Problem $\mathbf{1}$(a):

In the first diagram (1), the virtual photon has momentum $q = p_1' - p_1 = p_2 - p_2'$, hence $q^2 = t$. In the second diagram, the virtual photon's momentum is $\tilde{q} = p_1 + p_2 = p_1' + p_2'$, hence $\tilde{q}^2 = s$. Accordingly, the two diagrams are called the s-channel diagram and the t-channel diagram.

The t-channel diagram evaluates to

$$\begin{aligned} i\mathcal{M}_1 &= -\Big(\bar{v}(e^+)(ie\gamma_\mu)v(e^{+\prime})\Big) \times \Big(\bar{u}(e^{-\prime})(ie\gamma_\nu)u(e^-)\Big) \times \frac{-ig^{\mu\nu}}{q^2} \\ &= \frac{-ie^2}{t} \times \bar{v}(e^+)\gamma_\mu v(e^{+\prime}) \times \bar{u}(e^{-\prime})\gamma^\mu u(e^-) \end{aligned} \tag{S.1}$$

where the overall minus sign is due to the positron-out to positron-in fermionic line. And the s-channel diagram evaluates to

$$\begin{aligned} i\mathcal{M}_2 &= +\Big(\bar{v}(e^+)(ie\gamma_\mu)u(e^-)\Big) \times \Big(\bar{u}(e^{-\prime})(ie\gamma_\nu)v(e^{+\prime}\Big) \times \frac{-ig^{\mu\nu}}{\tilde{q}^2} \\ &= \frac{+ie^2}{s} \times \bar{v}(e^+)\gamma_\mu u(e^-) \times \bar{u}(e^{-\prime})\gamma^\mu v(e^{+\prime}) \end{aligned} \tag{S.2}$$

where the overall sign is plus because both fermionic lines have an incoming or outgoing electron at one end.

Problem $\mathbf{1}$(b):

Summing /averaging the $|\mathcal{M}_2|^2$ over spins works exactly as for the muon pair production discussed in class:

$$\begin{aligned} \sum_{\text{spins}} |\mathcal{M}_2|^2 &= \left(\frac{e^2}{s}\right)^2 \sum_{\text{spins}} \Big[\bar{v}(e^+)\gamma_\mu u(e^-) \times \bar{u}(e^-)\gamma_\nu v(e^+)\Big] \times \Big[\bar{u}(e^{-\prime})\gamma^\mu v(e^{+\prime}) \times \bar{v}(e^{+\prime})\gamma^\nu u(e^{-\prime})\Big] \\ &= \left(\frac{e^2}{s}\right)^2 \operatorname{tr}\left[(\not{p}_2 - m)\gamma_\mu(\not{p}_1 + m)\gamma_\nu\right] \times \operatorname{tr}\left[(\not{p}_1' - m)\gamma^\mu(\not{p}_2' - m)\gamma^\nu\right] \\ &\qquad \langle\!\langle \text{ neglecting the mass relative to the momenta } \rangle\!\rangle \\ &\approx \left(\frac{e^2}{s}\right)^2 \operatorname{tr}\left[\not{p}_2\gamma_\mu \not{p}_1\gamma_\nu\right] \times \operatorname{tr}\left[\not{p}_1'\gamma^\mu \not{p}_2'\gamma^\nu\right] \end{aligned} \tag{S.3}$$

$$
\begin{aligned}
&= \left(\frac{e^2}{s}\right)^2 \times 4\left[p_{2\mu}p_{1\nu} + p_{2\nu}p_{1\mu} - g_{\mu\nu}(p_2p_1)\right] \times 4\left[p_2'^{\mu}p_1'^{\nu} + p_2'^{\nu}p_1'^{\mu} - g^{\mu\nu}(p_2'p_1')\right] \\
&= 16\left(\frac{e^2}{s}\right)^2 \Big[2(p_2'p_2)(p_1'p_1) + 2(p_2'p_1)(p_1'p_2). \\
&\qquad - 2(p_2'p_1')(p_2p_1) - 2(p_2'p_1')(p_2p_1) + 4(p_2'p_1')(p_2p_1)\Big] \\
&= 32\left(\frac{e^2}{s}\right)^2 \left[(p_2'p_2)(p_1'p_1) + (p_2'p_1)(p_1'p_2)\right] \\
&= 8\left(\frac{e^2}{s}\right)^2 \left[t^2 + u^2\right] \qquad \text{(S.3)}
\end{aligned}
$$

where the last equality follows from the kinematic relations (4). Altogether,

$$
\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_2|^2 = 2e^4 \times \frac{t^2+u^2}{s^2}\,. \qquad (5)
$$

Problem **1**(c):

The two diagrams for Bhabha scattering are related by the *crossing symmetry*, so the amplitudes $\mathcal{M}_1$ and $\mathcal{M}_2$ are related to each other via analytic continuation of particle's momenta. In terms of the spin-summed $|\mathcal{M}|^2$ and Mandelstam variables,

$$
\sum_{\text{spins}} |\mathcal{M}_1(s,t,u)|^2 = \sum_{\text{spins}} |\mathcal{M}_2(t,s,u)|^2, \qquad \text{(S.4)}
$$

hence eq. (5) for the second amplitude implies a similar equation for the first amplitude, but with s and t exchanged with each other — *i.e.*, eq. (6).

