}(t_0)\frac{\partial \hat{A}(t)}{\partial t}\right)$$

$$= \delta(t - t_0) \times [\hat{A}(t), \hat{B}(t_0)] + T\left(\frac{\partial \hat{A}(t)}{\partial t} \hat{B}(t_0)\right).$$  \hfill (9)

Quod erat demonstrandum.

Now let’s prove that the propagator (1) is a Green’s function. In light of the lemma (6),

$$\frac{\partial}{\partial x^0} G_F(x - y) = \langle 0| \frac{\partial}{\partial x^0} \left(T\hat{\Phi}(x) \hat{\Phi}(y)\right) |0\rangle$$

$$= \langle 0| T\left(\partial_0 \hat{\Phi}(x) \times \hat{\Phi}(y)\right) |0\rangle + \delta(x^0 - y^0) \times \langle 0| \left[\hat{\Phi}(x), \hat{\Phi}(y)\right] |0\rangle.$$  \hfill (10)

In the second term on the bottom line here, the quantum fields $\hat{\Phi}(x)$ and $\hat{\Phi}(y)$ are at equal times $x^0 = y^0$, so they commute with each other. Consequently, the second term vanishes,
and we are left with
\[ \frac{\partial}{\partial x_0} G_F(x-y) = \langle 0 | T(\partial_0 \hat{\Phi}(x) \times \hat{\Phi}(y)) | 0 \rangle. \] (11)

Now let’s take another time derivative. Again, using the lemma (6), we obtain
\[ \partial_0^2 G_F(x-y) = \frac{\partial}{\partial x_0} \langle 0 | T(\partial_0 \hat{\Phi}(x) \times \hat{\Phi}(y)) | 0 \rangle
= \langle 0 | T(\partial_0^2 \hat{\Phi}(x) \times \hat{\Phi}(y)) | 0 \rangle + \delta(x^0-y^0) \times \langle 0 | [\partial_0 \hat{\Phi}(x), \hat{\Phi}(y)] | 0 \rangle. \] (12)

This time, in the second term on the bottom line, \( \partial_0 \hat{\Phi}(x) = \hat{\Pi}(x) \), and at equal times \( x^0 = y^0 \) it does not commute with the \( \hat{\Phi}(y) \). Instead,
\[ \delta(x^0-y^0) \times \langle 0 | [\partial_0 \hat{\Phi}(x), \hat{\Phi}(y)] | 0 \rangle = -i\delta^{(3)}(x-y), \] (13)

hence
\[ \delta(x^0-y^0) \times \langle 0 | [\partial_0 \hat{\Phi}(x), \hat{\Phi}(y)] | 0 \rangle = -i\delta^{(3)}(x-y) \times \delta(x^0-y^0) = -i\delta^{(4)}(x-y). \] (14)

Thus, eq. (12) reduces to
\[ \partial_0^2 G_F(x-y) = \langle 0 | T(\partial_0^2 \hat{\Phi}(x) \times \hat{\Phi}(y)) | 0 \rangle - i\delta^{(4)}(x-y). \] (15)

Now consider the space-derivative terms in the Klein-Gordon equation. Since the space derivatives commute with the time-ordering,
\[ \nabla_2 G_F(x-y) = \nabla_x^2 \langle 0 | (T \hat{\Phi}(x) \times \hat{\Phi}(y)) | 0 \rangle = \langle 0 | T(\nabla_x^2 \hat{\Phi}(x) \times \hat{\Phi}(y)) | 0 \rangle \] (16)

without any extra terms. Combining this formula with eq. (15), we obtain
\[ (\partial_0^2 - \nabla^2 + m^2) G_F(x-y) = \langle 0 | T((\partial_0^2 - \nabla^2 + m^2) \hat{\Phi}(x) \times \hat{\Phi}(y)) | 0 \rangle - i\delta^{(4)}(x-y). \] (17)

In the first term on the RHS here, the quantum field \( \hat{\Phi}(x) \) obeys the Klein–Gordon equation \( (\partial_0^2 - \nabla^2 + m^2) \hat{\Phi}(x) = 0 \), so the first term vanishes. The remaining second term is just the delta function, thus
\[ (\partial_0^2 - \nabla^2 + m^2) G_F(x-y) = -i\delta^{(4)}(x-y), \] (18)

which proves that \( G_F(x-y) \) is indeed a Green’s function of the Klein–Gordon equation. \textit{Quod erat demonstrandum.}
General Green’s functions and the Feynman’s choice

