Solutions for exercises in my notes about finite multiplets of the $\mathrm{Spin}(3,1) \cong SL(2,\mathbf{C})$ group, the double cover of the continuous Lorentz group $SO^+(3,1)$.

Problem **1**:

The lorentz generators $\hat{\mathbf{J}}$ and $\hat{\mathbf{K}}$ obey commutation relations

$$\left[\hat{J}^i, \hat{J}^j\right] = i\epsilon^{ijk}\hat{J}^k, \quad \left[\hat{J}^i, \hat{K}^j\right] = i\epsilon^{ijk}\hat{K}^k, \quad \left[\hat{K}^i, \hat{K}^j\right] = -i\epsilon^{ijk}\hat{J}^k. \tag{S.1}$$

Consequently,

$$\begin{aligned}
\left[\hat{J}^i_\pm, \hat{J}^j_\pm\right] &= \tfrac{1}{4}\left[\hat{J}^i, \hat{J}^j\right] \pm \tfrac{i}{4}\left[\hat{K}^i, \hat{J}^j\right] \pm \tfrac{i}{4}\left[\hat{J}^i, \hat{K}^j\right] - \tfrac{1}{4}\left[\hat{K}^i, \hat{K}^j\right] \\
&= \tfrac{1}{4}\, i\epsilon^{ijk}\hat{J}^k \pm \tfrac{i}{4}\, i\epsilon^{ijk}\hat{K}^k \pm \tfrac{i}{4}\, i\epsilon^{ijk}\hat{K}^k - \tfrac{1}{4}(-i)\epsilon^{ijk}\hat{J}^k \\
&= \epsilon^{ijk}\left(\tfrac{i}{2}\hat{J}^k \mp \tfrac{1}{2}\hat{K}^k\right) = i\epsilon^{ijk}\hat{J}^k_\pm\,, \\
\left[\hat{J}^i_\pm, \hat{J}^j_\mp\right] &= \tfrac{1}{4}\left[\hat{J}^i, \hat{J}^j\right] \pm \tfrac{i}{4}\left[\hat{K}^i, \hat{J}^j\right] \mp \tfrac{i}{4}\left[\hat{J}^i, \hat{K}^j\right] + \tfrac{1}{4}\left[\hat{K}^i, \hat{K}^j\right] \\
&= \tfrac{1}{4}\, i\epsilon^{ijk}\hat{J}^k \pm \tfrac{i}{4}\, i\epsilon^{ijk}\hat{K}^k \mp \tfrac{i}{4}\, i\epsilon^{ijk}\hat{K}^k + \tfrac{1}{4}(-i)\epsilon^{ijk}\hat{J}^k \\
&= 0.
\end{aligned} \tag{S.2}$$

Problem **2**:

In homework#6 [solutions], problem 2(a) we saw that in the Weyl basis for Dirac sponors, the Lorentz generators are represented by

$$\mathbf{J}_D = \begin{pmatrix} \frac{1}{2}\boldsymbol{\sigma} & 0 \\ 0 & \frac{1}{2}\boldsymbol{\sigma} \end{pmatrix}, \quad \mathbf{K}_D = \begin{pmatrix} -\frac{i}{2}\boldsymbol{\sigma} & 0 \\ 0 & +\frac{i}{2}\boldsymbol{\sigma} \end{pmatrix}. \tag{S.3}$$

Decomposing these Dirac-spinor representations of the Lorentz generators into the irreducible Weyl-spinor representations, we find that

$$\begin{aligned}
&\text{For the LH Weyl spinor,} \quad \mathbf{J} = \tfrac{1}{2}\boldsymbol{\sigma}, \quad \mathbf{K} = -\tfrac{i}{2}\boldsymbol{\sigma}, \\
&\qquad \text{exactly as for the } (j_+ = \tfrac{1}{2}, j_- = 0) \text{ multiplet,}
\end{aligned} \tag{S.4}$$

$$\begin{aligned}
&\text{For the RH Weyl spinor,} \quad \mathbf{J} = \tfrac{1}{2}\boldsymbol{\sigma}, \quad \mathbf{K} = +\tfrac{i}{2}\boldsymbol{\sigma}, \\
&\qquad \text{exactly as for the } (j_+ = 0, j_- = \tfrac{1}{2}) \text{ multiplet.}
\end{aligned} \tag{S.5}$$

Quod erat demonstrandum.

