

Problem 1(a):

The scalar potential part of the linear sigma model's Lagrangian (1) is

$$V(\phi) = \frac{\lambda}{8} \left(\sum_i \phi_i^2 - f^2 \right)^2 - \beta \lambda f^2 \times \phi_{N+1}, \quad (\text{S.1})$$

where the last term explicitly breaks the $O(N+1)$ symmetry of the first term down to $O(N)$. To find the minimum of this potential, let's first find the stationary points where all the first derivatives $\partial V / \partial \phi_i$ are zero:

$$\text{for } i = 1, \dots, N, \quad \frac{\partial V}{\partial \phi_i} = \frac{\lambda}{2} \left(\sum_j \phi_j^2 - f^2 \right) \times \phi_i = 0, \quad (\text{S.2})$$

$$\text{and } \frac{\partial V}{\partial \phi_{N+1}} = \frac{\lambda}{2} \left(\sum_j \phi_j^2 - f^2 \right) \times \phi_{N+1} - \beta \lambda f^2 = 0. \quad (\text{S.3})$$

From eq. (S.3) we immediately see that at any stationary point $(\sum \phi^2 - f^2) \neq 0$, hence eqs. (S.2) tell us that $\phi_1 = \dots = \phi_N = 0$. In other words, all the stationary points lie on the ϕ_{N+1} axis in the $(N+1)$ dimensional space of the scalar field values. And in this space, eq. (S.3) becomes a simple cubic equation

$$\phi_{N+1}^3 - f^2 \times \phi_{N+1} - 2\beta f^2 = 0. \quad (\text{S.4})$$

For small $\beta \ll f$, this cubic equation has 3 real solutions, approximately

$$\langle \phi_{N+1} \rangle_1 \approx -2\beta, \quad \langle \phi_{N+1} \rangle_2 \approx -f + \beta, \quad \langle \phi_{N+1} \rangle_3 \approx +f + \beta. \quad (\text{S.5})$$

Now let's find out which of the three stationary points is a minimum (or at least a local minimum) by looking at the second derivatives of the potential (S.1). Along the ϕ_{N+1} axis in the field space, the second derivatives amount to

$$\frac{\partial^2 V}{\partial \phi_i \partial \phi_j} = \frac{\lambda}{2} \times \begin{cases} (3\phi_{N+1}^2 - f^2) & \text{for } i = j = N + 1, \\ 0 & \text{for } i \leq N, j = N + 1 \text{ or } j \leq N, i = N + 1, \\ (\phi_{N+1}^2 - f^2) \times \delta_{ij} & \text{for } i, j \leq N. \end{cases} \quad (\text{S.6})$$

Evaluating these derivatives for the 3 stationary points (S.5) — while assuming small $\beta > 0$ —

gives us

$$\begin{aligned}
@ \langle \phi_{N+1} \rangle_1 : & \quad \frac{\partial^2 V}{(\partial \phi_{N+1})^2} < 0 \quad \text{while other} \quad \frac{\partial^2 V}{(\partial \phi_i)^2} < 0 \implies \text{maximum,} \\
@ \langle \phi_{N+1} \rangle_2 : & \quad \frac{\partial^2 V}{(\partial \phi_{N+1})^2} > 0 \quad \text{while other} \quad \frac{\partial^2 V}{(\partial \phi_i)^2} < 0 \implies \text{saddle point,} \\
@ \langle \phi_{N+1} \rangle_3 : & \quad \frac{\partial^2 V}{(\partial \phi_{N+1})^2} > 0 \quad \text{while other} \quad \frac{\partial^2 V}{(\partial \phi_i)^2} > 0 \implies \text{minimum.}
\end{aligned} \tag{S.7}$$

Thus, the potential (S.1) has a unique minimum at

$$\langle \phi_1 \rangle = \dots = \langle \phi_N \rangle = 0, \quad \langle \phi_{N+1} \rangle = +f + \beta + O(\beta^2/f). \tag{2}$$

Quod erat demonstrandum.

Problem 1(b):

Let's shift the fields as in eq. (3). In terms of the shifted fields,

$$T \stackrel{\text{def}}{=} \sum_i \phi_i^2 - f^2 = \underline{\pi}^2 + (\sigma + \langle \phi_{N+1} \rangle)^2 - f^2 = \underline{\pi}^2 + \sigma^2 + 2 \langle \phi_{N+1} \rangle \times \sigma + (\langle \phi_{N+1} \rangle^2 - f^2), \tag{S.8}$$

where $\underline{\pi}$ is a short-hand for N -vector (π^1, \dots, π^N) of the pion fields, thus $\underline{\pi}^2 = (\pi^1)^2 + \dots + (\pi^N)^2$.