Alternatively, we may sum the $|\mathcal{M}_1|^2$ over all the spins in the same way as we summed the $|\mathcal{M}_2|^2$ in part (b):

$$
\begin{aligned}
\sum_{\text{spins}} |\mathcal{M}_1|^2 &= \left(\frac{e^2}{t}\right)^2 \sum_{\text{spins}} \left[\bar{u}(e^{-\prime})\gamma^\mu u(e^-) \times \bar{u}(e^-)\gamma^\nu u(e^{-\prime})\right] \times \left[\bar{v}(e^+)\gamma_\mu v(e^{+\prime}) \times \bar{v}(e^{+\prime})\gamma_\nu v(e^+)\right] \\
&= \left(\frac{e^2}{t}\right)^2 \operatorname{tr}\left[(\not{p}_1' + m)\gamma^\mu(\not{p}_1 + m)\gamma^\nu\right] \times \operatorname{tr}\left[(\not{p}_2 - m)\gamma_\mu(\not{p}_2' - m)\gamma_\nu\right] \\
&\approx \left(\frac{e^2}{t}\right)^2 \operatorname{tr}\left[\not{p}_1'\gamma^\mu \not{p}_1\gamma^\nu\right] \times \operatorname{tr}\left[\not{p}_2\gamma_\mu \not{p}_2'\gamma_\nu\right] \qquad \text{(S.5)}
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{e^2}{t}\right)^2 \times 4\left[p_1'^{\mu}p_1^{\nu} + p_1'^{\nu}p_1^{\mu} - g^{\mu\nu}(p_1'p_1)\right] \times 4\left[p'_{2\mu}p_{2\nu} + p'_{2\nu}p_{2\mu} - g_{\mu\nu}(p_2'p_2)\right] \\
&= 16\left(\frac{e^2}{t}\right)^2 \Big[2(p_1'p_2')(p_1p_2) + 2(p_1'p_2)(p_1p_2') \\
&\qquad\qquad - 2(p_1'p_1)(p_2'p_2) - 2(p_1'p_1)(p_2'p_2) + 4(p_1'p_1)(p_2'p_2)\Big] \\
&= 32\left(\frac{e^2}{t}\right)^2 \left[(p_1'p_2')(p_1p_2) + (p_1'p_2)(p_1p_2')\right] \\
&= 8\left(\frac{e^2}{t}\right)^2 \left[s^2 + u^2\right] \qquad \text{(S.5)}
\end{aligned}
$$

and hence

$$
\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_1|^2 = 2e^4 \times \frac{s^2+u^2}{t^2}\,. \qquad (6)
$$

Problem **1**(d):

The interference term between the two diagrams is more complicated:

$$
\begin{aligned}
\mathcal{M}_1^* \times \mathcal{M}_2 &= -\frac{e^2}{t}\Big(\bar{u}(e^-)\gamma^\nu u(e^{-\prime}) \times \bar{v}(e^{+\prime})\gamma_\nu v(e^+)\Big) \times \\
&\quad \times \frac{e^2}{s}\Big(\bar{v}(e^+)\gamma_\mu u(e^-) \times \bar{u}(e^{-\prime})\gamma^\mu v(e^{+\prime})\Big) \\
&= -\frac{e^4}{st} \times \bar{u}(e^-)\gamma^\nu u(e^{-\prime}) \times \bar{u}(e^{-\prime})\gamma^\mu v(e^{+\prime}) \times \bar{v}(e^{+\prime})\gamma_\nu v(e^+) \times \bar{v}(e^+)\gamma_\mu u(e^-)
\end{aligned}
\qquad \text{(S.6)}
$$

where on the last line I have re-ordered the factors so that each $\bar{u}$ is followed by u of the same electron and each $\bar{v}$ is followed by v for the same positron. After summing over all the spins, each $u \times \bar{u}$ becomes $(\not{p} + m)$, each $v \times \bar{v}$ becomes $(\not{p} - m)$, and the whole product becomes a single big trace rather than a product of two traces,

$$
\begin{aligned}
\sum_{\text{spins}} \mathcal{M}_1^* \times \mathcal{M}_2 &= -\frac{e^4}{st} \times \mathrm{tr}\Big[(\not{p}_1 + m)\gamma^\nu(\not{p}_1' + m)\gamma^\mu(\not{p}_2' - m)\gamma_\nu(\not{p}_2 - m)\gamma_\mu\Big] \\
&\approx -\frac{e^4}{st} \times \mathrm{tr}\Big[\not{p}_1\gamma^\nu\,\not{p}_1'\gamma^\mu\,\not{p}_2'\gamma_\nu\,\not{p}_2\gamma_\mu\Big].
\end{aligned}
\qquad \text{(S.7)}
$$