Problem **3**:

A finite continuous Lorentz is a combination of a Boost and a rotation of space, hence in a Lorentz multiplet m it is represented by the matrix

$$M_m \;=\; \exp\bigl(-i\mathbf{a}\cdot\mathbf{J}_m \;-\; -i\mathbf{b}\cdot\mathbf{K}_m\bigr) \tag{S.6}$$

where $\mathbf{J}_m$ and $\mathbf{K}_m$ represent the Lorentz generators $\hat{\mathbf{J}}$ and $\hat{\mathbf{K}}$ in the multiplet m, while the 3-vectors $\mathbf{a}$ and $\mathbf{b}$ parametrize the rotation and the boost. In particular, for the LH and the RH Weyl spinor multiplets,

$$-i\mathbf{a}\cdot\mathbf{J} \;-\; i\mathbf{b}\cdot\mathbf{K} \;=\; \tfrac{1}{2}(-i\mathbf{a}\mp\mathbf{b})\cdot\boldsymbol{\sigma}, \tag{S.7}$$

hence

$$M_L \;=\; \exp\bigl(\tfrac{1}{2}(-i\mathbf{a}-\mathbf{b})\cdot\boldsymbol{\sigma}\bigr), \quad M_R \;=\; \exp\bigl(\tfrac{1}{2}(-i\mathbf{a}+\mathbf{b})\cdot\boldsymbol{\sigma}\bigr). \tag{S.8}$$

Since the Paili matrices are traceless, both of these matrix exponentials have unit determinants. Indeed, for any complex 3–vector $\mathbf{c}$, $\mathrm{tr}(\mathbf{c}\cdot\boldsymbol{\sigma})=0$, hence

$$\det\bigl(\exp(\mathbf{c}\cdot\boldsymbol{\sigma})\bigr) \;=\; \exp\bigl(\mathrm{tr}(\mathbf{c}\cdot\boldsymbol{\sigma})\bigr) \;=\; \exp(0) \;=\; 1. \tag{S.9}$$

Problem **4**:

Let's start with reality. The matrix $V=V^\mu\sigma_\mu$ is hermitian if and only if the 4-vector V^μ is real. For any matrix $M\in SL(2,\mathbf{C})$, the transform

$$V\to\; V' \;=\; MVM^\dagger \tag{S.10}$$

preserves hermiticity: if V is hermitian, then so is V'; indeed

$$\bigl(V'\bigr)^\dagger \;=\; \bigl(MVM^\dagger\bigr)^\dagger \;=\; \bigl(M^\dagger\bigr)^\dagger V^\dagger M^\dagger \;=\; MVM^\dagger \;=\; V'. \tag{S.11}$$

In terms of the 4-vectors, this means that if the V^μ is real than the $V'^\mu=L^\mu_{\ \nu}V^\nu$ is also real. In other words, the 4×4 matrix $L^\mu_{\ \nu}(M)$ is real.

Next, let's prove that $L^{\mu}_{\ \nu}(M) \in O(3,1)$ — it preserves the Lorentz metric $g^{\alpha\beta}$, or equivalently, for any V^{μ}, $g^{\alpha\beta}V'_{\alpha}V'_{\beta} = g^{\alpha\beta}V_{\alpha}V_{\beta}$. In terms of the 2×2 matrix $V = V^{\mu}\sigma_{\mu}$, the Lorentz square of the 4-vector becomes the determinant:

$$g^{\alpha\beta}V_{\alpha}V_{\beta} \;=\; \det\bigl(V = V_{\mu}\sigma^{\mu}\bigr). \tag{S.12}$$

Indeed, from the explicit form of the 4 matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \tag{S.13}$$

we have

$$V \;=\; V^{\mu}\sigma_{\mu} \;=\; \begin{pmatrix} V_0 + V_3 & V_1 - iV_2 \\ V_1 + iV_2 & V_0 - V_3 \end{pmatrix}$$

and hence

$$\det(V) \;=\; (V_0+V_3)(V_0-V_3) - (V_1-iV_2)(V_1+iV_2) \;=\; V_0^2 - V_3^2 - V_1^2 - V_2^2 \;=\; g^{\alpha\beta}V_{\alpha}V_{\beta}\,. \tag{S.14}$$

