Problem **1**(a):

Classically,

$$\mathcal{L} \;=\; \mathcal{L}_{\rm YM} \;+\; D^\mu\Phi^\dagger D_\mu\Phi \;-\; V(\Phi^\dagger,\Phi) \qquad\text{(S.1)}$$

where

$$D_\mu\Phi^i \;=\; \partial_\mu\Phi_i \;+\; igA^a_\mu(T^a_{(r)})^i{}_j\Phi^j\,, \qquad D^\mu\Phi^*_i \;=\; \partial^\mu\Phi^*_i \;-\; igA^{a\mu}\Phi^*_j(T^a_{(r)})^j{}_i, \qquad\text{(S.2)}$$

and $V(\Phi^\dagger,\Phi)$ is some kind of a G–invariant potential. For renormalizability's sake, V should be a polynomial of degree 4 (or less), and to keep my notations simple I assume that V has only the quadratic mass term and the quartic interaction term, thus

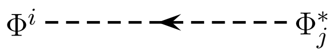
$$V \;=\; m^2\times\Phi^*_i\Phi^i \;+\; \tfrac14\lambda^{ij}_{k\ell}\times\Phi^*_i\Phi^*_j\Phi^k\Phi^\ell \qquad\text{(S.3)}$$

for a suitable G-invariant coupling array $\lambda^{ij}_{k\ell}$. For example, for scalars Φ^i in the fundamental $\mathbf{N}$ multiplet of the $SU(N)$ gauge group, $\lambda^{ij}_{k\ell}=\lambda\times\left(\delta^i_k\delta^j_\ell+\delta^i_\ell\delta^j_k\right)$. But the details of the $\lambda^{ij}_{k\ell}$ coupling are not germane for the present problem, so I'll keep them generic as long as they are G-invariant.

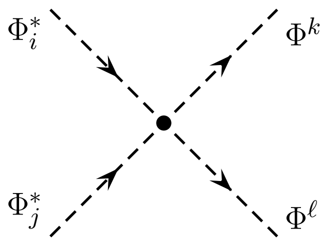
In the quantum field theory, the net bare Lagrangian comprises the classical terms (S.1) plus the ghost Lagrangian, the gauge fixing terms, and the whole slew of counterterms. Altogether, we have

$$\begin{aligned}
\mathcal{L} \;=\; & -\frac14(F^a_{\mu\nu})^2 \;-\; \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \;+\; \partial_\mu\bar c^a D^\mu c^a \\
&\quad +\; D_\mu\Phi^\dagger D^\mu\Phi \;-\; m^2\Phi^\dagger\Phi \;-\; \tfrac14\lambda^{ij}_{k\ell}\,\Phi^*_i\Phi^*_j\Phi^k\Phi^\ell \\
&-\; \frac{\delta_3}{4}(\partial_\mu A^a_\nu-\partial_\nu A^a_\mu)^2 \;+\; g\delta_1^{(3g)}f^{abc}A^b_\mu A^c_\nu\partial_\mu A^{a\nu} \;-\; \frac{g^2\delta_1^{(4g)}}{4}(f^{abc}A^b_\mu A^c_\nu)^2 \\
&+\; \delta_2^{(\rm gh)}\partial_\mu\bar c^a\partial^\mu c^a \;-\; g\delta_1^{(\rm gh)}f^{abc}\partial_\mu\bar c^a A^{b\mu}c^c \\
&+\; \delta_2^{(\phi)}\partial_\mu\Phi^\dagger\partial^\mu\Phi \;+\; ig\delta_1^{(\phi1g)}\,A^a_\mu\times\left(\partial^\mu\Phi^\dagger T^a_{(r)}\Phi \;-\; \Phi^\dagger T^a_{(r)}\partial^\mu\Phi\right) \\
&\quad +\; g^2\delta_1^{(\phi2g)}A^a_\mu A^b\mu\times\Phi^\dagger T^a_{(r)}T^b_{(r)}\Phi \\
&-\; \delta_m^{(\phi)}\Phi^\dagger\Phi \;-\; \tfrac14(\delta_\lambda)^{ij}_{k\ell}\,\Phi^*_i\Phi^*_j\Phi^k\Phi^\ell.
\end{aligned} \qquad\text{(S.4)}$$

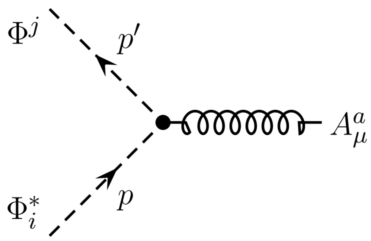
Note that all the terms in this bare Lagrangian which pertain only to the vector and ghost fields are exactly the same as in the fermionic QCD, *cf.* my notes on QCD Feynman rules. Consequently, in the Feynman rules of the present theory, the the 'gluon' propagator, the three-gluon and the four-gluon vertices, the ghost propagator and the ghost-gluon vertex are exactly as in my notes, and I don't need to repeat them here. But let me write doen the explicit Feynman rules pertaining to the scalar fields, in particular, the scalar propagator and the scalar vertices:



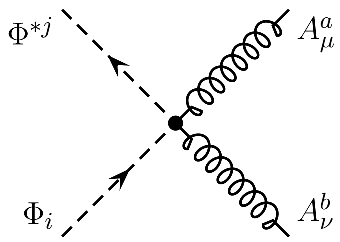
$$= \frac{i\delta^i_j}{p^2 - m^2 + i0}\,,$$



$$= -i\lambda^{ij}_{k\ell}\,,$$



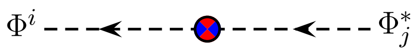
$$= ig(p+p')_\mu (T^a_{(r)})^j{}_i,$$



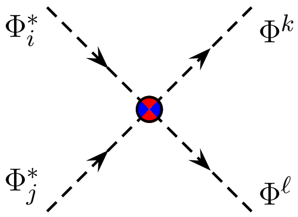
$$= ig^2 g_{\mu\nu}\{T^a_{(r)}, T^b_{(r)}\}^j{}_i.$$

Note the anticommutator of the group generators in the two-scalar two-gluon vertex: It follows from permutations of the two gluon lines.