This trace looks more complicated than it is, and we may drastically simplify it by summing over ν and μ before taking the trace. Back in homework#6 we saw that

$$\gamma^\alpha \not{a} \not{b} \not{c} \gamma_\alpha \;=\; -2\,\not{c}\not{b}\not{a} \quad \text{and} \quad \gamma^\alpha \not{a} \not{b} \gamma_\alpha \;=\; 4(ab). \tag{S.8}$$

For the problem at hand, this gives us $\gamma^\nu \not{p}_1' \gamma^\mu \not{p}_2' \gamma_\nu = -2\,\not{p}_2' \gamma^\mu \not{p}_1'$ and hence

$$\begin{aligned}
\operatorname{tr}\Big[\not{p}_1 \times \gamma^\nu \not{p}_1' \gamma^\mu \not{p}_2' \gamma_\nu \times \not{p}_2 \gamma_\mu\Big] \;&=\; -2\operatorname{tr}\Big[\not{p}_1 \times \not{p}_2' \gamma^\mu \not{p}_1' \times \not{p}_2 \gamma_\mu\Big] \;=\; -2\operatorname{tr}\Big[\not{p}_1 \not{p}_2' \times \gamma^\mu \not{p}_1' \not{p}_2 \gamma_\mu\Big] \\
&=\; -2\operatorname{tr}\Big[\not{p}_1 \not{p}_2' \times 4(p_1' p_2)\Big] \;=\; -8(p_1' p_2) \times \operatorname{tr}\Big[\not{p}_1 \not{p}_2'\Big] \\
&=\; -8(p_1' p_2) \times 4(p_1 p_2') \\
&=\; -8u^2.
\end{aligned} \tag{S.9}$$

Plugging this trace back into eq. (S.6), we arrive at

$$\frac{1}{4}\sum_{\text{spins}} \mathcal{M}_1^* \times \mathcal{M}_2 \;=\; +2e^4 \times \frac{u^2}{st}\,. \tag{7}$$

<u>Problem **1**(e):</u>

Assembling the spin sums / averages (5–7) together according to eq. (3), we get

$$\begin{aligned}
\overline{|\mathcal{M}|^2} \;&\stackrel{\text{def}}{=}\; \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_1 + \mathcal{M}_2|^2 \\
&=\; \frac{1}{4}\sum_{\text{spins}} \Big(|\mathcal{M}_1|^2 \;+\; |\mathcal{M}_2|^2 \;+\; 2\operatorname{Re}\mathcal{M}_1^*\mathcal{M}_2\Big) \\
&=\; 2e^4 \times \frac{s^2+u^2}{t^2} \;+\; 2e^4 \times \frac{t^2+u^2}{s^2} \;+\; 4e^4 \times \frac{u^2}{st} \\
&=\; 2e^4 \left(\frac{s^2}{t^2} \;+\; \frac{t^2}{s^2} \;+\; \frac{u^2}{s^2t^2} \times \Big(s^2+t^2+2st \;=\; (s+t)^2 \;=\; u^2\Big)\right) \\
&=\; 2e^4 \times \frac{s^4+t^4+u^4}{s^2 \times t^2}\,.
\end{aligned} \tag{S.10}$$

Consequently, the un-polarized partial cross-section for the Bhabha scattering is

$$\frac{d\sigma}{d\Omega_{\text{c.m.}}} \;=\; \frac{\overline{|\mathcal{M}|^2}}{64\pi^2\, s} \;=\; \frac{\alpha^2}{2s} \times \frac{s^4+t^4+u^4}{s^2 \times t^2}\,. \tag{8.a}$$