The determinant of a matrix product is the product of the individual matrices' determinants. Hence, for the transform (S.10),

$$\det(V') \;=\; \det(M)\times\det(V)\times\det(M^{\dagger}) \;=\; \det(V)\times\left|\det(M)\right|^2. \tag{S.15}$$

The M matrices of interest to us belong to the $SL(2,\mathbf{C})$ group — they are complex matrices with units determinants. There are no other restrictions, but $\det(M) = 1$ is enough to assure $\det(V') = \det(V)$, *cf.* eq. (S.15). Thanks to the relation (S.12), this means

$$g^{\alpha\beta}V'_{\alpha}V'_{\beta} \;=\; \det(V') \;=\; \det(V) \;=\; g^{\alpha\beta}V_{\alpha}V_{\beta} \tag{S.16}$$

— which proves that the matrix $L^{\mu}_{\ \nu}(M)$ is indeed Lorentzian.

To prove that the Lorentz transform $L^{\mu}{}_{\nu}(M)$ is orthochronous, we need to show that for any V_μ in the forward light cone — $V^2 > 0$ and $V_0 > 0$ — the V'_μ is also in the forward light cone. In matrix terms, $V^2 > 0$ and $V_0 > 0$ mean $\det(V) > 0$ and $\mathrm{tr}(V) > 0$; together, these two conditions means that the 2×2 hermitian matrix V is positive-definite. The transform (S.10) preserves positive definiteness: if for any complex 2-vector $\xi \neq 0$ we have $\xi^\dagger V \xi > 0$, then

$$\xi^\dagger V' \xi \;=\; \xi^\dagger M V M^\dagger \xi \;=\; (M^\dagger \xi)^\dagger V (M^\dagger \xi) \;>\; 0. \tag{S.17}$$

(Note that $M^\dagger \xi \neq 0$ for any $\xi \neq 0$ because $\det(M) \neq 0$.) Thus, for any $M \in SL(2,\mathbf{C})$ the Lorentz transform $V^\mu \to V'^\mu$ preserves the forward light cone — in other words, the $L^{\mu}{}_{\nu}(M)$ is orthochronous, $L^{\mu}{}_{\nu}(M) \in O^+(3,1)$.

Problem $\mathbf{4}^\star$:

The simplest proof that the Lorentz transform (8) is proper — $\det(L) = +1$ — for any $SL(2,\mathbf{C})$ matrix M is topological: The $SL(2,\mathbf{C})$ group manifold — which spans all matrices of the form

$$M \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{complex } a,b,c,d, \quad ad - bc \;=\; 1 \tag{S.18}$$

is connected, so all such matrices are continuously connected to the $\mathbf{1}_{2\times 2}$ matrix. It is easy to see that for $M = 1$, the Lorentz transform $L(1)$ is trivial, $L^{\mu}{}_{\nu} = \delta^{\mu}_{\nu}$, hence all Lorentz transfroms of the form (8) are continuously connected to the trivial transform. Consequently, they all must be continuous Lorentz transfroms and therefore proper and orthochronous.

There are other proofs not involving the topology, but they rely on the problems 5 and 6 of these notes, as well as problem 2(b) from the homework#6. I shall present one sich proof after the solutions to problems 5 and 6.

Problem $\mathbf{5}$:

Let $L_1 = L(M_1)$, $L_2 = L(M_2)$ and $L_{12} = L(M_2 M_1)$ be Lorentz transforms constructed according to eq. (8) for some $SL(2,\mathbf{C})$ matrices M_1 and M_2 and their product $M_2 M_1$. We want to prove that $L_{12} = L_2 L_1$, so consider how these transforms act on some 4–vector V^μ.

