

Textbook problem 10.2(a):

Let us start with the superficial degree of divergence  $\mathcal{D}$ . At large momenta, bosonic propagators behave as  $1/q^2$  while fermionic propagators behave as  $1/q$ , hence in 4 dimensions

$$\mathcal{D} = 4L - 2P_B - P_F. \quad (\text{S.1})$$

As in the  $\lambda\phi^4$  theory, we can relate this expression to the numbers of external legs using the vertex valences. Naively, the Yukawa theory has only one vertex type — involving one bosonic line and two fermionic lines — but we shall see that renormalization generally produces an additional four-boson vertex of the  $\lambda\phi^4$  type. Denoting the respected numbers of the two vertex types  $V_Y$  and  $V_\lambda$ , we have

$$\begin{aligned} 2P_F + E_F &= 2V_Y, \\ 2P_B + E_B &= V_Y + 4V_\lambda, \end{aligned} \quad (\text{S.2})$$

while the Euler formula says

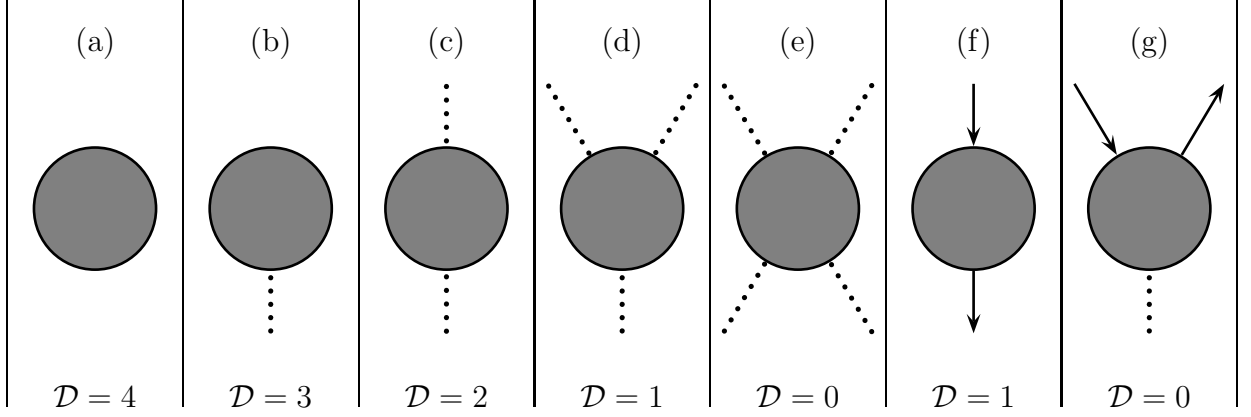
$$L - P + V \equiv L - P_B - P_F + V_Y + V_\lambda = 1, \quad (\text{S.3})$$

Combining these three equations, we obtain

$$\begin{aligned} \mathcal{D} &= 4L - 2P_B - P_F = 4(L - P_B - P_F) + 3P_F + 2P_B \\ &= 4(1 - V_Y - V_\lambda) + \frac{3}{2}(2V_Y - E_F) + (V_Y + 4V_\lambda - E_B) \\ &= 4 - \frac{3}{2}E_F - E_B. \end{aligned} \quad (\text{S.4})$$

Thus, the external legs of a diagram completely determine its superficial degree of divergence.

Consequently, for any number of loops, there are only seven superficially divergent amplitudes, namely



Furthermore, the amplitude (a) here is the vacuum energy while the amplitudes (b) and (d) vanish because of the parity symmetry. Indeed, the *pseudo*-scalar field  $\Phi$  is parity-odd, hence the amplitudes involving odd number of pseudoscalar particles and no fermions must have parity-odd dependence on the particles' momenta. But to construct a parity-odd Lorentz-invariant combination of the Lorentz vectors  $p_1^\alpha, p_2^\beta, \dots$ , one needs  $\epsilon$  tensors, *e.g.*  $\epsilon_{\alpha\beta\gamma\delta} p_1^\alpha p_2^\beta p_3^\gamma p_4^\delta$ , which requires at least 4 linearly independent momenta (in  $d = 4$  spacetime) and hence  $n \geq 5$  external legs. For the amplitudes (b) and (d) involving one or three pseudoscalars only and no fermions, such construction is not available and the amplitudes vanish identically.

Unlike QED, the Yukawa theory does not give rise to Ward identities, so any 1PI amplitude that can diverge generally does. Hence, expanding the 1PI amplitudes (c), (e), (f), and (g) in powers of relevant momenta we find the following independent divergences:

$$(c) \quad \Sigma_\phi(p^2) = O(\Lambda^2) \times \text{const} + O(\log \Lambda) \times p^2 + \text{finite};$$

$$(e) \quad V(k_1, \dots, k_4) = O(\log \Lambda) \times \text{const} + \text{finite};$$

$$(f) \quad \Sigma_\psi(\not{p}) = O(\Lambda^1) \times \text{const} + O(\log \Lambda) \times \not{p} + \text{finite};$$

$$(g) \quad \Gamma^5(p', p) = \gamma^5 \times O(\log \Lambda) \times \text{const} + \text{finite}.$$