Therefore, expanding the scalar potential (S.1) into powers of the shifted fields, we obtain

$$\begin{aligned}
V &= \frac{\lambda}{8} \times T^2 - \beta \lambda f^2 \times (\sigma + \langle \phi_{N+1} \rangle) \\
&= \frac{\lambda}{8} \times (\underline{\pi}^2 + \sigma^2)^2 + \frac{\lambda \langle \phi_{N+1} \rangle}{2} \times \sigma \times (\underline{\pi}^2 + \sigma^2) \\
&\quad + \frac{\lambda \langle \phi_{N+1} \rangle^2}{2} \times \sigma^2 + \frac{\lambda (\langle \phi_{N+1} \rangle^2 - f^2)}{4} \times (\underline{\pi}^2 + \sigma^2) \\
&\quad + \left(\frac{\lambda \langle \phi_{N+1} \rangle}{2} \times (\langle \phi_{N+1} \rangle^2 - f^2) - \beta \lambda f^2 \right) \times \sigma + \text{const.}
\end{aligned} \tag{S.9}$$

On the last line here, the coefficient of σ vanishes thanks to $\langle \phi_{N+1} \rangle$ obeying the cubic equation (S.4). For the same reason, the coefficient of $(\underline{\pi}^2 + \sigma^2)$ on the line before the last may be

simplified as

$$\frac{\lambda(\langle\phi_{N+1}\rangle^2 - f^2)}{4} = \frac{\beta\lambda f^2}{2\langle\phi_{N+1}\rangle}. \quad (\text{S.10})$$

Altogether, we have

$$V(\sigma, \underline{\pi}) = \frac{\lambda}{8} \times (\underline{\pi}^2 + \sigma^2)^2 + \frac{\kappa}{2} \times (\sigma^3 + \sigma \underline{\pi}^2) + \frac{M_\sigma^2}{2} \times \sigma^2 + \frac{M_\pi^2}{2} \times \underline{\pi}^2 + \text{const}, \quad (\text{S.11})$$

where

$$\begin{aligned} \text{quartic coupling} \quad \lambda &= \lambda, \\ \text{cubic coupling} \quad \kappa &= \lambda \times \langle\phi_{N+1}\rangle \approx \lambda(f + \beta), \\ \text{pion mass}^2 \quad M_\pi^2 &= \frac{\beta\lambda f^2}{\langle\phi_{N+1}\rangle} \approx \beta\lambda f, \\ \text{sigma mass}^2 \quad M_\sigma^2 &= M_\pi^2 + \lambda \langle\phi_{N+1}\rangle^2 \approx \lambda f(f + 3\beta). \end{aligned} \quad (\text{S.12})$$

Let's take a closer look at the pion's mass², $M_\pi^2 \approx \beta \times \lambda f$. In the $\beta = 0$ limit, the pions are massless in accordance with the Goldstone theorem. Indeed, for $\beta = 0$ the sigma model's Lagrangian has an exact $SO(N + 1)$ symmetry which is spontaneously broken down to an $SO(N)$ subgroup; there are N spontaneously broken generators, so there should be N massless Goldstone bosons. But for $\beta \neq 0$, the $SO(N + 1)$ symmetry of the Lagrangian is only approximate, and its *explicit* breaking by the $\beta\lambda f^2 \times \phi_{N+1}$ term spoils the Goldstone theorem. Thus, instead of exactly massless Goldstone bosons we should get light but not quite massless pseudo-Goldstone bosons; to the first order in β , their mass² should be proportional to β . And indeed, in the linear sigma model $M_\pi^2 \approx \beta \times \lambda f$.

Still, for $\beta \ll f$, the pions should be much lighter than the sigma particle. And indeed, according to eqs. (S.12),

$$\frac{M_\pi^2}{M_\sigma^2} \approx \frac{\beta\lambda f}{\lambda f^2} = \frac{\beta}{f} \ll 1. \quad (\text{S.13})$$

Problem 1(c): Back in [homework#8](#) (problem 2), we had a very similar setup to the *shifted* fields of the linear sigma models: $N + 1$ scalar fields $\sigma(x)$ and $\pi^i(x)$, with the Lagrangian

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}(\partial_\mu\pi)^2 - V, \\ V(\sigma, \pi) &= \frac{\lambda}{8}(\pi^2 + \sigma^2 + 2f \times \sigma)^2 \\ &= \frac{\lambda f^2}{2} \times \sigma^2 + \frac{\lambda f}{2} \times (\sigma^3 + \sigma\pi^2) + \frac{\lambda}{8} \times (\sigma^2 + \pi^2)^2.\end{aligned}\tag{9.2}$$

In particular, there is a mass term for the σ field but not for the pions, which are exactly massless — exactly as in the present sigma model with $\beta = 0$. The cubic and quartic terms in the potential (9.3) also have exactly the same form as in eq. (S.11), and the quatic coupling λ and the cubic coupling $\kappa = \lambda f$ are related to the σ field's mass as

$$\kappa^2 = \lambda \times M_\sigma^2.\tag{S.14}$$

For the present sigma model, we have exactly similar relation for $\beta = 0$. Indeed, according to eq. (S.12),

$$\text{for } \beta = 0, \quad \kappa = \lambda f, \quad M_\sigma^2 = \lambda f^2 \quad \implies \quad \kappa^2 = \lambda \times M_\sigma^2.\tag{S.15}$$