On one hand,

$$\big(L_{12}V\big)^\nu\,\sigma_\nu \;=\; (M_2M_1)\times(V^\nu\sigma_\nu)\times(M_2M_1)^\dagger \;=\; M_2M_1\times(V^\nu\sigma^\nu)\times M_1^\dagger M_2^\dagger. \qquad \text{(S.19)}$$

On the other hand,

$$\begin{aligned}\big(L_2L_1V\big)^\nu\,\sigma_\nu \;&=\; M_2\times\big((L_1V)^\nu\sigma_\nu\big)\times M_2^\dagger \;=\; M_2\times\big(M_1\times(V^\nu\sigma_\nu)\times M_1^\dagger\big)\times M_2^\dagger\\ \text{also}\;&=\; M_2M_1\times(V^\nu\sigma^\nu)\times M_1^\dagger M_2^\dagger.\end{aligned}$$

Thus we see that

$$\big(L_{12}V\big)^\nu\,\sigma_\nu \;=\; \big(L_2L_1V\big)^\nu\,\sigma_\nu \qquad \text{(S.20)}$$

and therefore

$$\big(L_{12}V\big)^\nu \;=\; \big(L_2L_1V\big)^\nu. \qquad \text{(S.21)}$$

Moreover, this holds true for any 4–vector V^μ, hence the Lorentz transforms $L_{12} = L(M_2M_1)$ and $L_2L_1 = L(M_2)L(M_1)$ must be equal to each other, *quod erat demonstrandum.*

Problem **6**:

Lemma 1: any $SL(2,\mathbf{C})$ matrix M may be written as a product $M = HU$ of an hermitian matrix H and a unitary matrix U, both having determinant $= 1$.

Proof: The matrix $MM^\dagger$ is hermitian, positive definite, and has determinant $= 1$ (since $\det(M) = 1$), so let H be the positive square root $H = \sqrt{MM^\dagger} \implies H = H^\dagger$ and $\det(H) = 1$. Let $U = H^{-1}M$, then

$$UU^\dagger \;=\; H^{-1}M\times M^\dagger H^{-1} \;=\; H^{-1}\times\big(MM^\dagger \;=\; H^2\big)\times H^{-2} \;=\; 1, \qquad \text{(S.22)}$$

so U is unitary, and also $\det(U) = \det(M)/\det(H) = 1$.

Lemma 2: For a unitary matrix U, the Lorentz transform $L(U)$ is a rotation of space which does not affect the time. Also, the Weyl spinor representations of this rotations are simply

$$M_L\big(L(U)\big) \;=\; M_R\big(L(U)\big) \;=\; U. \tag{S.23}$$

Proof: And $SL(2,\mathbf{C})$ matrix which happens to be unitary is an $SU(2)$ matrix, and any such $SU(2)$ matrix can be written as

$$U \;=\; \exp\left(-\tfrac{i}{2}\,\theta\,\mathbf{n}\cdot\boldsymbol{\sigma}\right) \tag{S.24}$$

for some angle θ and a unit 3–vector $\mathbf{n}$. Consequently, by the Cayley–Klein formula

$$U\times\sigma_i\times U^\dagger \;=\; R_i^{\,j}(\theta,\mathbf{n})\sigma_j \tag{S.25}$$

where $R_i^{\,j}(\theta,\mathbf{n})$ is the 3×3 matrix of a space rotation through angle θ around axis $\mathbf{n}$, while

$$U\times\sigma_0\times U^\dagger \;=\; UU^\dagger \;=\; 1 \;=\; \sigma_0. \tag{S.26}$$

Therefore, the Lorentz transform $L(U)$ defined according to eq. (8) is the purely spatial rotation $R(\theta,\mathbf{n})$. Moreover, as we saw in homework#6 (problem 2(b)), the Weyl spinor representations of this rotations are precisely

$$M_L(R) \;=\; M_R(R) \;=\; \exp\left(-\tfrac{i}{2}\,\theta\,\mathbf{n}\cdot\boldsymbol{\sigma}\right) \;=\; U, \tag{S.27}$$

quod erat demonstrandum.

Lemma 3: For an Hermitian matrix H, the transform $L(H)$ is a pure Lorentz boost (without a rotation). Also, the Weyl spinor representations of this boost are

$$M_L\big(L(H)\big) \;=\; H, \quad M_R\big(L(H)\big) \;=\; H^{-1}. \tag{S.28}$$

Proof: Suppose an $SL(2,\mathbf{C})$ matrix H happens to be Hermitian. Then the eigenvalues of H are either both positive or both negative (since their product is $\det(H)=1$), so either H

or $-H$ is positive definite. Let's take the log of the positive-definite $\pm H$, then this log is a traceless hermitian 2×2 matrix. Indeed,

$$\operatorname{tr}\bigl(\log(\pm H)\bigr) \;=\; \log\bigl(\det(\pm H)\bigr) \;=\; \log(1) \;=\; 0. \tag{S.29}$$