In general, the same differential equation may have many different Green’s functions, depending on the boundary conditions, etc. So let’s consider a generic Green’s function of the Klein–Gordon equation, that is, some function $G(x - y)$ satisfying

$$(\partial^2 + m^2)G(x - y) = -i\delta^{(4)}(x - y).$$

Let’s Fourier transform this function in all four dimensions, thus

$$G(x - y) = \int \frac{d^4 k}{(2\pi)^4} e^{-ik(x-y)} \times \tilde{G}(k).$$

In the 4–momentum space, eq. (19) becomes

$$(-k^2 + m^2) \times \tilde{G}(k) = -i,$$

hence naively

$$G(k) = \frac{i}{k^2 - m^2}$$

and therefore

$$G(x - y) = \int \frac{d^4 k}{(2\pi)^4} \frac{ie^{-ik(x-y)}}{k^2 - m^2}.$$  

The problem with this naive formula is that it integrates over the singularities of the integrand. Indeed, the denominator $k^2 - m^2 = k_0^2 - k^2 - m^2$ vanishes on the mass shells $k^0 = \pm\sqrt{k^2 + m^2}$, so we have two 3D families of poles. In general, an integral of a singular function over its pole is ill-defined, and we must regularize it to get a definite answer. For the Green’s function in question, we must regulate two 3D-families of poles, thus

$$G(x) = \int_{\text{reg}} \frac{d^4 k}{(2\pi)^4} \frac{ie^{-ik(x-y)}}{k^2 - m^2} = \int \frac{d^3 k}{(2\pi)^3} e^{i\mathbf{k} \cdot \mathbf{x}} \times \int_{\text{reg}} \frac{dk_0}{2\pi} \frac{ie^{-ik_0}}{k_0^2 - k^2 - m^2}.$$  

In other words, we integrate over the $k^0$ before we integrate over the $k$. In the $\int dk^0$ integral, we encounter two simple poles at $k^0 = \pm\omega_k$, and we must somehow regularize them to get a definite result. Only then we integrate that result over $k$; hopefully, that integral does not encounter any singularities.
Alas, the devil is in the details: There are many different ways to regularize an integral, and different regulators yield different regularized integrals — which eventually yield many different Green’s functions (24) of the same Klein–Gordon equation.

In these notes, we are going to use a particularly simple way to regulate an integral over a simple pole — move the pole away from the real axis into the complex plane,

$$\int_{\text{reg}} dx \frac{f(x)}{x-x_0} = \int dx \frac{f(x)}{x-(x_0 \pm i\epsilon)}$$  \hspace{1cm} (25)

for an infinitesimal $\epsilon \rightarrow +0$. Equivalently, we may leave the pole real but deform the integration contour slightly away from the real axis so that it bypasses the pole,

$$\bullet \ x_0 - i\epsilon$$

or

$$\bullet \ x_0 + i\epsilon$$

Note that the contour above the pole and the contour below the pole — or equivalently, shifting the pole below or above the real axis — makes for a different regulator which produces a different regularized integral:

$$\int dx \frac{f(x)}{x-(x_0 + i\epsilon)} - \int dx \frac{f(x)}{x-(x_0 - i\epsilon)} = 2\pi i \times f(x_0).$$  \hspace{1cm} (26)

In the context of the integral (24), there are two poles in the $\int dk^0$ for every $k$, so we must make our choices. For the sake of Lorentz invariance, we should use the same regulator for every $k$, which leaves us with $2 \times 2 = 4$ choices:

- Move the pole at $k^0 = +\omega_k$ to $+\omega_k + i\epsilon$ or to $+\omega_k - i\epsilon$.
- Move the pole at $k^0 = -\omega_k$ to $-\omega_k + i\epsilon$ or to $-\omega_k - i\epsilon$. 

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The 4 choices give rise to 4 distinct Lorentz-invariant Green’s functions, namely:

1. *Causal retarded Green’s function* $G_R$ for poles at $k_0 \pm \omega_k - i\epsilon$,

2. *Causal advanced Green’s function* $G_A$ for poles at $k_0 \pm \omega_k + i\epsilon$,

3. *Time-ordered Green’s function* $G_F$ for poles at $k_0 \pm (\omega_k - i\epsilon)$,

4. *Anti-time-ordered Green’s function* $G_{AT}$ for poles at $k_0 = \pm(\omega_k + i\epsilon)$,

This Green’s function is the Feynman’s propagator (1).