Therefore, the $\pi\pi \rightarrow \pi\pi$ scattering amplitudes in the linear sigma model for $\beta = 0$ come out to be exactly as [homework#8](#): At the tree level,

$$\begin{aligned}\mathcal{M}(\pi^j + \pi^k \rightarrow \pi^\ell + \pi^m) &= -\left(\lambda + \frac{\kappa^2}{s - M_\sigma^2}\right) \times \delta^{jk} \delta^{\ell m} - \left(\lambda + \frac{\kappa^2}{t - M_\sigma^2}\right) \times \delta^{j\ell} \delta^{km} \\ &\quad - \left(\lambda + \frac{\kappa^2}{u - M_\sigma^2}\right) \times \delta^{jm} \delta^{k\ell},\end{aligned}\tag{S.16}$$

which in light of the relation (S.14) becomes

$$\begin{aligned}\mathcal{M}(\pi^j + \pi^k \rightarrow \pi^\ell + \pi^m) &= -\frac{\lambda s}{s - M_\sigma^2} \times \delta^{jk} \delta^{\ell m} - \frac{\lambda t}{t - M_\sigma^2} \times \delta^{j\ell} \delta^{km} \\ &\quad - \frac{\lambda u}{u - M_\sigma^2} \times \delta^{jm} \delta^{k\ell}.\end{aligned}\tag{S.17}$$

When any of the 4 pions' energy becomes small, we get $s, t, u \ll M_\sigma^2$, and the scattering

amplitude becomes small as

$$\mathcal{M} \approx \frac{\lambda}{M_\sigma^2} \times \left(s \times \delta^{jk} \delta^{\ell m} + t \times \delta^{j\ell} \delta^{km} + u \times \delta^{jm} \delta^{k\ell} \right) = O\left(\frac{\lambda E_{\text{cm}}^2}{M_\sigma^2}\right). \quad (\text{S.18})$$

Problem 1(d): For $\beta \neq 0$, the quartic and the cubic couplings of the σ and π^i to each other has similar overall form to what we had back in [homework#8](#), but the overall coefficients λ and κ of those couplings are no longer related to the σ particle's mass by eq. (S.14). Instead, eq. (S.12) gives us

$$\kappa = \lambda \langle \phi_{N+!} \rangle, \quad M_\sigma^2 = M_\pi^2 + \lambda \langle \phi_{N+!} \rangle^2 \quad \implies \quad \kappa^2 = \lambda \times (M_\sigma^2 - M_\pi^2). \quad (\text{S.19})$$

Now consider the $\pi\pi \rightarrow \pi\pi$ scattering. At the tree level, we have exactly the same 4 diagrams for such scattering as in the [homework#8](#), namely

(S.20)

Altogether, these diagrams yield the scattering amplitude exactly as in eq. (S.16), but for the

β -modified couplings and masses. Consequently, in light of eq. (S.19) instead of (S.14), we have

$$\lambda + \frac{\kappa^2}{s - M_\sigma^2} = \frac{\lambda s - \lambda M_\sigma^2 + \kappa^2}{s - M_\sigma^2} = \frac{\lambda s - \lambda M_\pi^2}{s - M_\sigma^2} \quad (\text{S.21})$$

and likewise

$$\lambda + \frac{\kappa^2}{t - M_\sigma^2} = \frac{\lambda(t - M_\pi^2)}{t - M_\sigma^2} \quad \text{and} \quad \lambda + \frac{\kappa^2}{u - M_\sigma^2} = \frac{\lambda(u - M_\pi^2)}{u - M_\sigma^2}, \quad (\text{S.22})$$

so the amplitude (S.16) becomes

$$\begin{aligned} \mathcal{M}(\pi^j + \pi^k \rightarrow \pi^\ell + \pi^m) &= -\frac{\lambda(s - M_\pi^2)}{s - M_\sigma^2} \times \delta^{jk} \delta^{\ell m} - \frac{\lambda(t - M_\pi^2)}{t - M_\sigma^2} \times \delta^{j\ell} \delta^{km} \\ &\quad - \frac{\lambda(u - M_\pi^2)}{u - M_\sigma^2} \times \delta^{jm} \delta^{k\ell}. \end{aligned} \quad (\text{S.23})$$

When the pions' energies become low compared to M_σ — or in Lorentz-invariant terms, when $s, t, u \ll M_\sigma^2$ — we may simplify this amplitude by approximating all the denominators as $-M_\sigma^2$, thus

$$\mathcal{M}(\pi^j + \pi^k \rightarrow \pi^\ell + \pi^m) \approx \left(\frac{\lambda}{M_\sigma^2} \approx \frac{1}{f^2} \right) \times \left((s - M_\pi^2) \times \delta^{jk} \delta^{\ell m} + (t - M_\pi^2) \times \delta^{j\ell} \delta^{km} + (u - M_\pi^2) \times \delta^{jm} \delta^{k\ell} \right). \quad (4)$$