To cancel all these divergences *in situ* in the renormalized perturbation theory, we need four

counterterm-related Feynman vertices, namely

$$\begin{aligned}
\text{---} \bullet \text{---} &= -i\delta_m^\phi + ip^2 \delta_Z^\phi, \\
\text{---} \times \text{---} &= -i\delta_\lambda, \\
\text{---} \rightarrow \bullet \rightarrow &= -i\delta_m^\psi + i \not{p} \delta_Z^\psi, \\
\text{---} \bullet \text{---} &= -\delta_g \gamma^5
\end{aligned} \tag{S.5}$$

Clearly, all these vertices follow from local (in  $x$ ) counterterms in the renormalized Lagrangian, specifically

$$\mathcal{L}_{\text{terms}}^{\text{counter}} = \frac{1}{2}\delta_Z^\phi (\partial\Phi)^2 - \frac{1}{2}\delta_m^\phi \Phi^2 - \frac{1}{4!}\delta_\lambda \Phi^4 + i\delta_Z^\psi \bar{\Psi} \not{\partial} \Psi - \delta_m^\psi \bar{\Psi} \Psi - i\delta_g \Phi \bar{\Psi} \gamma^5 \Psi. \tag{S.6}$$

In order to produce such counterterms, one starts from the bare Lagrangian

$$\mathcal{L}_{\text{bare}} = \frac{1}{2}(\partial\Phi_b)^2 - \frac{1}{2}m_b^2 \Phi_b^2 - \frac{1}{4!}\lambda_b \Phi_b^4 + \bar{\Psi}_b(i\not{\partial} - M_b)\Psi_b - ig_b \Phi_b \bar{\Psi}_b \gamma^5 \Psi_b, \tag{S.7}$$

renormalizes the bare fields  $\Phi_b(x) = \sqrt{Z_\phi} \Phi_r(x)$ ,  $\Psi_b(x) = \sqrt{Z_\psi} \Psi_r(x)$ , and splits the Lagrangian into

$$\mathcal{L}_{\text{bare}} = \mathcal{L}^{\text{phys}} + \mathcal{L}_{\text{terms}}^{\text{counter}} \tag{S.8}$$

where

$$\mathcal{L}^{\text{phys}} = \frac{1}{2}(\partial\Phi_r)^2 - \frac{1}{2}m_{\text{phys}}^2 \Phi_r^2 - \frac{1}{4!}\lambda_{\text{phys}} \Phi_r^4 + \bar{\Psi}_r(i\not{\partial} - M_{\text{phys}})\Psi_r - ig_{\text{phys}} \Phi_r \bar{\Psi}_r \gamma^5 \Psi_r, \tag{S.9}$$

the counterterms are exactly as in eq. (S.6) (where  $\Phi \equiv \Phi_r$  and  $\Psi \equiv \Psi_r$ ), and the coefficients are

$$\begin{aligned}
\delta_Z^\phi &= Z_\phi - 1, & \delta_Z^\psi &= Z_\psi - 1, & \delta_m^\phi &= Z_\phi m_b^2 - m_{\text{phys}}^2, & \delta_m^\psi &= Z_\psi M_b - M_{\text{phys}}, \\
\delta_\lambda &= Z_\phi^2 \lambda_b - \lambda_{\text{phys}}, & \text{and } \delta_g &= Z_\psi Z_\phi^{1/2} g_b - g_{\text{phys}}.
\end{aligned}$$

In part (b) of this problem (to be assigned in the [next homework set](#)) we shall see that at the one-loop level of the theory we already need all of the counterterms (S.6). In particular,

even if we start with  $\lambda_{\text{phys}} = 0$ , we still need the  $\delta_\lambda$  counterterm to cancel the divergences of the fermionic loop diagrams

+ five similar. (S.10)

Thus, from the bare Lagrangian point of view,  $\lambda_{\text{phys}} = 0$  has no special meaning: the bare coupling  $\lambda_b \neq 0$ , so vanishing of a particular scattering amplitude we use to define the physical  $\lambda$  would be just an accident. In other words, we may *fine tune*  $\lambda_b$  to achieve  $\lambda_{\text{phys}} = 0$  just as we can fine tune  $\lambda_b$  to achieve any other experimental value of the physical coupling, but it would not have any special meaning for the theory itself.

This is an example of the general rule: *barring fine tuning of the coupling parameters, a renormalizable quantum field theory has all the renormalizable couplings consistent with the theory's symmetries.* For the theory at hand, we have a Dirac field  $\Psi$ , a real pseudoscalar field  $\Phi$ , and all the Lagrangian terms involving these fields should be invariant under Lorentz and parity transformations and have canonical dimensions  $\leq 4$  (for renormalizability's sake). There is only a finite number of such terms, and it is easy to see that the Lagrangian (S.9) comprises all such terms and no others. Consequently, the renormalized theory would not have any additional interactions.

Sometimes, in absence of some coupling the theory has an additional symmetry that would not be present otherwise. In such case, the extra symmetry would prevent such coupling from being restored by the renormalization procedure. For example, consider the Lagrangian (S.9) for  $g = 0$  (but  $\lambda \neq 0$ ): In the absence of the Yukawa coupling, the theory has an extra symmetry  $\Phi(x) \rightarrow -\Phi(x)$  (without parity), and this extra symmetry would prevent the renormalization procedure from restoring the Yukawa coupling. On the other hand, when  $\lambda = 0$  but  $g \neq 0$ , the theory does not have any additional symmetries it wouldn't have for  $\lambda \neq 0$ , and that's why the renormalization gives rise to the  $\lambda\Phi^4$  coupling even if it wasn't there to begin with.