**Feynman’s Choice**

Let’s focus on the Feynman’s choice of the poles at $+\omega_k - i\epsilon$ and $-\omega_k + i\epsilon$. Altogether, the denominator of the integrand in eq. (24) is

$$(k_0 - \omega_k + i\epsilon) \times (k_0 + \omega_k - i\epsilon) = k_0^2 - (\omega_k - i\epsilon)^2 \approx k_0^2 - \omega_k^2 + 2i\omega_k\epsilon = k_0^2 - k^2 - m^2 + i\epsilon \times 2\omega_k.$$  

(27)

In the last expression, we may replace $\epsilon \times 2\omega_k$ with simply $\epsilon$, since all we care about is $k$. 

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it’s a positive infinitesimal number → +0. Thus

\[ \text{the denominator} = k_0^2 - k^2 - m^2 + i\epsilon = k^2 - m^2 + i\epsilon, \]  

(28)

hence a manifestly Lorentz invariant expression for the Feynman’s Green’s function as

\[ G_F(x - y) = \int \frac{d^4k}{(2\pi)^4} \frac{ie^{-ikx}}{k^2 - m^2 + i\epsilon}. \]  

(29)

In this section of the notes, we shall see that this Green’s function is precisely the Feynman propagator (1). Without loss of generality, let’s set \( y = 0 \). In light of eq. (2), we expect two different cases according to the sign of the \( t = x^0 \). Let’s start with the \( t > 0 \) case and deal with \( t < 0 \) later.

We begin to evaluate the 4D integral (29) by integrating over the \( k_0 \) for a fixed \( \mathbf{k} \),

\[ I(t, \omega_k) = \int \frac{dk_0}{2\pi} \frac{ie^{-itk_0}}{k_0^2 - \omega_k^2 + i\epsilon}, \]  

(30)

then \( G_F(x, t) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{x} \cdot \mathbf{k}} \times I(t, \omega_k) \)  

(31)

In the integral (30), the integration contour is the real axis, while the two poles lie near the axis — but not quite on it — as on the following diagram

\[ \begin{array}{c}
\hspace{1cm} \\
\hspace{1cm} \end{array} \]  

(32)

Outside the real axis, the exponential \( e^{-itk_0} \) — with positive \( t \) — rapidly decreases for large negative \( \text{Im}(k_0) \). Consequently, we may close the integration contour by adding to it a large semicircular arc in the negative \( \text{Im}(k_0) \) half of the complex plane. Thus,

\[ I(t, \omega_k) = \oint \frac{dk_0}{2\pi} \frac{ie^{-itk_0}}{(k_0 - \omega_k + i\epsilon)(k_0 + \omega_k - i\epsilon)} \]  

(33)
The closed-contour integrals like (33) may be evaluated in terms of residues at the poles 

surrounded by the contour. For the contour (34) at hand, the pole at \( +\omega_k - i\epsilon \) lies inside the contour while the other pole lies outside the contour. Consequently,

\[
I(t, \omega_k) = -2\pi i \times \text{Residue at } k_0 = +\omega_k - i\epsilon,
\]

where the overall \(-2\pi i\) factor is due to clockwise direction of the contour. Specifically,

\[
\begin{align*}
I(t, \omega_k) &= -2\pi i \times \left( \frac{ie^{-itk_0}}{2\pi \times (k_0 - \omega_k + i\epsilon) \times (k_0 + \omega_k - i\epsilon)} \right)_{k_0 = +\omega_k - i\epsilon} \\
&= +\frac{\exp(-it(\omega_k - i\epsilon))}{2(\omega_k - i\epsilon)} \\
&= +\frac{e^{-it\omega_k}}{2\omega_k}.
\end{align*}
\]

Plugging this result into eq. (31), we have

\[
G_F(x) = \int \frac{d^3k}{(2\pi)^3} e^{ix\cdot k} \times e^{-it\omega_k} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \exp(i x \cdot k - it\omega_k) = D(x),
\]

in perfect agreement with the Feynman propagator (1) for \( t > 0 \), cf. eq. (2).
Now let’s turn to the $t < 0$ case. Again, we need to take the integral

$$I(t, \omega_k) = \int \frac{dk_0}{2\pi} \frac{ie^{-itk_0}}{k_0^2 - \omega_k^2 + i\epsilon}$$

(30)

along the real axis, bypassing the poles according to

$$\cdots \cdots \cdots$$

(32)