To complete the problem, let's work out the kinematics in the center of mass frame:

$$\begin{aligned} s &= 4E^2 \approx 4\mathbf{p}^2, \\ t &= -(\mathbf{p}'_1 - \mathbf{p}_1)^2 = -2\mathbf{p}^2(1-\cos\theta), \\ u &= -(\mathbf{p}'_2 - \mathbf{p}_1)^2 = -2\mathbf{p}^2(1+\cos\theta), \end{aligned} \tag{S.11}$$

hence

$$\begin{aligned} \frac{s^4+t^4+u^4}{s^2\,t^2} &= \frac{(4\mathbf{p}^2)^4 + (2\mathbf{p}^2)^4\times(1-\cos\theta)^4 + (2\mathbf{p}^2)^4\times(1+\cos\theta)^4}{(4\mathbf{p}^2)^2\times(2\mathbf{p}^2)^2(1-\cos\theta)^2} \\ &= \frac{16+(1-\cos\theta)^4+(1+\cos\theta)^4}{4\times(1-\cos\theta)^2} = \frac{18+12\cos^2\theta+2\cos^4\theta}{4\times(1-\cos\theta)^2} \\ &= \frac{(3+\cos^2\theta)^2}{2(1-\cos\theta)^2}\,. \end{aligned} \tag{S.12}$$

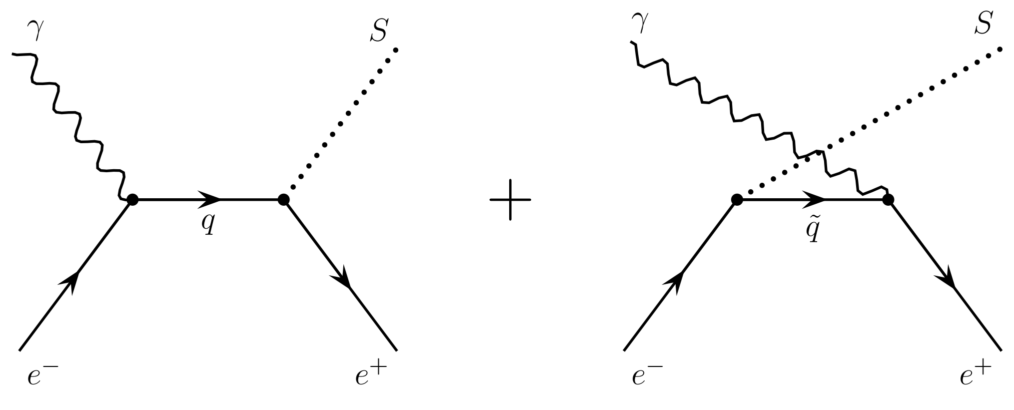
Plugging this formula into eq. (8.a), we finally obtain

$$\frac{d\sigma^{\rm Bhabha}}{d\Omega_{\rm cm}} = \frac{\alpha^2}{4s}\times\frac{(3+\cos^2\theta)^2}{(1-\cos\theta)^2}\,. \tag{8.b}$$

Quod erat demonstrandum.

Problem **3**(a):

There are two tree diagrams for the $e^-e^+ \to S\gamma$ process, namely



(S.13)

These two diagrams are related by $t \leftrightarrow u$ crossing, and also by the charge conjugation (which exchanges the initial e^- and e^+). The net tree-level amplitude is

$$\mathcal{M}_{\rm tree} = \mathcal{E}^{*\mu}_{\mathbf{k},\lambda}(\gamma)\times\mathcal{M}_\mu\,, \tag{S.14.a}$$

$$
\begin{aligned}
\mathcal{M}^\mu \;&=\; \mathcal{M}_1^\mu \;+\; \mathcal{M}_2^\mu\,, && \text{(S.14.b)}\\
\mathcal{M}_1^\mu \;&=\; -i\,\bar v(e^+)(-ig)\frac{i}{\not{q}-m_e}\,(ie\gamma^\mu)\,u(e^-)\\
&=\; \frac{eg}{t-m^2}\times\bar v(\not{q}+m_e)\gamma^\mu u\,, && \text{(S.14.c)}\\
\mathcal{M}_2^\mu \;&=\; -i\,\bar v(e^+)\,(ie\gamma^\mu)\,\frac{i}{\not{\tilde q}-m_e}(-ig)u(e^-)\\
&=\; \frac{eg}{u-m^2}\times\bar v\gamma^\mu(\not{\tilde q}+m_e)u\,, && \text{(S.14.d)}
\end{aligned}
$$

where

$$
\begin{aligned}
&q \;=\; p_- - k_\gamma \;=\; k_s - p_+\,,\quad q^2 \;=\; t,\\
\text{and}\quad &\tilde q \;=\; p_- - k_s \;=\; k_\gamma - p_+\,,\quad \tilde q^2 \;=\; u.
\end{aligned}
\qquad \text{(S.15)}
$$

<u>Problem **3**(b)</u>:

The Ward identity for the one-photon amplitude (S.14.a) says $k_\gamma^\mu\times\mathcal{M}_\mu=0$. To verify it, let's start with the first diagram:

$$
\begin{aligned}
k_\gamma^\mu\times\bar v(\not{q}+m_e)\gamma_\mu u \;&=\; \bar v(\not{q}+m_e)\,\not{k}_\gamma u\\
&=\; \bar v(\not{p}_- - \not{k}_\gamma+m_e)\,\not{k}_\gamma u\\
&=\; \bar v(\not{p}_- + m_e)\,\not{k}_\gamma u \qquad \langle\!\langle\,\text{because } \not{k}_\gamma\,\not{k}_\gamma = k_\gamma^2 = 0\,\rangle\!\rangle\\
&=\; \bar v\Big(2(p_-k_\gamma) - \not{k}_\gamma(\not{p}_- - m_e)\Big)u\\
&=\; 2(p_-k_\gamma)\times\bar v u \;-\; 0 \qquad \langle\!\langle\,\text{because } (\not{p}_- - m_e)\times u(e^-)=0\,\rangle\!\rangle\\
&=\; (m_e^2-t)\times\bar v u\,,
\end{aligned}
\qquad \text{(S.16)}
$$

and hence

$$
k_\gamma^\mu\times\mathcal{M}_{1\mu} \;=\; -eg\times\bar v u\,. \qquad \text{(S.17)}
$$

We see that *by itself*, the first diagram does not satisfy the Ward entity. Instead, we need to add the second diagram's contribution

$$\begin{aligned}
k_\gamma^\mu \times \bar{v}\gamma_\mu(\tilde{\not{q}} + m_e)u &= \bar{v}\,\not{k}_\gamma(\tilde{\not{q}} + m_e)u \\
&= \bar{v}\,\not{k}_\gamma(\not{k}_\gamma - \not{p}_+ + m_e)u \\
&= \bar{v}\,\not{k}_\gamma(-\not{p}_+ + m_e)u \qquad \langle\!\langle \text{ because } \not{k}_\gamma\,\not{k}_\gamma = k_\gamma^2 = 0 \,\rangle\!\rangle \\
&= \bar{v}\Big(-2(p_+k_\gamma) + \not{k}_\gamma(\not{p}_+ + m_e)\Big)u \\
&= -2(p_+k_\gamma) \times \bar{v}u \;+\; 0 \qquad \langle\!\langle \text{ because } \bar{v})e^+) \times (\not{p}_+ + m_e) = 0 \,\rangle\!\rangle \\
&= (u - m_e^2) \times \bar{v}u\,,
\end{aligned} \tag{S.18}$$

and hence

$$k_\gamma^\mu \times \mathcal{M}_{2\mu} \;=\; +eg \times \bar{v}u\,. \tag{S.19}$$

Again, the second diagram does not satisfy the Ward identity *by itself*, but the net amplitude does:

$$k_\gamma^\mu \times (\mathcal{M}_\mu \;=\; \mathcal{M}_{1\mu} \;+\; \mathcal{M}_{2\mu}) \;=\; 0. \tag{S.20}$$

Problem **3**(c):

Thanks to the Ward identity, summing $|\mathcal{M}|^2$ over the photon's polarizations is easy:

$$\begin{aligned}
\sum_\lambda |\mathcal{M}|^2 &= -\mathcal{M}^\mu\mathcal{M}_\mu^* \qquad \langle\!\langle \text{ see my notes on Ward identities } \rangle\!\rangle \\
&= -\mathcal{M}_1^\mu\mathcal{M}_{1\mu}^* \;-\; \mathcal{M}_2^\mu\mathcal{M}_{2\mu}^* \;-\; 2\,\mathrm{Re}\left(\mathcal{M}_1^\mu\mathcal{M}_{2\mu}^*\right) \\
&= -\frac{e^2g^2}{(t-m_e^2)^2} \times \bar{v}(\not{q} + m_e)\gamma^\mu u \times \bar{u}\gamma_\mu(\not{q} + m_e)v \\
&\quad - \frac{e^2g^2}{(u-m_e^2)^2} \times \bar{v}\gamma^\mu(\tilde{\not{q}} + m_e)u \times \bar{u}(\tilde{\not{q}} + m_e)\gamma_\mu v \\
&\quad - \frac{2e^2g^2}{(t-m_e^2)(u-m_e^2)} \times \mathrm{Re}\Big(\bar{v}(\not{q} + m_e)\gamma^\mu u \times \bar{u}(\tilde{\not{q}} + m_e)\gamma_\mu v\Big).
\end{aligned} \tag{S.21}$$