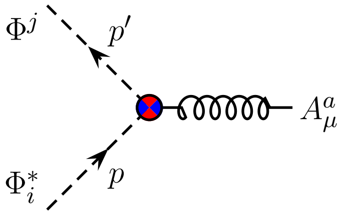
In addition, there are several counterterm vertices involving the scalar fields. Although such vertices are not germane to the present exercise, let me list them here for the completeness sake:



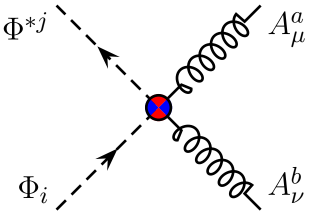
$$= \delta^i_j\bigl(i\delta^{(\phi)}_m - i\delta^{(\phi)}_2 p^2\bigr)$$



$$= \; -i\big(\delta_\lambda\big)^{ij}_{k\ell}\,,$$



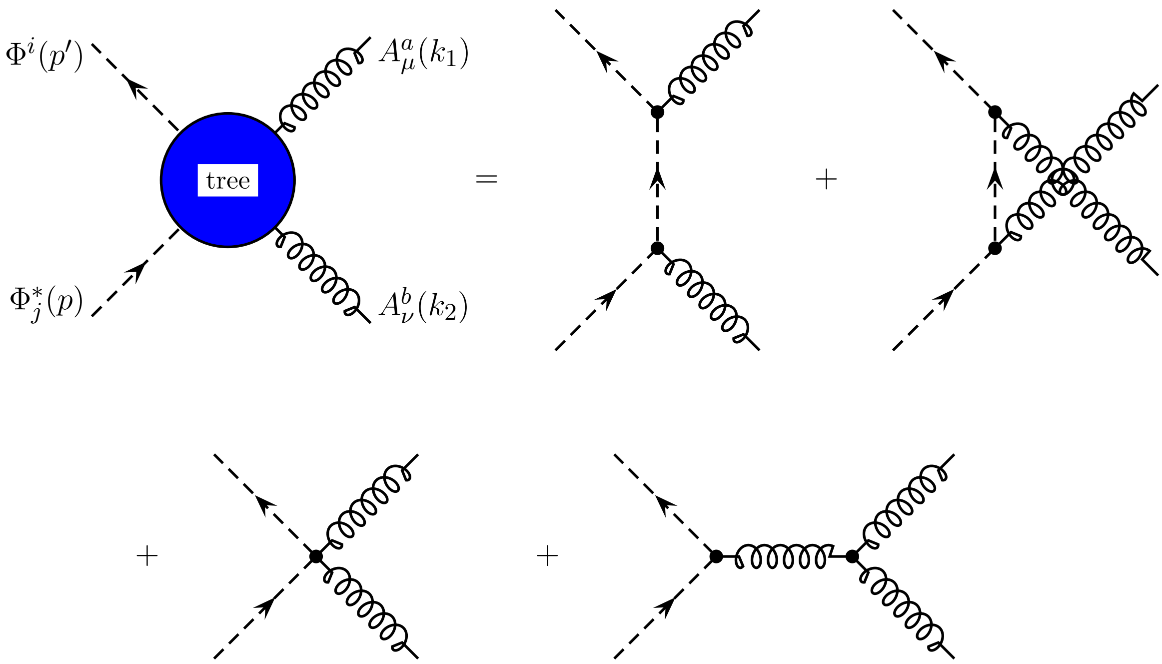
$$= \; ig\delta_1^{(\phi 1g)} \times (p+p')_\mu (T^a_{(r)})^j{}_i,$$



$$= \; ig^2\delta_1^{(\phi 2g)} \times g_{\mu\nu}\{T^a_{(r)}, T^b_{(r)}\}^j{}_i.$$

Problem **1**(b):

At the tree level there are four diagrams for the $\Phi + \Phi^* \to g + g$ annihilation process, namely



(S.5)

The amplitude stemming from each of these 4 diagrams has form

$$\mathcal{M}[\text{diagram}\#n] \;=\; e^*_{1\mu}\, e^*_{2\nu} \times \mathcal{M}^{\mu\nu}_n \tag{S.6}$$

where

$$\mathcal{M}^{\mu\nu}_1 \;=\; \frac{-g^2}{(p-k_1)^2 - m^2}\,(2p-k_1)^\mu (k_2 - 2p')^\nu \times (T^b T^a)^i{}_j\,, \tag{S.7}$$

$$\mathcal{M}^{\mu\nu}_2 \;=\; \frac{-g^2}{(p'-k_1)^2 - m^2}\,(k_1-2p')^\mu (2p - k_2)^\nu \times (T^a T^b)^i{}_j\,, \tag{S.8}$$

$$\mathcal{M}^{\mu\nu}_3 \;=\; +g^2 g^{\mu\nu} \times \{T^a, T^b\}^i{}_j\,, \tag{S.9}$$

$$\begin{aligned}\mathcal{M}^{\mu\nu}_4 \;=\; & -\frac{ig^2}{(k_1+k_2)^2}\,(p-p')_\lambda\,(T^c)^i{}_j \\ & \times f^{abc}\big(g^{\mu\nu}(k_1-k_2)^\lambda \,+\, g^{\nu\lambda}(2k_2+k_1)^\mu \,+\, g^{\lambda\mu}(-2k_1-k_2)^\nu\big),\end{aligned} \tag{S.10}$$

so the net tree-level amplitude is

$$\mathcal{M}_{\text{net}} \;=\; e^*_{1\mu}\, e^*_{2\nu} \times \mathcal{M}^{\mu\nu}_{\text{net}} \;=\; e^*_{1\mu}\, e^*_{2\nu} \times \big(\mathcal{M}^{\mu\nu}_1 \,+\, \mathcal{M}^{\mu\nu}_2 \,+\, \mathcal{M}^{\mu\nu}_3 \,+\, \mathcal{M}^{\mu\nu}_4\big) \tag{S.11}$$

Problem **1**(c):

Our task is to verify that the net amplitude (S.11) satisfies

$$k^\mu_1 e^\nu_2 \mathcal{M}_{\mu\nu} \;=\; 0 \tag{S.12}$$

provided $e^\nu_2 k_{2\nu} = 0$ and all external momenta are on shell. Let's start by calculating the $k_{1\mu}\mathcal{M}^{\mu\nu}_n$ for each of the 4 diagrams. For the first diagram's amplitude (S.7),

$$k_{1\mu}\mathcal{M}^{\mu\nu}_1 \;=\; -g^2(k_2 - 2p')^\nu \times \frac{2(pk_1) - k_1^2}{(p-k_1)^2 - m^2} \times (T^b T^a)^j{}_i \tag{S.13}$$

where for the on-shell momenta $p^2 = p'^2 = m^2$, $k_1^2 = k_2^2 = 0$,

$$\frac{2(pk_1) - k_1^2}{(p-k_1)^2 - m^2} \;=\; -1 \tag{S.14}$$

and hence

$$k_{1\mu}\mathcal{M}_1^{\mu\nu} \;=\; +g^2(k_2-2p')^\nu \times (T^bT^a)^j{}_i\,. \tag{S.15}$$

Likewise, for the second diagram's amplitude (S.8),

$$\begin{aligned} k_{1\mu}\mathcal{M}_2^{\mu\nu} \;&=\; -g^2(2p-k_2)^\nu \times \frac{k_1^2-2(p'k_1)}{(p'-k_1)^2-m^2} \times (T^aT^b)^j{}_i \\ &\qquad \langle\!\langle \text{ for the on-shell momenta } \rangle\!\rangle \\ &=\; +g^2(k_2-2p)^\nu \times 1 \times (T^aT^b)^i{}_j\,. \end{aligned} \tag{S.16}$$