However, for a negative $t$, the exponential $e^{-itk_0}$ decreases for large positive $\text{Im}(k_0)$ (rather than large negative $\text{Im}(k_0)$ as we had for positive $t$), so to close the integration contour (32) we should add a large semicircular arc in the positive half of the complex plane. Thus,

$$I(t, \omega_k) = \oint_{\Gamma'} \frac{dk_0}{2\pi} \frac{ie^{-itk_0}}{(k_0 - \omega_k + i\epsilon)(k_0 + \omega_k - i\epsilon)}$$

(38)

where

$$\Gamma' = \cdots \cdots \cdots$$

(39)

Unlike the contour (34) which we have used for positive $t$, the contour (39) surrounds the
negative-frequency pole at \( k_0 = -\omega_k + i\epsilon \). It is also counterclockwise, hence

\[
I(t, \omega_k) = +2\pi i \times \text{Residue at } k_0 = -\omega_k + i\epsilon \\
= +2\pi i \times \left( \frac{ie^{-itk_0}}{2\pi \times (k_0 - \omega_k + i\epsilon) \times (k_0 + \omega_k - i\epsilon)} \right)_{k_0 = -\omega_k + i\epsilon} \\
= -\frac{\exp(-it(-\omega_k + i\epsilon))}{2(-\omega_k + i\epsilon)} \\
\langle \text{taking the } \epsilon \to +0 \text{ limit, which is non-singular} \rangle \\
= +\frac{e^{it\omega_k}}{2\omega_k}.
\] (40)

Plugging this \( k_0 \) integral into the \( \int d^3k \) integral (31), we obtain

\[
G_F(x, t) = \int \frac{d^3k}{(2\pi)^3} e^{+ix \cdot k} \times \frac{e^{it\omega_k}}{2\omega_k} = D(+x, -t).
\]

At first blush, this is not quite the answer we want, but fortunately \( D \) is invariant under orthochronous Lorentz transformation, and in particular under any rotations of the 3D space. Consequently

\[
D(+x, -t) = D(-x, -t),
\] (41)

and therefore

\[
\text{for } t < 0, \quad G_F(x) = D(-x),
\] (42)

in perfect agreement with eq. (2).

Altogether, eqs. (37) and (42) tell us that the Feynman’s Green’s function

\[
G_F(x - y) = \int \frac{d^4k}{(2\pi)^4} \frac{ie^{-ikx}}{k^2 - m^2 + i\epsilon} = \begin{cases} 
D(x - y) & \text{when } x^0 > y^0 \\
D(y - x) & \text{when } x^0 < y^0 
\end{cases} = \langle 0 | \hat{\Phi}(x)\hat{\Phi}(y) | 0 \rangle
\] (43)

is precisely the time-ordered correlation function of two free scalar fields.


Other Green’s functions

Besides the Feynman’s time-ordered Green’s function, there are other useful Green’s functions (of the same Klein-Gordon equation) which obtain for other choices of regularizing the poles. Of particular interest is the causal retarded Green’s function

\[
G_R(x - y) = \int \frac{d^3\mathbf{x}}{(2\pi)^3} e^{i(x-y)\mathbf{k}} \times \int \frac{dk_0}{2\pi} \frac{i e^{-i(x^0-y^0)k_0}}{(k_0 - \omega_k + i\epsilon)(k_0 + \omega_k + i\epsilon)},
\]

which obtains by shifting both poles below the real axis,

As before, we close this contour by adding a large semicircular arc in the lower or upper half of the complex plane, depending on the sign of the time difference \( t = x^0 - y^0 \). In particular, for \( t < 0 \) we close the contour above the real axis,

which puts both poles outside the contour. Consequently, the contour integral vanishes altogether, thus

\[
G_R(x - y) = 0 \quad \text{when} \quad x^0 - y^0 < 0.
\]

This is why this Green’s function is called \textit{retarded}: time-wise, the point \( x \) must follow the
point $y$, hence in the context of a source $j(y)$ and the induced field

$$\phi(x) = \int d^4y G_R(x - y) \times j(y), \quad (48)$$

the source at point $y$ affects the field $\phi(x)$ only at later times $x^0 > y^0$ than the source.