Consequently, this traceless log is a real linear combination of the Pauli matrices,

$$\log(\pm H) \;=\; -\tfrac{1}{2}\,\mathbf{r}\cdot\boldsymbol{\sigma} \quad\Longrightarrow\quad H \;=\; \pm\exp(-\tfrac{1}{2}\,\mathbf{r}\cdot\boldsymbol{\sigma}) \tag{S.30}$$

for some real 3–vector $\mathbf{r}$. Let r be its magnitude while $\mathbf{n}$ is its direction, then

$$H \;=\; \pm\cosh(r/2) \;\mp\; \sinh(r/2)\,(\mathbf{n}\cdot\boldsymbol{\sigma}). \tag{S.31}$$

Consequently, after some algebra we obtain

$$H\times\sigma_0 H \;=\; \cosh(r)\,\sigma_0 \;+\; \sinh(r)\,(\mathbf{n}\cdot\boldsymbol{\sigma}), \tag{S.32}$$

$$H\times(\mathbf{n}\cdot\boldsymbol{\sigma})\times H \;=\; \cosh(r)\,(\mathbf{n}\cdot\boldsymbol{\sigma})\sigma_0 \;+\; \sinh(r)\,\sigma_0, \tag{S.33}$$

while for $\mathbf{v}\perp\mathbf{n}$,

$$H\times(\mathbf{v}\cdot\boldsymbol{\sigma})\times H \;=\; (\mathbf{v}\cdot\boldsymbol{\sigma}), \tag{S.34}$$

which means that the Lorentz transform $L(H)$ defined according to eq. (8) is a pure boost of rapidity r in the direction $\mathbf{n}$. Moreover, as we saw in homework#6 (problem 2(b)), the Weyl spinor representations of this boost are precisely

$$\begin{aligned} M_L(R) \;&=\; \exp\bigl(-\tfrac{1}{2}\,\theta\,\mathbf{n}\cdot\boldsymbol{\sigma}\bigr) \;=\; H,\\ M_M(R) \;&=\; \exp\bigl(+\tfrac{1}{2}\,\theta\,\mathbf{n}\cdot\boldsymbol{\sigma}\bigr) \;=\; H^{-1}, \end{aligned} \tag{S.35}$$

quod erat demonstrandum.

Summary: Now that we have proven all these lemmas, the proof for a most general $SL(2,\mathbf{C})$ matrix M follows via the group law $L(M_2M_1)=L(M_2)L(M_1)$, *cf.* problem 5.

Indeed, by Lemma 1, any $M\in SL(2,\mathbf{C})$ is a product $M=HU$ of an hermitian matrix H and a unitary matrix U. By the group law,

$$L(M) \;=\; L(H)\times L(U), \tag{S.36}$$

hence in any Lorentz multiplet m,

$$M_m\bigl(L(M)\bigr) \;=\; M_m\bigl(L(H)\bigr)\times M_m\bigl(L(U)\bigr). \tag{S.37}$$

Moreover, by Lemmas 2 and 3, $L(U)$ is a rotation while $L(H)$ is a boost, and their LH Weyl spinor representations are simply

$$M_L\bigl(L(U)\bigr) \;=\; U, \quad M_L\bigl(L(H)\bigr) \;=\; H. \tag{S.38}$$

Consequently

$$M_L\bigl(L(M)\bigr) \;=\; H\times U \;=\; M. \tag{S.39}$$

Likewise, the RH spinor representations of the $L(U)$, $L(H)$ and $L(M)$ matrices are

$$M_R\bigl(L(U)\bigr) \;=\; U, \quad M_R\bigl(L(H)\bigr) \;=\; H^{-1}, \tag{S.40}$$

and hence

$$M_R\bigl(L(M)\bigr) \;=\; H^{-1}U. \tag{S.41}$$

Finally, in problem 2(d) of the homework#6 we saw that

$$\sigma_2U^*\sigma_2 \;=\; \bigl(U^{-1}\bigr)^\dagger \;=\; U, \quad \sigma_2H^*\sigma_2 \;=\; \bigl(H^{-1}\bigr)^\dagger \;=\; H^{-1}, \tag{S.42}$$

and hence

$$\overline{M} \;\stackrel{\rm def}{=}\; \sigma_2M^*\sigma_2 \;=\; \sigma_2H^*\sigma_2\times\sigma_2U^*\sigma_2 \;=\; H^{-1}\times U, \tag{S.43}$$

therefore

$$M_R\bigl(L(M)\bigr) \;=\; H^{-1}U \;=\; \overline{M}. \tag{S.44}$$

Quod erat demonstrandum.