For the third diagram's amplitude (S.9) we have

$$k_{1\mu}\mathcal{M}_3^{\mu\nu} \;=\; +g^2k_1^\nu \times \{T^a,T^b\}^i{}_j \;=\; g^2k_1^\nu \times (T^aT^b)^i{}_j \;+\; g^2k_1^\nu \times (T^bT^a)^i{}_j\,, \tag{S.17}$$

so adding the first three diagrams together, we ontain

$$\begin{aligned} k_{1\mu}\times\mathcal{M}_{1+2+3}^{\mu\nu} \;&=\; g^2(T^aT^b)^i{}_j \times \bigl((k_2-2p')\,+\,k_1\bigr)^\nu \;+\; g^2(T^bT^a)^i{}_j \times \bigl((k_2-2p)\,+\,k_1\bigr)^\nu \\ &\qquad \langle\!\langle \text{ using momentum conservation } k_1+k_2=p+p' \,\rangle\!\rangle \\ &=\; g^2(T^aT^b)^i{}_j \times (p-p')^\nu \;+\; g^2(T^bT^a)^i{}_j \times (p'-p)^\nu \\ &=\; g^2(p-p')^\nu \times (T^aT^b\,-\,T^bT^a)^i{}_j \\ &=\; g^2(p-p')^\nu \times if^{abc}(T^c)^i{}_j\,. \end{aligned} \tag{S.18}$$

As to the fourth diagram's amplitude (S.10),

$$\begin{aligned} k_{1\mu}\mathcal{M}_4^{\mu\nu} \;&=\; g^2(p'-p)_\lambda \times if^{abc}(T^c)^i{}_j \times \frac{1}{(k_1+k_2)^2} \times \\ &\qquad \times k_{1\mu}\bigl[g^{\mu\nu}(k_1-k_2)^\lambda \;+\; g^{\nu\lambda}(2k_2+k_1)^\mu \;+\; g^{\lambda\mu}(-2k_1-k_2)^\nu\bigr], \end{aligned} \tag{S.19}$$

where the expression on the second line is exactly similar to its analogue in the fermionic QCD, *cf.* eqs. (28–29) on page 7 of my notes on QCD Feynman rules and Ward identities.

Just as in my notes, for the on-shell photon momenta

$$k_{1\mu} \times [\cdots] \;=\; (k_1+k_2)^2 g^{\nu\lambda} \;-\; (k_1+k_2)^\nu (k_1+k_2)^\lambda \;+\; k_2^\nu k_2^\lambda\,. \tag{S.20}$$

hence plugging each of the 3 terms here into eq. (S.19), we obtain

$$k_{1\mu}\mathcal{M}_4^{\mu\nu} \;=\; k_{1\mu}\mathcal{M}_{4,a}^{\mu\nu} \;+\; k_{1\mu}\mathcal{M}_{4,a}^{\mu\nu} \;+\; k_{1\mu}\mathcal{M}_{4,a}^{\mu\nu}\,, \tag{S.21}$$

where

$$k_{1\mu}\mathcal{M}_{4,a}^{\mu\nu} \;=\; g^2(p'-p)^\nu \times if^{abc}(T^c)^i{}_j\,, \tag{S.22}$$

$$k_{1\mu}\mathcal{M}_{4,b}^{\mu\nu} \;=\; -g^2(k_1-k_2)^\nu \times \frac{(p'-p)_\lambda)(k_1+k_2)^\lambda}{(k_1+k_2)^2} \times if^{abc}(T^c)^i{}_j\,, \tag{S.23}$$

$$k_{1\mu}\mathcal{M}_{4,c}^{\mu\nu} \;=\; g^2 k_2^\nu \times \frac{(p'-p)_\lambda)k_2^\lambda}{(k_1+k_2)^2} \times if^{abc}(T^c)^i{}_j\,. \tag{S.24}$$

By instection of eqs. (S.22) and (S.18), the first term's contribution precisely cancels the combined contriibutions of the diagrams 1, 2, and 3,

$$k_{1\mu}\mathcal{M}_{4,a}^{\mu\nu} \;+\; k_{1\mu}\mathcal{M}_{1+2+3}^{\mu\nu} \;=\; 0. \tag{S.25}$$

As to the second term's contribution (S.23), it vanishes for the on-shell scalars' momenta $p^2 = p'^2 = m^2$; indeed,

$$(p'-p)_\lambda)(k_1+k_2)^\lambda \;=\; (p'-p)_\lambda(p+p')^\lambda \;=\; p'^2 - p^2 \;=\; m^2 - m^2 \;=\; 0 \quad\Longrightarrow\quad k_{1\mu}\mathcal{M}_{4,b}^{\mu\nu} \;=\; 0. \tag{S.26}$$

As to the thir term's contribution (S.24), it does not vanish but its ν index belongs to the k_2^ν factor, thus

$$k_{1\mu}\mathcal{M}_{\text{net}}^{\mu\nu} \;=\; k_{1\mu}\mathcal{M}_{4,c}^{\mu\nu} \;=\; [\text{stuff}] \times k_2^\nu\,. \tag{S.27}$$

Consequently, when the net amplitude is contracted with the polarization vector $e^*_{2\nu}$ of the second gluon, it vanishes when the second gluon is transversely polarized, $k_2 e_2^* = 0$, but not if the other gluon's polarization is longitudinal. And this is in accordance to the weak form of Ward Identity: *On-shell amplitudes involving one longitudinal gluon vanish, but only if all the other gluons are transverse.*

Problem **2**(a):

Let us evaluate the trace of the Casimir operator C_2 over an irreducible multiplet (r). On one hand,

$$\begin{aligned}
\operatorname{tr}_{(r)}\left(C_2 \stackrel{\text{def}}{=} \sum_a T^a T^a\right) &= \sum_a \operatorname{tr}_{(r)}\left(T^a T^a\right) \ = \sum_a \operatorname{tr}\left(T^a_{(r)} T^b_{(r)}\right) \\
\langle\!\langle \text{by eq. (1)} \rangle\!\rangle \quad &= \sum_a R(r) \times \left(\delta^{aa} = 1\right) \ = \ R(r) \times \dim(G)
\end{aligned} \tag{S.28}$$

where $\dim(G) \stackrel{\text{def}}{=} \dim(\mathrm{Adj}(G))$ is the number of the generators of the symmetry group G — which is also the dimension of the adjoint representation of G, hence the notation. On the other hand,

$$\operatorname{tr}_{(r)}(C_2) \ = \ \operatorname{tr}_{(r)}\left(C_2\big|_{(r)}\right) \ = \ \operatorname{tr}\left(C(r) \times \mathbf{1}_{(r)}\right) \ = \ C(r) \times \dim(r). \tag{S.29}$$

Together, eqs. (S.28) and (S.29) immediately imply eq. (3), *Quod erat demonstrandum.*.