Now let’s see what $G_R(x - y)$ looks like for $t = x^0 - y^0 > 0$. This time, we close the contour (45) below the real axis,

$$\Gamma = \text{contour below the real axis}, \quad (49)$$

so both poles are inside the contour. Consequently,

$$I_R(t, \omega) = \int dk_0 \frac{i e^{-ikt_0}}{2\pi} \left( \frac{i e^{-ikt_0}}{(k_0 - \omega + i\epsilon)(k_0 + \omega + i\epsilon)} \right)_{k_0 = \pm \omega - i\epsilon}$$

$$= -2\pi i \times \text{Residue @} (k_0 = \pm \omega + i\epsilon) - 2\pi i \times \text{Residue @} (k_0 = \mp \omega + i\epsilon)$$

$$= -\frac{2\pi i}{2\pi} \times \left( \frac{i e^{-ikt_0}}{(k_0 - \omega + i\epsilon)(k_0 + \omega + i\epsilon)} \right)_{k_0 = \omega - i\epsilon}$$

$$+ \frac{-2\pi i}{2\pi} \times \left( \frac{i e^{-ikt_0}}{(k_0 - \omega + i\epsilon)(k_0 + \omega + i\epsilon)} \right)_{k_0 = -\omega - i\epsilon}$$

$$= +\frac{e^{-it(\omega-i\epsilon)}}{2(\omega-i\epsilon)} + \frac{e^{-it(-\omega-i\epsilon)}}{2(-\omega-i\epsilon)} + \frac{e^{-it\omega}}{2\omega} - \frac{e^{+it\omega}}{2\omega}. \quad (50)$$
Plugging this result into the $\int d^3k$ integral, we obtain

$$\text{For } x^0 > y^0, \quad G_R(x-y) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} \times \frac{e^{-it\omega_k} - e^{+it\omega_k}}{2\omega_k}$$

$$= D(x - y; t) - D(x - y; -t) \quad (51)$$

$$= D(x - y; t) - D(y - x; -t)$$

$$= D(x - y) - D(y - x).$$

Note that the bottom line here vanishes for spacelike $(x-y)$, which makes the Green’s function $G_R$ not only retarded but also causal: it vanishes unless $x$ lies in the future light cone from $y$.

Similar to the causal retarded Green’s function $G_R(x-y)$ we can make the causal advanced Green’s function $G_A(x-y)$ by shifting both poles above the real axis,

\[
\begin{align*}
  G_A(x - y) &= \int \frac{d^3x}{(2\pi)^3} e^{i(x-y)\mathbf{k}} \times \int \frac{dk_0}{2\pi} \frac{ie^{-i(x^0 - y^0)k_0}}{k_0 - \omega_k - i\epsilon(k_0 + \omega_k - i\epsilon)} \quad (53) \\
  &= \begin{cases} 
    0 & \text{when } x^0 > y^0, \\
    D(y-x) - D(x-y) & \text{when } x^0 < y^0. 
  \end{cases} \quad (54)
\end{align*}
\]

This Green’s function vanishes for $x$ being later than $y$ or at spacelike separation from $y$; in other words, it vanishes unless unless $y$ is in the past light cone from $x$. That’s why it’s called the causal advanced Green’s function.

Finally, the fourth choice of regularized poles

\[
\begin{align*}
  G_{AT}(x - y) &= \int \frac{d^4k}{(2\pi)^4} \frac{ie^{-ik(x-y)}}{k^2 - m^2 - i\epsilon} \quad (56) \\
  &= \begin{cases} 
    -D(y-x) & \text{when } x^0 > y^0, \\
    -D(x-y) & \text{when } y^0 > x^0. 
  \end{cases}
\end{align*}
\]

This Green’s function vanishes for $x$ being later than $y$ or at spacelike separation from $y$; in other words, it vanishes unless unless $y$ is in the past light cone from $x$. That’s why it’s called the causal advanced Green’s function.
Propagators for non-scalar fields

Let me conclude these notes with a few words about propagators for the non-scalar relativistic fields — the vector fields, the tensor fields, the spinor fields, etc., etc. For all such fields, the Feynman propagator is the time-ordered correlation function of two free fields in the vacuum state, for example

\[ G_F^{\mu\nu}(x - y) = \langle 0 | T^* \hat{A}_\mu(x) \times \hat{A}_\nu(y) | 0 \rangle \]  

for the massive vector fields (see homework set #5 for details), or

\[ S_F^{\alpha\beta}(x - y) = \langle 0 | T\hat{\Psi}_\alpha(x) \times \hat{\Psi}_\beta(y) | 0 \rangle \]  

for the Dirac spinor field \( \hat{\Psi}_\beta(x) \) and its conjugate \( \hat{\Psi}_\alpha(x) \) (to be explained in future classes).