What happens to this amplitude when one of the pions' momentum becomes very small? Alas, for $\beta \neq 0$ the pions are massive, so we cannot take all 4 components of a pion's p^μ to zero. The best we can do is to take $\mathbf{p} \rightarrow 0$ while $p^0 \rightarrow m$, which is the *non-relativistic limit*. However, if only one pion is non-relativistic while the other 3 pions have $E \gg M_\pi$ (but $E \ll M_\sigma$), we generally have $s, t, u = O(E \times M_\pi) \gg M_\pi^2$ (albeit $s, t, u \ll M_\sigma^2$), and the scattering amplitude becomes

$$\mathcal{M} = O\left(\frac{E \times M_\pi}{f^2}\right) \not\rightarrow 0. \quad (\text{S.24})$$

The strongest low-energy limit we can take for massive pions is to make all four pions non-relativistic. In this limit, $s = E_{\text{cm}}^2 \approx 4M_\pi^2$ while $u, t = O(\mathbf{p}^2) \ll M_\pi^2$, so the scattering

amplitude (5) becomes

$$\mathcal{M}(\pi^j + \pi^k \rightarrow \pi^\ell + \pi^m) \approx \left(\frac{\lambda M_\pi^2}{M_\sigma^2} \approx \frac{\beta \lambda}{f} \right) \times \left(3\delta^{jk}\delta^{\ell m} - \delta^{j\ell}\delta^{km} - \delta^{jm}\delta^{k\ell} \right). \quad (5)$$

This amplitude is suppressed by the factor β/f , but it does not vanish! And even if all 4 pions belong to the same species, the scattering amplitude does not vanish in the non-relativistic limit,

$$\mathcal{M}(\pi^1 + \pi^1 \rightarrow \pi^1 + \pi^1) \approx \frac{\lambda\beta}{f} \neq 0, \quad (\text{S.25})$$

unlike what we had back in [homework#8](#).

Problem 2(a):

$$\text{given } \Phi \rightarrow e^{+i\theta} U_L \Phi U_R^\dagger, \quad (8)$$

$$\text{we have } \Phi^\dagger \rightarrow e^{-i\theta} U_R \Phi^\dagger U_L^\dagger, \quad (\text{S.26})$$

$$\text{hence } \Phi^\dagger \Phi \rightarrow U_R (\Phi^\dagger \Phi) U_R^\dagger, \quad (\text{S.27})$$

$$(\Phi^\dagger \Phi)^2 \rightarrow U_R \Phi^\dagger \Phi U_R^\dagger U_R \Phi^\dagger \Phi U_R^\dagger = U_R (\Phi^\dagger \Phi)^2 U_R^\dagger, \quad (\text{S.28})$$

$$\text{likewise } (\Phi^\dagger \Phi)^n \rightarrow U_R (\Phi^\dagger \Phi)^n U_R^\dagger \quad \forall n = 1, 2, 3, \dots, \quad (\text{S.29})$$

and therefore

$$\text{all traces } \text{tr} \left((\Phi^\dagger \Phi)^n \right) \text{ are invariant under symmetries (8),} \quad (\text{S.30})$$

thanks to the cyclic invariance rule for traces, $\text{tr}(U_R X U_R^\dagger) = \text{tr}(X U_R^\dagger U_R) = \text{tr}(X)$ for any $X = (\Phi^\dagger \Phi)^n$. Consequently, the scalar potential (7) is invariant under symmetries (8).

For the global symmetries where $e^{i\theta}$, U_L , and U_R do not depend on x , the kinetic term

in (6) is also invariant. Indeed,

for constant $e^{i\theta}$, U_L , U_R ,

$$\begin{aligned}
\partial_\mu \Phi &\rightarrow e^{+i\theta} U_L (\partial_\mu \Phi) U_R^\dagger, \\
\partial_\mu \Phi^\dagger &\rightarrow e^{-i\theta} U_R (\partial_\mu \Phi^\dagger) U_L^\dagger, \\
\partial^\mu \Phi^\dagger \partial_\mu \Phi &\rightarrow U_R (\partial^\mu \Phi^\dagger \partial_\mu \Phi) U_R^\dagger,
\end{aligned} \tag{S.31}$$

and $\text{tr}(\partial^\mu \Phi^\dagger \partial_\mu \Phi)$ is invariant.

Altogether, the whole Lagrangian (6) is invariant, $\mathcal{Q.E.D.}$

Problem 2(★):

The kinetic term in (6) and the last two terms in the potential (7) have a much bigger symmetry than $G = SU(N) \times SU(N) \times U(1)$, namely the $SO(2N^2)$ which does not care for the matrix structure of the $\Phi(x)$ and treats it as $2N^2$ real component fields. Indeed,

$$\text{tr}(\Phi^\dagger \Phi) = \sum_{i,j} |\Phi_i^j|^2 = \sum_{i,j} \left((\text{Re } \Phi_i^j)^2 + (\text{Im } \Phi_i^j)^2 \right) \tag{S.32}$$

is invariant under all $SO(2N^2)$ “rotations” of the components, and so is the kinetic term.