Problem $\mathbf{4}^*$, a non-topological solution:

Let's use the Lemmas 1–3 from the above solutions to problem 6. By Lemma 1, any $SL(2,\mathbf{C})$ matrix M decomposes into $M = H \times U$ where U is unitary and H is hermitian, and by Lemmas 2 and 3, $L(H)$ is a pure boost while $L(U)$ is a pure rotation of space. Both boost and rotations are proper Lorentz transforms, hence

$$L(M) \;=\; L(H) \times L(U) \tag{S.45}$$

must also be a proper Lorentz transform.

Problem $\mathbf{7}$:

For any Lie algebra equivalent to an angular momentum or its analytic continuation, the product of two doublets comprises a triplet and a singlet, $\mathbf{2}\otimes\mathbf{2}=\mathbf{3}\oplus\mathbf{1}$, or in (j) notations, $(\frac{1}{2})\otimes(\frac{1}{2}) = (1)\oplus(0)$. Furthermore, the triplet $\mathbf{3}=(1)$ is symmetric with respect to permutations of the two doublets while the singlet $\mathbf{1}=(0)$ is antisymmetric.

For two separate and independent types of angular momenta $\mathbf{J}_+$ and $\mathbf{J}_-$ we combine the j_+ quantum numbers independently from the j_- and the j_- quantum numbers independently from the j_+. For two bi-spinors, this gives us

$$(\tfrac{1}{2},\tfrac{1}{2})\otimes(\tfrac{1}{2},\tfrac{1}{2}) \;=\; (1,1)\oplus(1,0)\oplus(0,1)\oplus(0,0). \tag{S.46}$$

Furthermore, the symmetric part of this product should be either symmetric with respect to both the j_+ and the j_- indices or antisymmetric with respect to both indices, thus

$$\left[(\tfrac{1}{2},\tfrac{1}{2})\otimes(\tfrac{1}{2},\tfrac{1}{2})\right]_{\rm sym} \;=\; (1,1)\oplus(0,0). \tag{S.47}$$

Likewise, the antisymmetric part is either symmetric with respect to the j_+ but antisymmetric with respect to the j_- or the other way around, thus

$$\left[(\tfrac{1}{2},\tfrac{1}{2})\otimes(\tfrac{1}{2},\tfrac{1}{2})\right]_{\rm antisym} \;=\; (1,0)\oplus(0,1). \tag{S.48}$$

From the $SO^+(3,1)$ point of view, the bi-spinor $(\frac{1}{2},\frac{1}{2})$ is the Lorentz vector. A general 2-index Lorentz tensor transforms like a product of two such vectors, so from the $SL(2,\mathbf{C})$

point of view it's a product of two bi-spinors, which decomposes to irreducible multiplets according to eq. (S.46).

The Lorentz symmetry respects splitting of a general 2-index tensor into a symmetric tensor $T^{\mu\nu} = +T^{\nu\mu}$ and an asymmetric tensor $F^{\mu\nu} = -F^{\nu\mu}$. The symmetric tensor corresponds to a symmetrized square of a bi-spinor, which decomposes into irreducible multiplets according to eq. (S.47). The singlet $(0,0)$ component is the Lorentz-invariant trace T^{μ}_{μ} while the $(1,1)$ irreducible multiplet is the traceless part of the symmetric tensor.

Likewise, the antisymmetric Lorentz tensor $F^{\mu\nu} = -F^{\nu\mu}$ decomposes according to eq. (S.48). Here, the irreducible components $(1,0)$ and $(0,1)$ are complex but conjugate to each other; individually, they describe antisymmetric tensors subject to complex duality conditions $\frac{1}{2}\epsilon^{\kappa\lambda\mu\nu}F_{\mu\nu} = \pm iF^{\kappa\lambda}$, or in 3D terms, $\mathbf{E} = \pm i\mathbf{B}$.