All such propagators are Green’s functions of the equations of motion for the appropriate fields. For example, the free massive vector fields obey

\[ \left( g_{\mu\nu}(\partial^2 + m^2) - \partial_\mu \partial_\nu \right) A^\nu = 0, \]  

so the propagator is a Green’s function of the differential operator here,

\[ \left( g_{\mu\nu}(\partial^2 + m^2) - \partial_\mu \partial_\nu \right) G_F^{\mu\lambda} = -i\delta^\lambda_\mu \times \delta^{(4)}(x - y). \]  

(The proof is part of homework set #5.) Likewise, the free Dirac spinor fields \( \Psi^\alpha(x) \) obey the Dirac equation

\[ (i\gamma^\mu \partial_\mu - m)_{\alpha\beta} \Psi^\beta(x) = 0, \]  

so the Dirac propagator is a Green’s function of the Dirac equation,

\[ (i\gamma^\mu \partial_\mu - m)_{\alpha\beta} S_F^{\alpha\beta}(x - y) = -i\delta^\delta_\alpha \times \delta^{(4)}(x - y). \]  

(I shall prove this in class in a few weeks.)
Moreover, all such Green’s functions involve momentum integrals over poles along both mass shells $k_0 = \pm \omega \mathbf{k}$, and those poles must be regularized. For the Feynman propagators, the poles are always regularized just as we did for the scalar field, the pole at $k_0 = +\omega \mathbf{k}$ shifts below the real axis to $+\omega \mathbf{k} - i\epsilon$ while the pole at $k_0 = -\omega \mathbf{k}$ shifts above the real axis to $-\omega \mathbf{k} + i\epsilon$. Consequently, all the Feynman propagators have momentum-space form of

$$(\text{propagator})^{indices}(x - y) = \int \frac{d^4k}{(2\pi)^4} \frac{ie^{-ik(x-y)}}{k^2 - m^2 + i\epsilon} \times F^{indices}(k) \quad (63)$$

for some simple — and hopefully non-singular — function $F^{indices}(k)$. For example, for the massive vector field

$$G^{\mu\nu}(x - y) = \int \frac{d^4k}{(2\pi)^4} \frac{ie^{-ik(x-y)}}{k^2 - m^2 + i\epsilon} \times (-g^{\mu\nu} - \frac{k^\mu k^\nu}{m^2}), \quad (64)$$

while for the Dirac spinor field

$$S^{\alpha\beta}_F(x - y) = \int \frac{d^4k}{(2\pi)^4} \frac{ie^{-ik(x-y)}}{k^2 - m^2 + i\epsilon} \times (k^\mu \gamma^\mu + m)^{\alpha\beta}. \quad (65)$$

In general, for any kind of a field other than a gauge field, the function $F^{indices}(k)$ is simply a polynomial of $k$ of degree $2 \times \text{Spin}$. For the gauge fields — and other massless gauge-like fields, such as the (linearized) gravitational fields or the gravitino fields — we have propagators of the form (63) with a polynomial $F(k)$ for the gauge-invariant tension fields. For example, for the EM tension fields,

$$\langle 0 | F^{\mu\nu}(x) F^{\rho\sigma}(y) | 0 \rangle = \int \frac{d^4k}{(2\pi)^4} \frac{ie^{-ik(x-y)}}{k^2 - m^2 + i\epsilon} \times F^{\mu\nu,\rho\sigma}(k) \quad (66)$$

where $F^{\mu\nu,\rho\sigma}(k) = k^\mu k^\rho g^{\nu\sigma} - k^\nu k^\rho g^{\mu\sigma} - k^\mu k^\sigma g^{\nu\rho} + k^\nu k^\sigma g^{\mu\rho}$, a quadratic polynomial.

But the propagators $\langle 0 | \mathbf{F}^{\mu}(x) \mathbf{F}^{\nu}(y) | 0 \rangle$ of the potential fields themselves are more complicated: although we may always write them in the form (63), but this time the function $F^{\mu,\nu}(k)$ is non-polynomial. Worse, it’s specific form depends on a particular gauge-fixing condition for the potentials $\hat{A}^{\mu}(x)$.

I shall explain the EM propagator in detail later in class, probably in November. Meanwhile, you may read my notes on QED (pages 1–8).