On the other hand, the $\text{tr}(\Phi^\dagger \Phi \Phi^\dagger \Phi)$ in the potential does depend on the packing of $2N^2$ real components into a complex $N \times N$ matrix, and it is this term which reduces the internal symmetry group of the theory to $G = SU(N) \times SU(N) \times U(1)$.

Proving that all the $SO(2N^2)/G$ symmetries are broken by the quartic trace term is a non-trivial exercise in group theory rather than field theory. You do not have to do it as part of this homework set, and I am not writing down the proof here.

Problem 2(b):

Given the eigenvalues $(\kappa_1, \dots, \kappa_N)$ of the $\Phi^\dagger \Phi$ matrix, the invariant traces (S.30) obtain as

$$\text{tr} \left((\Phi^\dagger \Phi)^n \right) = \sum_{i=1}^N \kappa_i^n. \tag{S.33}$$

Consequently, the scalar potential is

$$V = \frac{\alpha}{2} \sum_i \kappa_i^2 + \frac{\beta}{2} \left(\sum_i \kappa_i \right)^2 + m^2 \sum_i \kappa_i. \quad (\text{S.34})$$

Now let's minimize this potential. Since the matrix $\Phi^\dagger \Phi$ cannot have any negative eigenvalues, we are looking for a minimum of $V(\kappa_1, \dots, \kappa_N)$ *under constraints* $\kappa_i \geq 0$. This requires

$$\forall i = 1, \dots, N, \quad \text{either } \kappa_i \geq 0 \text{ and } \frac{\partial V}{\partial \kappa_i} = 0, \quad \text{or else } \kappa_i = 0 \text{ and } \frac{\partial V}{\partial \kappa_i} > 0, \quad (\text{S.35})$$

where

$$\frac{\partial V}{\partial \kappa_i} = \alpha \kappa_i + m^2 + \beta \sum_j \kappa_j. \quad (\text{S.36})$$

These derivatives are linear functions of the eigenvalues κ_i , so all the non-zero eigenvalues must obey the same linear equation

$$\alpha \times \kappa_i = -m^2 - \beta \times \sum_j \kappa_j, \quad \text{same for all } \kappa_i \neq 0,$$

which means that all non-zero κ_i have the same value. Thus, up to a permutation of eigenvalues,

$$\kappa_1 = \dots = \kappa_k = C^2, \quad \kappa_{k+1} = \dots = \kappa_N = 0, \quad (\text{S.37})$$

for some $k = 0, 1, 2, \dots, N$, and C^2 obtains from

$$\alpha \times C^2 + m^2 + \beta \times kC^2 = 0 \quad \longrightarrow \quad C^2 = \frac{-m^2}{\alpha + k\beta}. \quad (\text{S.38})$$

To make sure that the solution (S.37) is a minimum rather than a maximum or a saddle point, we need

$$\begin{aligned} C^2 &= \frac{-m^2}{\alpha + k\beta} > 0 \quad \text{unless } k = 0, \\ m^2 + \beta k C^2 &= \frac{\alpha m^2}{\alpha + k\beta} > 0 \quad \text{unless } k = N. \end{aligned} \quad (\text{S.39})$$

Depending on the signs of α , β and m^2 parameters, this limits the solutions to the following:

- For $\alpha > 0$, $\beta > 0$, and $m^2 > 0$, the only solution is $k = 0$, which means $\kappa_1 = \dots \kappa_N = 0$ and hence $\langle \Phi \rangle = 0$.
- For $\alpha > 0$, $\beta > 0$, and $m^2 < 0$, the only solutions is $k = N$, which means

$$\kappa_1 = \dots = \kappa_N = C^2 = \frac{-m^2}{\alpha + N\beta} > 0, \quad (10)$$

and hence $\langle \Phi \rangle = C \times$ a unitary matrix. We shall focus on this regime through the rest of this problem.

- For $\alpha < 0$ or $\beta < 0$, the situation is more complicated:
 - * When $\alpha + \beta < 0$ or $\alpha + N\beta < 0$, the scalar potential (7) is unbounded from below and the theory is sick.
 - * When $\alpha > 0$ and $\beta < 0$ but $\alpha + N\beta > 0$, the solutions are similar to the $\beta > 0$ case: For $m^2 > 0$ all $\kappa_i = 0$, while for $m^2 < 0$ the κ_i are as in eq. (9).
 - * When $\beta > 0$ and $\alpha < 0$ but $\alpha + \beta > 0$: for $m^2 > 0$ the only solution is $k = 0$, meaning $\langle \Phi \rangle = 0$, but for $m^2 < 0$ all the solutions (S.37) with $k = 1, 2, \dots, N$ are good local minima.