For the special case of $G = SU(2)$, an irreducible multiplets of isospin I has $C = \mathbf{I}^2 = I(I+1)$ and dimension $2I+1$, hence

$$R(I) \ = \ C(I) \times \frac{\dim(I)}{\dim(G)} \ = \ I(I+1) \times \frac{2I+1}{3}\,. \tag{S.30}$$

Problem **2**(b):

Unlike the Casimir value $C(r)$, the index $R(r)$ is well defined for any complete multiplet (r), irreducible or otherwise. For a reducible multiplet

$$(r) \ = \ \bigoplus_{i=1}^{n} (r_i) \ \equiv \ (r_1) \oplus (r_2) \oplus \cdots \oplus (r_n)$$

one has

$$\begin{aligned}
\operatorname{tr}_{(r)}\left(T^a T^b\right) \ &= \ \operatorname{tr}\left(T^a T^b\Big|_{\bigoplus_{i=1}^n (r_i)}\right) \ = \sum_{i=1}^{n} \operatorname{tr}\left(T^a T^b\Big|_{(r_i)}\right) \\
&= \sum_{i=1}^{n} \left(R(r_i) \times \delta^{ab}\right) \ = \ \delta^{ab} \times \sum_{i=1}^{n} R(r_i)
\end{aligned} \tag{S.31}$$

and thus

$$R(r) \;=\sum_{i=1}^{n} R(r_i). \tag{S.32}$$

In particular, a reducible multiplet

$$(r) \;=\; \bigoplus_{i=1}^{n} (I_i)$$

of the isospin group $SU(2)$ has index

$$R(r) \;=\sum_{i=1}^{n} \tfrac{1}{3} I(I+1)(2I+1). \tag{S.33}$$

Now consider a bigger symmetry group G which contains the 'isospin' $SU(2)$ as a subgroup. Then any complete multiplet (r) of G is automatically a complete multiplet of the $SU(2) \subset G$. However, irreducible multiplets of G usually become reducible from the $SU(2)$ point of view, $(r) = (I_1) \oplus (I_2) \oplus \cdots \oplus (I_n)$; for example, the adjoint multiplet of $SU(3)$ decomposes into $(0) \oplus (\frac{1}{2}) \oplus (\frac{1}{2}) \oplus (1)$ of the $SU(2) \subset SU(3)$. Let T^1, T^2, and T^3 be generators of the $SU(2)$ subgroup of G. Then according to eq. (S.33),

$$\text{for } a, b = 1, 2, 3, \quad \operatorname{tr}_{(r)} \left(T^a T^b \right) \;=\; \delta^{ab} \times \sum_{i=1}^{n} \tfrac{1}{3} I(I+1)(2I+1). \tag{S.34}$$

Now, let us suppose that the Lie group G is *simple*, that is, all its generators are related to each other by the symmetry G itself. In this case, for any complete multiplet (r) of G

$$\operatorname{tr}_{(r)} \left(T^a T^b \right) \;=\; R(r) \times \delta^{ab}, \quad \text{same } R(r)\ \forall a, b = 1, \ldots, \dim(G). \tag{S.35}$$

Combining this formula with eq. (S.34) we immediately obtain

$$R(r) \;=\sum_{i=1}^{n} \tfrac{1}{3} I(I+1)(2I+1), \tag{4}$$

Quod erat demonstrandum..

Caveat: We have silently assumed that $T^{1,2,3}$ have the same normalization as generators of G as they have as generators of the $SU(2) \subset G$. This assumption is correct for the $SU(2) \subset SU(N)$ discussed in parts (c) and (d) of this problem, but it would fail for a different (*i.e.*, inequivalent) $SU(2)$ subgroup. In general, properly normalized $SU(2) \subset G$ generators $I^{1,2,3}$ are related to the properly normalized generators of G as

$$I^a \;=\; T^{(a)} \times \sqrt{k} \tag{S.36}$$

where $T^{(1)}$, $T^{(2)}$, and $T^{(3)}$ are 3 generators of G which happen to satisfy $[T^{(a)}, T^{(b)}] = i\epsilon^{abc}T^{(c)}/\sqrt{k}$. The k here is always a positive integer; it's called *the level of embedding of the $SU(2)$ into G*. For example, consider the $SU(2)$ subgroup of $SU(3)$ which acts on the fundamental triplet as a real $SO(3)$ rotation. This subgroup is generated by the

$$I^1 = \sqrt{4}\times T^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & +i \\ 0 & -i & 0 \end{pmatrix}, \quad I^2 = \sqrt{4}\times T^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ +i & 0 & 0 \end{pmatrix}, \quad I^3 = \sqrt{4}\times T^2 = \begin{pmatrix} 0 & +i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{S.37}$$

(note $T^a = \frac{1}{2}\lambda^a$), so its embedding level is $k = 4$.

When you decompose a multiplet (r) of G into irreducible multiplets of an $SU(2)$ subgroup, you should take into account the level at which this $SU(2)$ is embedded into G. As written, eq. (4) works only for the $k = 1$ subgroups; for other embedding levels,

$$R(r) \;=\; \frac{1}{k}\sum_{i=1}^{n} \tfrac{1}{3} I(I+1)(2I+1). \tag{S.38}$$

Note that the decomposition of the G multiplet (r) into $SU(2)$ multiplets depends on the $SU(2)$ embedding into G. For example, under the $k = 1$ subgroup $SU(2) \subset SU(3)$

$$\text{triplet} \;=\; (\tfrac{1}{2}) \oplus (0), \qquad \text{octet} \;=\; (1) \oplus (\tfrac{1}{2}) \oplus (\tfrac{1}{2}) \oplus (0), \tag{S.39}$$

while under the $k = 4$ subgroup (S.37)

$$\text{triplet} \;=\; (1), \qquad \text{octet} \;=\; (1) \oplus (2). \tag{S.40}$$

In both cases, eq. (S.38) produces the same index R for each $SU(3)$ multiplet, for example

$R(\text{triplet}) = \frac{1}{2}$ and $R(\text{octet}) = 3$, but only if you remember the $1/k$ factor in front of the sum.

Problem **2**(c):

From the $SU(2) \subset SU(N)$ point of view, the fundamental representation $\mathbf{N}$ of the $SU(N)$ decomposes into one doublet plus $(N-2)$ singlets,

$$\mathbf{N} \;=\; \mathbf{2} \;+\; (N-2)\times\mathbf{1} \;\equiv\; (I=\tfrac{1}{2}) \;+\; (N-2)\times(I=0), \tag{S.41}$$

hence according to eq. (4),

$$R(\mathbf{N}) \;=\; R(I=\tfrac{1}{2}) \;+\; (N-2)\times R(I=0) \;=\; \tfrac{1}{2} \;+\; (N-2)\times 0 \;=\; \tfrac{1}{2}$$

and consequently

$$C(\mathbf{N}) \;=\; R(\mathbf{N})\times\frac{\dim(G)}{\dim(\mathbf{N})} \;=\; \frac{1}{2}\times\frac{N^2-1}{N} \;=\; \frac{N^2-1}{2N} \tag{4}$$

Now consider the adjoint representation of the $SU(N)$. Let us form a tensor product of the fundamental representation $\mathbf{N}$ and the conjugate (anti-fundamental) representation $\overline{\mathbf{N}}$. Given the transformation laws

$$\begin{aligned} \Psi \;&\to\; U\Psi, \qquad &&\textit{i.e.}\quad \Psi'_j \;=\; U_j^{\,k}\Psi_k\,, \\ \overline{\Psi} \;&\to\; \overline{\Psi}U^\dagger, \qquad &&\textit{i.e.}\quad \overline{\Psi}'^{\ell} \;=\; \overline{\Psi}^m\, U^{*\ell}_{\;m}\,, \end{aligned}$$

it follows that the tensor product is a hermitian $N\times N$ matrix Φ_j^k which transforms as

$$\Phi' \;=\; U\Phi U^\dagger \qquad \textit{i.e.}\quad \Phi'^{\ell}_j \;=\; U_j^{\,k}\Phi_k^{\,m}U^{*\ell}_{\;m}\,. \tag{5}$$