Consequently, averaging this formula over the electron's and the positron's spins yields

$$\overline{|\mathcal{M}|^2} \;=\; \frac{1}{4}\sum_{s_-,s_+}\sum_\lambda |\mathcal{M}|^2 \;=\; e^2g^2\left(\frac{A_{11}}{(t-m_2^2)^2} \;+\; \frac{A_{22}}{(u-m_e^2)^2} \;+\; \frac{2\,\mathrm{Re}\,A_{12}}{(t-m_e^2)(u-m_e^2)}\right) \tag{10}$$

where

$$\begin{aligned} A_{11} &= \frac{1}{4}\sum_{s_-,s_+} \bar{v}(\not{q}+m_e)\gamma^\mu u \times \bar{u}\gamma_\mu(\not{q}+m_e)v, \\ A_{22} &= \frac{1}{4}\sum_{s_-,s_+} \bar{v}\gamma^\mu(\tilde{\not{q}}+m_e)u \times \bar{u}(\tilde{\not{q}}+m_e)\gamma_\mu v, \\ A_{12} &= \frac{1}{4}\sum_{s_-,s_+} \bar{v}(\not{q}+m_e)\gamma^\mu u \times \bar{u}(\tilde{\not{q}}+m_e)\gamma_\mu v. \end{aligned} \tag{S.22}$$

At this point, we use the spin sums

$$\sum_{s_-} u \times \bar{u} \;=\; (\not{p}_- + m_e), \quad \sum_{s_+} v \times \bar{v} \;=\; (\not{p}_+ - m_e) \tag{S.23}$$

to convert eqs. (S.22) to Dirac traces (11):

$$\begin{aligned} A_{11} &= \frac{1}{4}\,\mathrm{Tr}\left(\left(\sum\nolimits_{s_+} v\times\bar{v}\right)(\not{q}+m_e)\gamma^\mu\left(\sum\nolimits_{s_-} u\times\bar{u}\right)\gamma_\mu(\not{q}+m_e)\right) \\ &= \frac{1}{4}\,\mathrm{Tr}\Big((\not{p}_+ - m_e)(\not{q}+m_e)\gamma^\mu(\not{p}_- + m_e)\gamma_\mu(\not{q}+m_e)\Big), \end{aligned} \tag{S.24}$$

and likewise

$$\begin{aligned} A_{22} &= \frac{1}{4}\,\mathrm{Tr}\left(\left(\sum\nolimits_{s_+} v\times\bar{v}\right)\gamma^\mu(\tilde{\not{q}}+m_e)\left(\sum\nolimits_{s_-} u\times\bar{u}\right)(\tilde{\not{q}}+m_e)\gamma_\mu\right) \\ &= \frac{1}{4}\,\mathrm{Tr}\Big((\not{p}_+ - m_e)\gamma^\mu(\tilde{\not{q}}+m_e)(\not{p}_- + m_e)(\tilde{\not{q}}+m_e)\gamma_\mu\Big), \end{aligned} \tag{S.25}$$

$$\begin{aligned} A_{12} &= \frac{1}{4}\,\mathrm{Tr}\left(\left(\sum\nolimits_{s_+} v\times\bar{v}\right)(\not{q}+m_e)\gamma^\mu\left(\sum\nolimits_{s_-} u\times\bar{u}\right)(\tilde{\not{q}}+m_e)\gamma_\mu\right) \\ &= \frac{1}{4}\,\mathrm{Tr}\Big((\not{p}_+ - m_e)(\not{q}+m_e)\gamma^\mu(\not{p}_- + m_e)(\tilde{\not{q}}+m_e)\gamma_\mu\Big). \end{aligned} \tag{S.26}$$

Quod erat demonstrandum.

Problem **3**(d):

Evaluating the Dirac traces (11) is straightforward but tedious. Fortunately, it becomes much simpler when we neglect the electron's mass. In that limit, the first trace becomes

$$
\begin{aligned}
A_{11} &\approx -\tfrac{1}{4}\,\mathrm{Tr}(\not{p}_+\,\not{q}\gamma^\mu\,\not{p}_-\gamma_\mu\,\not{q}) \\
&= +\tfrac{1}{2}\,\mathrm{Tr}(\not{p}_+\,\not{q}\,\not{p}_-\,\not{q}) \qquad \langle\!\langle\, \text{using } \gamma^\mu\,\not{p}_-\gamma_\mu = -2\,\not{p}_-\,\rangle\!\rangle \\
&= 4(p_+q)(p_-q) \;-\; 2(p_+p_-)\,q^2 \\
&\approx (M_s^2 - t)\times t \;-\; s\times t \;=\; (M_s^2 - t - s)\times t \\
&\approx u\times t\,,
\end{aligned}
\tag{S.27}
$$

where the last two lines follow from

$$
\begin{aligned}
q^2 &= t, \\
p_+p_- &= \tfrac{1}{2}(p_- + p_+)^2 - \cancel{m_e^2} \;\approx\; \frac{s}{2}\,, \\
p_-q &= p_-(p_- - k_\gamma) \;=\; \tfrac{1}{2}(p_- - k_\gamma)^2 + \cancel{\tfrac{1}{2}m_e^2} \;\approx\; \frac{t}{2}\,, \\
p_+q &= p_+(k_S - p_+) \;=\; -\tfrac{1}{2}(p_+ - k_S)^2 \,+\, \tfrac{1}{2}M_s^2 \,-\, \cancel{\tfrac{1}{2}m_e^2} \;\approx\; \frac{M_s^2 - t}{2}\,, \\
s + t + u &= M_s^2 + \cancel{2m_e^2} \;\approx\; M_s^2\,.
\end{aligned}
\tag{S.28}
$$