To find the global minimum, we compare the potentials at the local minima,

$$\begin{aligned} V(\text{minimum}\#k) &= \frac{\alpha}{2} \times kC^4 + \frac{\beta}{2} \times (kC^2)^2 + m^2 \times kC^2 \\ &= \frac{k\alpha + k^2\beta}{2} \times \frac{m^4}{(\alpha + k\beta)^2} + km^2 \times \frac{-m^2}{(\alpha + k\beta)} \\ &= -\frac{m^4}{2} \times \frac{k}{k\beta + \alpha}. \end{aligned} \quad (S.40)$$

Since $\alpha < 0$ but $\alpha + \beta > 0$, the deepest minimum obtains for $k = 1$, thus

$$\kappa_1 = \frac{-m^2}{\alpha + \beta}, \quad \kappa_2 = \dots = \kappa_N = 0. \quad (S.41)$$

Problem 2(c):

Let's act with some $SU(N)_L \times SU(N)_R \times U(1)$ symmetry (8) on the vacuum expectation values (10):

$$\langle \Phi \rangle = C \times \mathbf{1}_{N \times N} \rightarrow e^{i\theta} U_L \langle \Phi \rangle U_R^\dagger = C \times e^{i\theta} U_L U_R^\dagger. \quad (\text{S.42})$$

Clearly, to keep the VEVs $\langle \Phi \rangle$ invariant, we need

$$e^{i\theta} U_L U_R^\dagger = \mathbf{1}_{N \times N} \quad (\text{S.43})$$

and hence

$$U_R = e^{i\theta} \times U_L. \quad (\text{S.44})$$

Moreover, since the U_L and U_R matrices have unit determinants, this requires

$$\det \left(e^{i\theta} \times \mathbf{1}_{N \times N} \right) = 1 \implies N \times \theta = 0 \pmod{2\pi}. \quad (\text{S.45})$$

Such a phase can be absorbed into the $U_L \in SU(N)$, so without loss of generality we need

$$e^{i\theta} = 1 \quad \text{and} \quad U_L = U_R \in SU(N). \quad (\text{S.46})$$

In other words, the *unbroken* symmetry group is $SU(N)$ which acts on the scalar fields as

$$\Phi(x) \rightarrow U \Phi(x) U^\dagger, \quad U \in SU(N). \quad (\text{S.47})$$

Problem 2(d):

In terms of the shifted fields (11),

$$\partial_\mu \Phi = \frac{1}{\sqrt{2}} (\partial_\mu \varphi_1 + i \partial_\mu \varphi_2), \quad \partial_\mu \Phi^\dagger = \frac{1}{\sqrt{2}} (\partial_\mu \varphi_1 - i \partial_\mu \varphi_2), \quad (\text{S.48})$$

hence the kinetic term in the Lagrangian becomes

$$\text{tr}(\partial_\mu \Phi^\dagger \partial^\mu \Phi) = \frac{1}{2} \text{tr}(\partial_\mu \varphi_1 \partial^\mu \varphi_1) + \frac{1}{2} \text{tr}(\partial_\mu \varphi_2 \partial^\mu \varphi_2). \quad (\text{S.49})$$

As to the potential terms, we have

$$\begin{aligned} \Phi^\dagger \Phi &= C^2 \times \mathbf{1}_{N \times N} + C \times (\delta \Phi^\dagger + \delta \Phi) + \delta \Phi^\dagger \delta \Phi \\ &= C^2 \times \mathbf{1}_{N \times N} + \sqrt{2} C \times \varphi_1 + \frac{1}{2} \varphi_1^2 + \frac{1}{2} \varphi_2^2 + \frac{i}{2} [\varphi_1, \varphi_2] \end{aligned} \quad (\text{S.50})$$

and consequently

$$\text{tr}(\Phi^\dagger \Phi) = NC^2 + \sqrt{2} C \text{tr}(\varphi_1) + \frac{1}{2} \text{tr}(\varphi_1^2) + \frac{1}{2} \text{tr}(\varphi_2^2), \quad (\text{S.51})$$

$$\begin{aligned} \text{tr}^2(\Phi^\dagger \Phi) &= N^2 C^4 + 2\sqrt{2} NC^3 \text{tr}(\varphi_1) + 2C^2 \text{tr}^2(\varphi_1) + NC^2 (\text{tr}(\varphi_1^2) + \text{tr}(\varphi_2^2)) \\ &\quad + \sqrt{2} C \text{tr}(\varphi_1) \times (\text{tr}(\varphi_1^2) + \text{tr}(\varphi_2^2)) + \frac{1}{4} (\text{tr}(\varphi_1^2) + \text{tr}(\varphi_2^2))^2 \end{aligned} \quad (\text{S.52})$$

$$\begin{aligned} \text{tr} \left((\Phi^\dagger \Phi)^2 \right) &= NC^4 + 2\sqrt{2} C^3 \text{tr}(\varphi_1) + 3C^2 \text{tr}(\varphi_1^2) + C^2 \text{tr}(\varphi_2^2) \\ &\quad + \sqrt{2} C \text{tr}(\varphi_1^3) + \sqrt{2} C \text{tr}(\varphi_1 \varphi_2^2) \\ &\quad + \frac{1}{4} \text{tr}(\varphi_1^4) + \frac{1}{4} \text{tr}(\varphi_2^4) + \frac{3}{2} \text{tr}(\varphi_1^2 \varphi_2^2) - \frac{1}{2} \text{tr}(\varphi_1 \varphi_2 \varphi_1 \varphi_2). \end{aligned} \quad (\text{S.53})$$