This matrix is a reducible multiplet $\mathrm{Adj}+\mathbf{1}$ of the $SU(N)$: The trace $\mathrm{tr}(\Phi)$ is an invariant singlet, while the traceless part $\Phi_i^{\,j} - \delta_i^j \times \mathrm{tr}(\Phi)/N$ forms the adjoint multiplet, *cf.*

(homework#11). In other words,

$$\mathbf{N}\otimes\overline{\mathbf{N}}\ =\ \mathrm{Adj}\oplus\mathbf{1} \qquad \text{(S.42)}.$$

In $SU(2)$ $\bar{\mathbf{2}}=\mathbf{2}$, so from the $SU(2)\subset SU(N)$ point of view, both the fundamental and the anti-fundamental multiplets of the $SU(N)$ decompose into similar sets of one doublet and $N-2$ singlets. Therefore,

$$\begin{aligned}
[\mathrm{Adj}\ +\ \mathbf{1}]_{SU(N)}\ &=\ \left[\mathbf{N}\otimes\overline{\mathbf{N}}\right]_{SU(N)}\\
&=\ [\mathbf{2}\ +\ (N-2)\times\mathbf{1}]_{SU(2)}\ \otimes\ [\mathbf{2}\ +\ (N-2)\times\mathbf{1}]_{SU(2)}\\
&=\ \left[(\mathbf{2}\otimes\mathbf{2})\ +\ 2(N-2)\times(\mathbf{2}\otimes\mathbf{1})\ +\ (N-2)^2\times(\mathbf{1}\otimes\mathbf{1})\right]_{SU(2)}\\
&=\ \left[\mathbf{3}\ +\ \mathbf{1}\ +\ 2(N-2)\times\mathbf{2}\ +\ (N-2)^2\times\mathbf{1}\right]_{SU(2)}\,,\\
i.e.,\quad [\mathrm{Adj}]_{SU(N)}\ &=\ \left[\mathbf{3}\ +\ 2(N-2)\times\mathbf{2}\ +\ (N-2)^2\times\mathbf{1}\right]_{SU(2)}\,,
\end{aligned} \qquad \text{(S.43)}$$

and consequently

$$\begin{aligned}
R(\mathrm{Adj})\ &=\ R_{SU(2)}(\mathbf{3})\ +\ 2(N-2)\times R_{SU(2)}(\mathbf{2})\ +\ (N-2)^2\times R_{SU(2)}(\mathbf{1})\\
&=\ 2\ +\ 2(N-2)\times\tfrac{1}{2}\ +\ (N-2)^2\times 0\ =\ N.
\end{aligned} \qquad \text{(S.44)}$$

Finally,

$$C(G)\ \stackrel{\text{def}}{=}\ C(\mathrm{Adj}(G))\ =\ R(\mathrm{Adj})\times\frac{\dim(G)}{\dim(G)}\ =\ R(\mathrm{Adj})\ =N. \qquad \text{(S.45)}$$

Problem **2**(d):

Consider the two-index symmetric tensor $S_{(ij)}$ representation of the $SU(N)$ symmetry group. Denote the index $i=\alpha$ if $i=1,2$ or $i=\mu$ if $i=3,4,\ldots,N$ and likewise $j=\beta$ if $j=1,2$ and $j=\nu$ if $j=3,4,\ldots,N$. Thus, the complete set of independent $S_{(ij)}$ decomposes into $S_{(\alpha\beta)}$, $S_{\alpha,\mu}\equiv S_{\mu,\alpha}$ and $S_{(\mu\nu)}$. The $SU(2)\subset SU(N)$ acts on indices $\alpha,\beta=1,2$ and ignores indices $\mu,\nu=3,4,\ldots,N$, so from the $SU(2)$ point of view, $S_{(\alpha\beta)}$ is a triplet, $S_{\alpha,\mu}$ are $N-2$

separate doublets, and $S_{(\mu\nu)}$ are $(N-2)(N-1)/2$ singlets. Consequently,

$$\begin{aligned} R(S) &= R_{SU(2)}(\mathbf{3}) \;+\; (N-2)\times R_{SU(2)}(\mathbf{2}) \;+\; \tfrac{1}{2}(N-1)(N-2)\times R_{SU(2)}(\mathbf{1}) \\ &= 2 \;+\; (N-2)\times\tfrac{1}{2} \;+\; \tfrac{1}{2}(N-1)(N-2)\times 0 \;=\; \tfrac{1}{2}(N+2), \end{aligned} \tag{S.46}$$

and hence

$$C(S) = R(S)\times\frac{\dim(G)}{\dim(S)} = \frac{N+2}{2}\times\frac{N^2-1}{\frac{1}{2}N(N+1)} = \frac{N^2+N-2}{N}. \tag{S.47}$$

Similarly, the two-index anti-symmetric tensor $A_{[ij]}$ decomposes into $A_{[\alpha\beta]}$, $A_{\alpha,\mu}$, and $A_{[\mu\nu]}$. In $SU(2)$, the $A_{[\alpha\beta]}$ is equivalent to the trivial singlet $A\times\epsilon_{[\alpha\beta]}$, the $A_{\alpha,\mu}$ are $N-2$ doublets, and $A_{[\mu\nu]}$ are $(N-2)(N-3)/2$ singlets. Altogether

$$(A) \;=\; (N-2)\times\mathbf{2} \;+\; \text{singlets},$$

therefore

$$R(A) \;=\; (N-2)\times\tfrac{1}{2} \;+\; 0 \;=\; \tfrac{1}{2}(N-2) \tag{S.48}$$

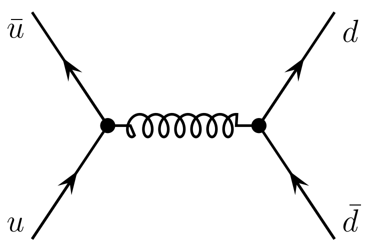
and

$$C(A) = R(A)\times\frac{\dim(G)}{\dim(A)} = \frac{N-2}{2}\times\frac{N^2-1}{\frac{1}{2}N(N-1)} = \frac{N^2-N-2}{N}. \tag{S.49}$$

<u>Problem **3**</u>:

At the tree level of QCD,

$$i\mathcal{M}(u\bar{u}\to d\bar{d}) \;=$$



$$= \frac{ig^2}{s}\times\bar{v}(\bar{u})\gamma^\mu u(u)\,(T^a)^i{}_j\times\bar{u}(d)\gamma_\mu v(d)\,(T^a)^k{}_\ell \tag{S.50}$$

where $s=E^2_{\text{c.m.}}$, the quarks and the antiquarks have color indices i,j,k,ℓ, the virtual gluon has adjoint color index a, and the summation over a is implicit. Except for the color indices, the $u\bar{u} \to d\bar{d}$ process in QCD is completely analogous to the $e^-e^+ \to \mu^-\mu^+$ pair production in QED. In particular, summing / averaging $|\mathcal{M}|^2$ over the fermion's spins yields

$$\begin{aligned} \tfrac{1}{4} \sum_{\text{all spins}} |\bar{v}(\bar{u})\gamma^\mu u(u)\, \bar{u}(d)\gamma_\mu v(d)|^2 \;&\approx\; \tfrac{1}{4}\, \mathrm{tr}(\not{p}_{\bar{u}}\gamma^\mu \not{p}_u \gamma^\nu) \times \mathrm{tr}(\not{p}_{\bar{d}}\gamma_\mu \not{p}_d \gamma_\nu) \\ &=\; 2(t^2+u^2) \;=\; s^2(1+\cos^2\theta_{\text{c.m.}}) \end{aligned} \tag{S.51}$$

where the approximation is neglecting the quark masses m_u and m_d.

The new part of this exercise is summing / averaging over the color indices. By hermiticity of the Lie Algebra matrices T^a, we have

$$\Big((T^a)^i{}_j\,(T^a)^k{}_\ell\Big)^* \;=\; (T^a)^j{}_i\,(T^a)^\ell{}_k \;=\; (T^b)^j{}_i\,(T^b)^\ell{}_k \tag{S.52}$$

— note the implicit summation over a or b — and hence

$$\begin{aligned} \sum_{i,j,k,\ell} \Big|(T^a)^i{}_j\,(T^a)^k{}_\ell\Big|^2 \;&=\; \sum_{i,j,k,\ell} (T^a)^i{}_j\,(T^a)^k{}_\ell \times (T^b)^j{}_i\,(T^b)^\ell{}_k \\ &= \sum_{ij} (T^a)^i{}_j\,(T^b)^j{}_i \times \sum_{k,\ell} (T^a)^k{}_\ell\,(T^b)^\ell{}_k \\ &=\; \mathrm{tr}(T^aT^b) \times \mathrm{tr}(T^aT^b) \end{aligned} \tag{S.53}$$

For the moment, let us consider 'quarks' belonging to some generic multiplet (r) of some generic gauge group G. In such a generic case, $\mathrm{tr}(T^aT^b) = R(r) \times \delta^{ab}$ where $R(r)$ is the index of the quark multiplet, *cf.* problem 2, and therefore

$$\sum_{a,b} \mathrm{tr}(T^aT^b) \times \mathrm{tr}(T^aT^b) \;=\; R^2(r) \times \sum_{a,b} \delta^{ab}\delta^{ab} \;=\; R^2(r) \times \dim(G). \tag{S.54}$$

Thus,

$$\sum_{i,j,k,\ell} \Big|\sum_a (T^a)^i{}_j\,(T^a)^k{}_\ell\Big|^2 \;=\; R^2(r) \times \dim(G),$$

or, for the average over the initial 'colors' i and j,

$$\frac{1}{\dim^2(r)} \sum_{i,j}\sum_{k,\ell} \Big|\sum_a (T^a)^i{}_j\,(T^a)^k{}_\ell\Big|^2 \;=\; \frac{R^2(r)\,\dim(G)}{\dim^2(r)} \;=\; \frac{C^2(r)}{\dim(G)}\,. \tag{S.55}$$

Specializing to the 'quarks' in the fundamental representation of an $SU(N)$ gauge group, we have $R(r) = \frac{1}{2}$, $\dim(r) = N$ and $\dim(G) = N^2 - 1$, hence eq. (S.55) evaluates to $(N^2-1)/(4N^2)$; for the actual QCD $N = 3$ and the color sum / average (S.55) gives 2/9.

Altogether, $|\mathcal{M}|^2$ summed / averaged over both spins and colors of all the fermions is

$$\frac{2}{9} \times g^4(1 + \cos^2\theta_{\text{c.m.}}) \tag{S.56}$$

and hence the cross section

$$\frac{d\sigma(u\bar{u} \to d\bar{d})}{d\Omega_{\text{cm}}} = \frac{2\alpha_{\text{QCD}}^2(1+\cos^2\theta_{\text{cm}})}{9E_{\text{c.m.}}^2}\,. \tag{S.57}$$

Problem **4**(a):

Let's go back to problem 1(b). There are four tree-level diagrams contributing to the quark+ antiquark → twoglons process, and the net amplitude is given in eqs. (S.7) through (S.11). When both gluons have transverse polarizations vectors $e^*_{1\mu}$ and $e^*_{2\nu}$ — and in particular $(k_1e^*_1) = (k_2e^*_2) = 0$, — the net amplitude simplifies to

$$\begin{aligned}
\mathcal{M}_{\text{net}} &= \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 + \mathcal{M}_4\,, &\text{(S.58)}\\
\mathcal{M}_1 &= \frac{4g^2}{t-m^2} \times (pe^*_1)(p'e^*_2) \times (T^bT^a)^i{}_j\,, &\text{(S.59)}\\
\mathcal{M}_2 &= \frac{4g^2}{u-m^2} \times (p'e^*_1)(pe^*_2) \times (T^aT^b)^i{}_j\,, &\text{(S.60)}\\
\mathcal{M}_3 &= g^2 \times (e^*_1e^*_2) \times \{T^bT^a\}^i{}_j\,, &\text{(S.61)}\\
\mathcal{M}_4 &= -\frac{g^2}{s} \times X \times [T^a,T^b]^i{}_j\,, &\text{(S.62)}
\end{aligned}$$

where s, t, and u are the Mandelstam variables and

$$\begin{aligned}
X &= (p-p')_\lambda e^*_{1\mu}e^*_{2\nu} \times \left[g^{\mu\nu}(k_1-k_2)^\lambda + g^{\nu\lambda}(2k_2+k_1)^\mu + g^{\lambda\mu}(-2k_1-k_2)^\nu\right]\\
&= (u-t) \times (e^*_1e^*_2) + 2(e^*_1k_2) \times \big(e^*_2(p-p')\big) - 2(e^*_2k_1) \times \big(e^*_1(p-p')\big).
\end{aligned} \tag{S.63}$$

The color dependences of amplitudes (S.61) and (S.62) are already in the form (9), while the amplitudes (S.59) and (S.60) can be brought into this form using

$$\begin{aligned}(T^aT^b) &= \tfrac{1}{2}\{T^a,T^b\} + \tfrac{1}{2}[T^a,T^b]\,,\\ (T^bT^a) &= \tfrac{1}{2}\{T^a,T^b\} - \tfrac{1}{2}[T^a,T^b]\,,\end{aligned}\tag{S.64}$$