Likewise, the second trace becomes

$$
\begin{aligned}
A_{22} &\approx -\tfrac{1}{4}\,\mathrm{Tr}(\not{p}_+\gamma^\mu\,\not{\tilde{q}}\,\not{p}_-\,\not{\tilde{q}}\gamma_\mu) \\
&= -\tfrac{1}{4}\,\mathrm{Tr}(\gamma_\mu\,\not{p}_+\gamma^\mu\,\not{\tilde{q}}\,\not{p}_-\,\not{\tilde{q}}) \\
&= +\tfrac{1}{2}\,\mathrm{Tr}(\not{p}_+\,\not{\tilde{q}}\,\not{p}_-\,\not{\tilde{q}}) \qquad \langle\!\langle\, \text{using } \gamma_\mu\,\not{p}_+\gamma^\mu = -2\,\not{p}_+\,\rangle\!\rangle \\
&= 4(p_+\tilde{q})(p_-\tilde{q}) \;-\; 2(p_+p_-)\,\tilde{q}^2 \\
&\approx (M_s^2 - u)\times u \;-\; s\times u \;=\; (M_s^2 - u - s)\times u \\
&\approx t\times u\,,
\end{aligned}
\tag{S.29}
$$

where the last two lines follow from (S.28) and

$$
\begin{aligned}
\tilde{q}^2 &= u, \\
p_+\tilde{q} &= p_+(k_\gamma - p_+) \;=\; -\tfrac{1}{2}(k_\gamma - p_+)^2 - \cancel{\tfrac{1}{2}m_e^2} \;\approx\; -\frac{u}{2}\,, \\
p_-\tilde{q} &= p_-(p_- - k_s) \;=\; \tfrac{1}{2}(p_- - k_s)^2 \,-\, \tfrac{1}{2}M_s^2 \,+\, \cancel{\tfrac{1}{2}m_e^2} \;\approx\; \frac{u - M_s^2}{2}\,.
\end{aligned}
\tag{S.30}
$$

Finally, the third trace becomes

$$\begin{aligned} A_{22} &\approx -\tfrac{1}{4}\,\mathrm{Tr}(\not{p}_+\,\not{q}\gamma^\mu\,\not{p}_-\,\tilde{\not{q}}\gamma_\mu) \\ &= -(p_-\tilde{q})\times\mathrm{Tr}(\not{p}_+\,\not{q})\langle\!\langle\,\text{using }\gamma^\mu\,\not{p}_-\,\tilde{\not{q}}\gamma^\mu = +4(p_-\tilde{q})\,\rangle\!\rangle \\ &= -4(p_-\tilde{q})(p_+q) \\ &\approx +(u-M_s^2)(t-M_s^2). \end{aligned} \tag{S.31}$$

Quad erat demonstrandum.

Problem **3**(e):

Now let's evaluate eq. (10) for the spin summed/averaged $\overline{|\mathcal{M}|^2}$. Neglecting the m_e^2 terms in the denominators and plugging in eqs. (12) for the A_{11}, A_{22}, and A_{12}, we have

$$\begin{aligned} \overline{|\mathcal{M}|^2} &= e^2g^2\left(\frac{tu}{t^2}\,+\,\frac{ut}{u^2}\,+\,\frac{2(t-M_s^2)(u-M_s^2)}{tu}\right) \\ &= \frac{e^2g^2}{tu}\times\Big(u^2\,+\,t^2\,+\,2(t-M_s^2)(u-M_s^2)\Big) \\ &= \frac{e^2g^2}{tu}\times\Big((t+u-M_s^2)^2\,+\,M_s^4\Big) \\ &= e^2g^2\times\frac{s^2+M_s^4}{tu}\,. \end{aligned} \tag{S.32}$$