Plugging all these formulae into the potential (8) and expanding in powers of φ_1 and φ_2 , we obtain

$$V = \text{const} + V_1 + V_2 + V_3 + V_4 \quad (\text{S.54})$$

where

$$\begin{aligned} V_1 &= \sqrt{2} C \times (m^2 + \beta NC^2 + \alpha C^2) \times \text{tr}(\varphi_1) = 0 \\ &\quad \langle\langle \text{because } m^2 + (\alpha + N\beta)C^2 = 0 \rangle\rangle \end{aligned} \quad (\text{S.55})$$

$$\begin{aligned}
V_2 &= \beta C^2 \times \text{tr}^2(\varphi_1) + \frac{1}{2}(m^2 + \beta N C^2 + 3\alpha C^2) \times \text{tr}(\varphi_1^2) \\
&\quad + \frac{1}{2}(m^2 + \beta N C^2 + \alpha C^2) \times \text{tr}(\varphi_2^2) \\
&= \beta C^2 \times \text{tr}^2(\varphi_1) + \alpha C^2 \text{tr}(\varphi_1^2) + 0,
\end{aligned} \tag{S.56}$$

$$V_3 = \frac{\beta C}{\sqrt{2}} \times \text{tr}(\varphi_1) \times \left(\text{tr}(\varphi_1^2) + \text{tr}(\varphi_2^2) \right) + \frac{\alpha C}{\sqrt{2}} \times \left(\text{tr}(\varphi_1^3) + \text{tr}(\varphi_1 \varphi_2^2) \right), \tag{S.57}$$

$$\begin{aligned}
V_4 &= \frac{\beta}{8} \left(\text{tr}(\varphi_1^2) + \text{tr}(\varphi_2^2) \right)^2 \\
&\quad + \frac{\alpha}{8} \left(\text{tr}(\varphi_1^4) + \text{tr}(\varphi_2^4) + 6 \text{tr}(\varphi_1^2 \varphi_2^2) - 2 \text{tr}(\varphi_1 \varphi_2 \varphi_1 \varphi_2) \right).
\end{aligned} \tag{S.58}$$

Combining the quadratic part (S.56) of this potential with the kinetic terms (S.49), we arrive at

$$\mathcal{L}_2 = \frac{1}{2} \text{tr}(\partial_\mu \varphi_1 \partial^\mu \varphi_1) - \beta C^2 \times \text{tr}^2(\varphi_1) - \alpha C^2 \text{tr}(\varphi_1^2) + \frac{1}{2} \text{tr}(\partial_\mu \varphi_2 \partial^\mu \varphi_2) \tag{S.59}$$

which gives us the mass spectrum of the theory: the $\varphi_1(x)$ matrix of fields is massive and the $\varphi_2(x)$ matrix is massless. Each matrix is $N \times N$ and hermitian, so it contains N^2 independent real scalar fields, which give rise to N^2 particles. Altogether, the spectrum comprises:

- N^2 massless particles from the $\varphi_2(x)$ matrix.
- $N^2 - 1$ massive particles with $M^2 = 2\alpha C^2$ from the traceless part of the $\varphi_1(x)$ matrix.
- One more massive particle with $M^2 = 2(\alpha + N\beta)C^2 = -2m^2$ from the trace of $\varphi_1(x)$.

To see where the values of the masses come from, let's decompose the φ_1 matrix into the pure trace plus the traceless part,

$$\xi(x) \stackrel{\text{def}}{=} \frac{\text{tr}(\varphi_1(x))}{\sqrt{N}} \quad \text{and} \quad \tilde{\varphi}_1(x) \stackrel{\text{def}}{=} \varphi_1(x) - \frac{\xi(x)}{\sqrt{N}} \times \mathbf{1}_{N \times N} \implies \text{tr}(\tilde{\varphi}_1) \equiv 0. \tag{S.60}$$

Consequently,

$$\text{tr}^2(\varphi_1) = N \times \xi^2, \quad \text{tr}(\varphi_1^2) = \xi^2 + \text{tr}(\tilde{\varphi}_1^2), \tag{S.61}$$

likewise

$$\text{tr}(\partial_\mu \varphi_1 \partial^\mu \varphi_1) = \partial_\mu \xi \partial^\mu \xi + \text{tr}(\partial_\mu \tilde{\varphi}_1 \partial^\mu \tilde{\varphi}_1),$$

so the free Lagrangian (S.59) becomes

$$\begin{aligned} \mathcal{L}_2 = & \frac{1}{2}(\partial_\mu \xi)^2 - 2(\beta C^2 N + \alpha C^2) \times \frac{1}{2}\xi^2 \\ & + \frac{1}{2} \text{tr}((\partial_\mu \tilde{\varphi}_1)^2) - 2\alpha C^2 \times \frac{1}{2} \text{tr}(\tilde{\varphi}_1^2) \\ & + \frac{1}{2} \text{tr}((\partial_\mu \varphi_2)^2) - 0 \times \frac{1}{2} \text{tr}(\varphi_2^2) \end{aligned} \quad (\text{S.62})$$

where all the masses are manifest.