Thus altogether,

$$\mathcal{M}_{\rm net}(\Phi^i+\Phi^*_j\to g^a+g^b) \;=\; F\times\{T^bT^a\}^i{}_j \;+\; iG\times[T^a,T^b]^i{}_j \tag{8}$$

for

$$F \;=\; \frac{2g^2}{t-m^2}\times(pe^*_1)(p'e^*_2) \;+\; \frac{2g^2}{u-m^2}\times(p'e^*_1)(pe^*_2) \;+\; g^2\times(e^*_1e^*_2) \tag{S.65}$$

and

$$G \;=\; \frac{2ig^2}{t-m^2}\times(pe^*_1)(p'e^*_2) \;-\; \frac{2ig^2}{u-m^2}\times(p'e^*_1)(pe^*_2) \;+\; \frac{ig^2}{s}\times X. \tag{S.66}$$

<u>Problem **4**(b)</u>:
First, let us average over the scalar particles' color indices $i,j=1,2,\ldots,\dim(r)$. For fixed gauge bosons a and b, let

$$M \;=\; F\{T^a_{(r)},T^b_{(r)}\} \;+\; iG\,[T^a_{(r)},T^b_{(r)}] \tag{S.67}$$

be a matrix (in the representation (r) of the gauge group) whose elements $M^i{}_j$ are annihilation amplitudes (8) for the scalar particles Φ^i and Φ^*_j of specific colors i,j. Then averaging over those colors gives

$$\frac{1}{\dim^2(r)}\sum_{i,j}\left|M^i{}_j\right|^2 \;=\; \frac{1}{\dim^2(r)}\sum_{i,j}M^i{}_j(M^\dagger)^j{}_i = \frac{1}{\dim^2(r)}\,{\rm tr}(MM^\dagger). \tag{S.68}$$

For the specific form (S.67) of the matrix M, we write

$$\begin{aligned}M &= (F+iG)\,T^a_{(r)}T^b_{(r)} \;+\; (F-iG)\,T^b_{(r)}T^a_{(r)}\,,\\ M^\dagger &= (F+iG)^*\,T^b_{(r)}T^a_{(r)} \;+\; (F-iG)^*\,T^a_{(r)}T^b_{(r)}\,,\end{aligned}\tag{S.69}$$

and therefore

$$
\begin{aligned}
\operatorname{tr}(MM^\dagger) \;&=\; |F+iG|^2\,\operatorname{tr}_{(r)}(T^aT^bT^bT^a) \;+\; (F-iG)(F+iG)^*\,\operatorname{tr}_{(r)}(T^bT^aT^bT^a) \\
&\qquad +\; (F+iG)(F-iG)^*\,\operatorname{tr}_{(r)}(T^aT^bT^aT^b) \;+\; |F-iG|^2\,\operatorname{tr}_{(r)}(T^bT^aT^aT^b) \\
&\qquad\langle\!\langle\text{ using the cyclic symmetry of the trace }\rangle\!\rangle \\
&=\; \operatorname{tr}_{(r)}(T^aT^aT^bT^b)\times\Big(|F+iG|^2 \;+\; |F-iG|^2 \;=\; 2|F|^2+2|G^2\Big) \\
&\qquad +\; \operatorname{tr}_{(r)}(T^aT^bT^aT^b)\times\begin{pmatrix}(F-iG)(F+iG)^* \;+\; (F+iG)(F-iG)^* \\ \qquad\qquad =\; 2|F^2| \;-\; 2|G^2|\end{pmatrix} \\
&=\; 2(|F|^2+|G|^2)\times\operatorname{tr}_{(r)}(T^aT^aT^bT^b) \\
&\qquad +\, 2(|F|^2-|G|^2)\times\begin{pmatrix}\operatorname{tr}_{(r)}(T^aT^bT^aT^b) \;=\; \operatorname{tr}_{(r)}(T^aT^aT^bT^b) \\ \qquad\qquad -\; \operatorname{tr}_{(r)}(T^a[T^a,T^b]T^b)\end{pmatrix} \\
&=\; 4|F|^2\times\operatorname{tr}_{(r)}(T^aT^aT^bT^b) \;+\; 2(|G|^2-|F|^2)\times\operatorname{tr}_{(r)}(T^a[T^a,T^b]T^b).
\end{aligned}
\tag{S.70}
$$

Our next step is to sum over the color indices a and b of the gauge bosons. In the context of eq. (S.70), we have

$$
\sum_{a,b}\operatorname{tr}_{(r)}(T^aT^aT^bT^b) \;=\; \operatorname{tr}_{(r)}\left(\Big(\sum_a T^aT^a\Big)\Big(\sum_b T^bT^b\Big)\right) \;=\; \operatorname{tr}_{(r)}\left(C_2\,C_2\right) \;=\; C^2(r)\times\dim(r)
\tag{S.71}
$$

and

$$
\begin{aligned}
\sum_{a,b}\operatorname{tr}_{(r)}(T^a[T^a,T^b]T^b) \;&=\sum_{a,b}\sum_c if^{abc}\operatorname{tr}_{(r)}(T^aT^cT^b) \\
&=\; \tfrac12\sum_{a,b,c} if^{abc}\operatorname{tr}_{(r)}(T^aT^cT^b - T^aT^bT^c) \\
&=\; \tfrac12\sum_{a,b,c} if^{abc}\sum_d if^{cbd}\operatorname{tr}_{(r)}(T^aT^d) \\
&=\; \tfrac12\sum_{a,b,c,d}(if^{abc})(if^{dbc})\times R(r)\delta^{ad} \\
&=\; \tfrac12 R(r)\sum_a\sum_{b,c}\Big(if^{abc}=(T^a_{(\mathrm{adj})})^{bc}\Big)\Big(if^{acb}=(T^a_{(\mathrm{adj})})^{cb}\Big)
\end{aligned}
\tag{S.72}
$$

$$
\begin{aligned}
&= \tfrac{1}{2} R(r) \sum_a \operatorname{tr}\Big(T^a_{(\mathrm{adj})} T^a_{(\mathrm{adj})} \Big) \\
&= \tfrac{1}{2} R(r) \times C(G) \dim(G) \\
&= \tfrac{1}{2} C(G) C(r) \dim(r).
\end{aligned} \tag{S.72}
$$

Therefore,

$$
\sum_{a,b} \operatorname{tr}(MM^\dagger) \;=\; C(r)\dim(r) \times \big[4C(r)|F|^2 \;+\; C(G)\big(|G|^2 - |F|^2\big)\big] \tag{S.73}
$$

and hence in light of eq. (S.68),

$$
\frac{1}{\dim^2(r)} \sum_{ij} \sum_{ab} |\mathcal{M}|^2 \;=\; \frac{C(r)}{\dim(r)} \times \big(4C(r)\,|F|^2 \,+\, C(\mathrm{adj})\,(|G|^2 - |F|^2) \big)\,. \tag{10}
$$

Eq. (11) follows from this as a special case.