Now let's work out the kinematics in the center of mass frame. The initial electron and positron have 4–momenta $p^\mu_\mp = (E_e, \pm\mathbf{p})$ where $E_e \approx |\mathbf{p}|$. But since the scalar and the photon produced in the collision have different masses, they have equal and opposite 3-momenta (in the CM frame) but different energies: $k^\mu_\gamma = (\omega, +\mathbf{k})$ while $k^\mu_S = (E_s, -\mathbf{k})$, where $\omega = |\mathbf{k}| \neq E_s = \sqrt{\mathbf{k}^2 + M_s^2}$. By energy conservation

$$\omega\,+\,E_s\;=\;2E_e\;=\;\sqrt{s}\,. \tag{S.33}$$

To solve this equation, we rewrite it as

$$\omega^2\,+\,M_s^2\;=\;E_s^2\;=\;(\sqrt{s}\,-\omega)^2\;=\;s\,-\,2\sqrt{s}\times\omega\,+\omega^2, \tag{S.34}$$

which gives us

$$\omega \;=\; \frac{s-M_s^2}{2\sqrt{s}} \quad\Longrightarrow\quad E_s \;=\; \frac{s+M_s^2}{2\sqrt{s}}\,. \tag{S.35}$$

Given all these momenta, Mandelstam's t and u obtain as

$$\begin{aligned} t \;&\approx\; -2(p_-k_\gamma) \;=\; -2E_e\omega \,+\, 2\mathbf{p}\cdot\mathbf{k} \approx\; -2E_e\omega\times(1-\cos\theta) \\ &=\; -\tfrac{1}{2}(s-M_s^2)\times(1-\cos\theta), \end{aligned} \tag{S.36}$$

$$\begin{aligned} u \;&\approx\; -2(p_+k_\gamma) \;=\; -2E_e\omega \,-\, 2\mathbf{p}\cdot\mathbf{k} \approx\; -2E_e\omega\times(1+\cos\theta) \\ &=\; -\tfrac{1}{2}(s-M_s^2)\times(1+\cos\theta). \end{aligned} \tag{S.37}$$

Hence, plugging these values into eq. (S.32) gives us

$$\overline{|\mathcal{M}|^2} \;=\; 4e^2g^2\times\frac{s^2+M_S^4}{(s-M_S^2)^2}\times\frac{1}{\sin^2\theta}\,. \tag{S.38}$$

Finally, the partial cross-section for a 2 *particles* $\to$ 2 *particles* inelastic scattering in the CM frame is given by

$$\frac{d\sigma}{d\Omega_{\rm cm}} \;=\; \frac{|\mathcal{M}|^2}{64\pi^2 s}\times\frac{|\mathbf{p}'|}{|\mathbf{p}|}\,. \tag{S.39}$$

For the problem at hand, the inelasticity factor $|\mathbf{p}'|/|\mathbf{p}|$ is

$$\frac{|\mathbf{k}|}{|\mathbf{p}|} \;\approx\; \frac{\omega}{E_e} \;=\; \frac{s-M_s^2}{s}\,. \tag{S.40}$$

Combining this factor with eq. (S.38), we finally arrive at the following formula for the partial cross-section:

$$\frac{d\sigma(e^-e^+\to\gamma S)}{d\Omega_{\rm c.m.}} \;=\; \frac{\alpha g^2}{4\pi}\times\frac{s^2+M_s^4}{s^2(s-M_s^2)}\times\frac{1}{\sin^2\theta}\,. \tag{S.41}$$

Note the forward-backward symmetry $\theta\leftrightarrow\pi-\theta$ of this cross section. Physically, it is due to the charge-conjugation symmetry which exchanges the initial electron and positron.

As usual for annihilation processes in the ultra-relativistic limit, the cross-section (S.41) has divergent peaks in forward and backward directions, $\theta\to 0$ or $\theta\to\pi$. The divergence here is an artefact of the $m_e^2=0$ approximation, which becomes inaccurate at very small angles $\theta\lesssim(m_e/E)$ (or $\pi-\theta\lesssim(m_e/E)$).

A more careful analysis — which was not a required part of this homework — leads to

$$\text{for } \theta \lesssim \gamma^{-1}, \quad \frac{d\sigma(e^-e^+ \to \gamma S)}{d\Omega_{\text{c.m.}}} \approx \frac{\alpha g^2}{4\pi s} \times \left(\frac{s - M_s^2}{s} \times \frac{1}{\theta^2 + \gamma^{-2}} + \frac{M_s^2}{s - M_s^2} \times \frac{2\theta^2}{(\theta^2 + \gamma^{-2})^2} \right) \tag{S.42}$$

— where $\gamma^{-1} = m_e/E \ll 1$ — instead of eq. (S.41). Consequently, the total cross-section turns out to be finite rather then divergent, namely

$$\sigma_{\text{tot}}(e^-e^+ \to \gamma S) = \alpha g^2 \times \frac{(s^2 + M_s^4)}{s^2(s - M_s^2)} \left(\log \frac{2E_e}{m_e} - \frac{sM_s^2}{s^2 + M_s^4} + O\left(\frac{m_e^2}{E_e^2} \right) \right). \tag{S.43}$$