Problem 2(e):

The unbroken $SU(N)$ symmetry acts on the scalar fields according to

$$\Phi(x) \rightarrow U \times \Phi(x) \times U^\dagger. \quad (\text{S.47})$$

and since the VEV (10) is invariant, the shifted fields $\delta\Phi(x) = \Phi(x) - \langle \Phi \rangle$ also transform according to

$$\delta\Phi(x) \rightarrow U \times \delta\Phi(x) \times U^\dagger. \quad (\text{S.63})$$

Moreover, unitary transforms like these preserve hermiticity, so when we decompose $\delta\Phi(x)$ into a hermitian matrix $\varphi_1(x)$ and an antihermitian matrix $i\varphi_2(x)$, the transforms (S.63) do not mix the φ_1 and φ_2 with each other. Instead, they transform like

$$\varphi_1(x) \rightarrow U \varphi_1(x) U^\dagger, \quad \varphi_2(x) \rightarrow U \varphi_2(x) U^\dagger, \quad (\text{S.64})$$

which means that φ_1 and φ_2 comprise separate $SU(N)$ multiplets. Furthermore, the transforms (S.64) preserve traces $\text{tr}(\varphi_1)$ and $\text{tr}(\varphi_2)$, so to make the $SU(N)$ multiplet structure manifest, let's decompose both φ_1 and φ_2 into their traceless parts and pure traces along the

lines of eq. (S.60),

$$\varphi_1(x) = \frac{\xi_1(x)}{\sqrt{N}} \times \mathbf{1}_{N \times N} + \tilde{\varphi}_1(x), \quad \varphi_2(x) = \frac{\xi_2(x)}{\sqrt{N}} \times \mathbf{1}_{N \times N} + \tilde{\varphi}_2(x), \quad \text{tr}(\tilde{\varphi}_1) \equiv \text{tr}(\tilde{\varphi}_2) \equiv 0. \quad (\text{S.65})$$

With this decomposition, the ξ_1 and the ξ_2 are both invariant under the $SU(N)$ — which puts each of them into its own singlet multiplet — while the each of the traceless parts $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ makes each own *adjoint multiplet*.

This multiplet structure agrees with the masses we obtained in part (d). Indeed, all $N^2 - 1$ members of the adjoint multiplet $\tilde{\varphi}_1$ have the same mass $2\alpha C^2$, while the singlet ξ_1 has a different mass $2(\alpha + N\beta)C^2$.

On the other hand, both the adjoint multiplet $\tilde{\varphi}_2$ and the singlet ξ_2 are massless. The reason for this degeneracy goes beyond the un-broken $SU(N)$ symmetry; instead, both the $\tilde{\varphi}_2$ and the ξ_2 are Goldstone bosons of the spontaneously broken symmetries in

$$G/H = \left(SU(N)_L \times SU(N)_R \times U(1) \right) / SU(N). \quad (\text{S.66})$$

Specifically, the singlet ξ_2 is the Goldstone boson of the broken $U(1)$ symmetry. Indeed, the $U(1)$'s generator commutes with all the other generators, so it belongs in its own singlet of the symmetry, and the corresponding Goldstone particle should also be a singlet.

Now consider the non-abelian generators. Generators T_L^a of the $SU(N)_L$ form an adjoint multiplet of the $SU(N)_L$, but are invariant under the $SU(N)_R$. Likewise, generators T_R^a of the $SU(N)_R$ form an adjoint multiplet of the $SU(N)_R$, but are invariant under the $SU(N)_L$. In other words, under an $(U_L, U_R) \in SU(N)_L \times SU(N)_R$ they transform as

$$T_L^a \rightarrow U_L T_L^a U_L^\dagger, \quad T_R^a \rightarrow U_R T_R^a U_R^\dagger. \quad (\text{S.67})$$

When the $SU(N)_L \times SU(N)_R$ is broken down to a single $SU(N)$ spanning $U_L = U_R = U$, both T_L^a and T_R^a transform as

$$T_L^a \rightarrow U T_L^a U^\dagger, \quad T_R^a \rightarrow U T_R^a U^\dagger, \quad (\text{S.68})$$

which puts them into *two adjoint multiplets* of the unbroken $SU(N)$. Equivalently, we may

form two adjoint multiplets out of

$$T_V^a = T_L^a + T_R^a \quad \text{and} \quad T_A^a = T_L^a - T_R^a, \quad (\text{S.69})$$

which act on the scalar fields according to

$$T_V^a \Phi = \frac{i}{2} [\lambda^a, \Phi], \quad T_A^a \Phi = \frac{i}{2} \{\lambda^a, \Phi\}. \quad (\text{S.70})$$

The T_V^a generate the unbroken $SU(N)$ symmetry, *cf.* eq. (S.47). The T_A^a generators are spontaneously broken, hence there should be an adjoint multiplet of massless Goldstone bosons. And indeed there is — the $\tilde{\varphi}_2$.