Problem **4**(c):

Let us take a closer look at eqs. (S.65) and (S.66). In the center of mass frame, $\mathbf{p}' = -\mathbf{p}$, $\mathbf{k}_2 = -\mathbf{k}_1$, and the vector bosons' polarizations $e^\mu_{1,2}$ are purely spatial and transverse, $e^0_{1,2} = 0$ and $\mathbf{k}_{1,2} \cdot \mathbf{e}_{1,2} = 0$. Consequently, eqs. (S.65) and (S.66) simplify to

$$
\begin{aligned}
F &= 2g^2(\mathbf{e}_1^*\mathbf{p})(\mathbf{e}_2^*\mathbf{p}) \left(\frac{1}{t-m^2} + \frac{1}{u-m^2} \right) \;+\; g^2(\mathbf{e}_1^*\mathbf{e}_2^*), \\
G &= 2ig^2(\mathbf{e}_1^*\mathbf{p})(\mathbf{e}_2^*\mathbf{p}) \left(\frac{1}{t-m^2} - \frac{1}{u-m^2} \right) \;-\; ig^2\,\frac{u-t}{s}\,(\mathbf{e}_1^*\mathbf{e}_2^*).
\end{aligned} \tag{S.74}
$$

Furthermore, in the center of mass frame $E = E' = \omega_1 = \omega_2$, $|\mathbf{k}| = \omega = E$, $|\mathbf{p}| = \beta E$,

$$
s \;=\; 4E^2, \qquad t - m^2 \;=\; -2E^2(1 - \beta\cos\theta), \qquad u - m^2 \;=\; -2E^2(1 + \beta\cos\theta),
$$

hence

$$
\begin{aligned}
\frac{u-t}{s} &= \beta\cos\theta, \\
\frac{1}{t-m^2} + \frac{1}{u-m^2} &= \frac{-1}{m^2 + \mathbf{p}^2 \sin^2\theta}\,, \\
\frac{1}{t-m^2} - \frac{1}{u-m^2} &= \frac{-\beta\cos\theta}{m^2 + \mathbf{p}^2 \sin^2\theta}\,,
\end{aligned}
$$

and therefore

$$\begin{aligned} F &= g^2\left((\mathbf{e}_1^*\mathbf{e}_2^*)\;-\;\frac{2(\mathbf{e}_1^*\mathbf{p})(\mathbf{e}_2^*\mathbf{p})}{m^2+\mathbf{p}^2\sin^2\theta}\right), \\ G &= -ig^2\left((\mathbf{e}_1^*\mathbf{e}_2^*)\;+\;\frac{2(\mathbf{e}_1^*\mathbf{p})(\mathbf{e}_2^*\mathbf{p})}{m^2+\mathbf{p}^2\sin^2\theta}\right)\times\beta\cos\theta. \end{aligned} \tag{S.75}$$

Now consider the gluons' polarization vectors. For the problem at hand it is easier to use linear polarizations for which the $\mathbf{e}_1$ and $\mathbf{e}_2$ are real unit vectors. Specifically, for each gluon there is a choice of two transverse $\mathbf{e}$, one parallel to the $(\mathbf{p},\mathbf{k})$ plane and one perpendicular to it. In the coordinate system where

$$\mathbf{p} \;=\; \beta E(0,0,1) \qquad \text{and} \quad \mathbf{k} \;=\; E(\sin\theta,0,\cos\theta), \tag{S.76}$$

the two polarization vectors are

$$\mathbf{e}_{\|} \;=\; (-\cos\theta,0,+\sin\theta) \quad \text{and} \quad \mathbf{e}_{\perp} \;=\; (0,1,0). \tag{S.77}$$

For these vectors

$$(\mathbf{p}\mathbf{e}_{\|}) \;=\; \beta E\sin\theta, \qquad (\mathbf{p}\mathbf{e}_{\perp}) \;=\; 0, \tag{S.78}$$

so according to eqs. (S.75),

$$\text{for } \mathbf{e}_1=\mathbf{e}_2=\mathbf{e}_{\perp}\,, \quad F \;=\; g^2 \quad \text{and} \quad G \;=\; -ig^2\beta\cos\theta, \tag{S.79}$$

$$\text{for } \mathbf{e}_1=\mathbf{e}_2=\mathbf{e}_{\|}\,, \quad F \;=\; g^2(1-2A) \quad \text{and} \quad G \;=\; -ig^2(1+2A)\beta\cos\theta \tag{S.80}$$

where

$$A \;=\; \frac{\mathbf{p}^2\sin^2\theta}{m^2+\mathbf{p}^2\sin^2\theta}\,, \tag{S.81}$$

and finally

$$\text{for } \mathbf{e}_1=\mathbf{e}_{\perp},\, \mathbf{e}_2=\mathbf{e}_{\|} \text{ or } \textit{vice verse}, \quad F \;=\; G \;=0. \tag{S.82}$$

Problem **4**(d):

According to eq. (S.82), the two gauge bosons produced in the $\Phi\Phi^*$ annihilation must have similar polarizations: either both are polarized $\parallel$ to the $(\mathbf{p},\mathbf{k})$ plane of scattering or both are polarized $\perp$ to the plane. Consequently, there are only two polarized partial cross sections to consider, namely

$$\begin{aligned}
\left(\frac{d\sigma(\perp)}{d\Omega}\right)_{\rm c.m.} &= \frac{g^4}{64\pi^2 E_{\rm c.m.}^2\beta}\,\frac{C(r)}{\dim(r)} \times \Bigl(4C(r)\,-\,(1-\beta^2\cos^2\theta)C(G)\Bigr),\\
\left(\frac{d\sigma(\parallel)}{d\Omega}\right)_{\rm c.m.} &= \frac{g^4}{64\pi^2 E_{\rm c.m.}^2\beta}\,\frac{C(r)}{\dim(r)} \times \left(\begin{array}{l} 4C(r)\times(1-2A) \\ +\,C(G)\times\Bigl((1-2A)^2-\beta^2(1+2A)^2\cos^2\theta\Bigr)\end{array}\right).
\end{aligned} \tag{S.83}$$

Note that the angular dependence of the $\parallel$ polarized partial cross section is more complicated than it looks because A is θ–dependent according to eq. (S.81).

In the limit of non-relativistic scalar particles, $\beta \ll 1$ leads to $A \ll 1$ and hence to the expected isotropy and polarization independence of the annihilation cross-section,

$$\left(\frac{d\sigma(\parallel)}{d\Omega}\right)_{\rm c.m.} \approx \left(\frac{d\sigma(\perp)}{d\Omega}\right)_{\rm c.m.} \approx \frac{g^4}{256\pi^2 m^2\beta}\,\frac{C(r)\bigl(4C(r)-C(G)\bigr)}{\dim(r)}\,. \tag{S.84}$$

In the opposite limit of ultra-relativistic scalars, $\beta \approx 1$ leads to $A \approx 1$ (except for $\theta \approx 0$) and therefore

$$\begin{aligned}
\left(\frac{d\sigma(\perp)}{d\Omega}\right)_{\rm c.m.} &\approx \frac{g^4}{16\pi^2 E_{\rm c.m.}^2}\,\frac{C^2(r)}{\dim(r)}\,,\\
\left(\frac{d\sigma(\parallel)}{d\Omega}\right)_{\rm c.m.} &\approx \frac{g^4}{16\pi^2 E_{\rm c.m.}^2}\,\frac{C(r)}{\dim(r)}\left[C(r)\,+\,C(G)\,\frac{9\cos^2\theta-1}{4}\right].
\end{aligned} \tag{S.85}$$