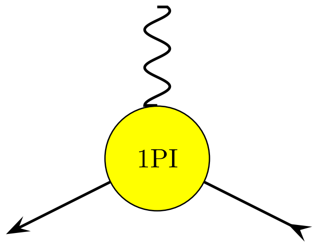
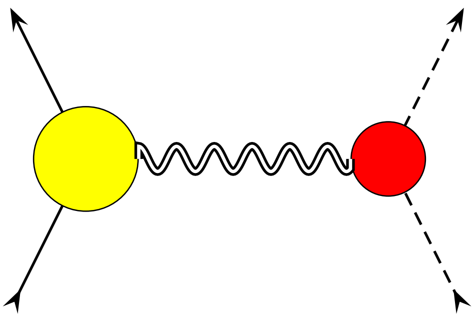
QED Vertex Correction

Consider the dressed electron-electron-photon vertex in QED,

$$ie\Gamma^{\mu}(p',p) \;=\; \tag{1}$$



We are interested in this vertex in the context of elastic Coulomb scattering,



(2)

so we take the incoming and the outgoing electrons to be on-shell, $p^2 = p'^2 = m^2$, but the photon is off-shell, $q^2 \neq 0$. Moreover, we put the vertex in the context of the complete electron line — including the external line factors, thus $\bar{u}(p') \times ie\Gamma^{\mu} \times u(p)$. As discussed in class, this simplifies the Lorentz and Dirac structure of the vertex and allows us to write it as

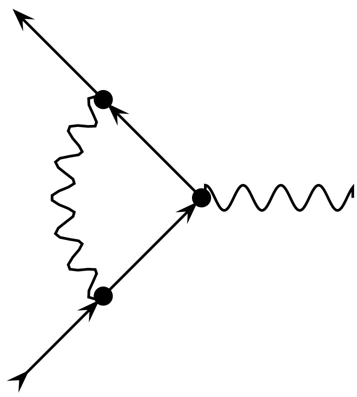
$$\Gamma^{\mu}(p',p) \;=\; F_{\rm el}(q^2) \times \frac{(p'+p)^{\mu}}{2m} \;+\; F_{\rm mag}(q^2) \times \frac{i\sigma^{\mu\nu}q_{\nu}}{2m} \;=\; F_1(q^2) \times \gamma^{\mu} \;+\; F_2(q^2) \times \frac{i\sigma^{\mu\nu}q_{\nu}}{2m}. \tag{3}$$

At the tree level, the electron is a point-like spin $= \frac{1}{2}$ particle obeying Dirac equations, hence $F_1(q^2) \equiv 1$ and $F_2(q^2) \equiv 0$. But the quantum corrections in QED mix the elementary electron state with the composite states like $|e^-\gamma\rangle$, $|e^-\gamma\gamma\rangle$, or $|e^-e^-e^+\rangle$, and this leads to the non-trivial q^2–dependent form-factors.

In these notes we shall calculate the $F_1(q^2)$ and the $F_2(q^2)$ form-factors to the one-loop order in QED.

Working Through the Algebra

Fortunately, there is only one 1PI one-loop diagram contributing to the dressing-up of the electron-electron-photon vertex, namely

 (4)

Using the Feynman gauge for the internal photon's propagator, this diagram evaluates to

$$\begin{aligned} ie\Gamma^\mu_{1\,\text{loop}}(p',p) &= \int\limits_{\text{reg}} \frac{d^4k}{(2\pi)^4}\,\frac{-ig^{\nu\lambda}}{k^2+i0} \times ie\gamma_\nu \times \frac{i}{\not p' + \not k \ - m + i0} \times ie\gamma^\mu \times \frac{i}{\not p + \not k \ - m + i0} \times ie\gamma_\lambda \\ &= e^3 \int\limits_{\text{reg}} \frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2+i0} \times \gamma^\nu \times \frac{\not p' + \not k + m}{(p'+k)^2 - m^2 + i0} \times \gamma^\mu \times \frac{\not p + \not k + m}{(p+k)^2 - m^2 + i0} \times \gamma_\nu \\ &= e^3 \int\limits_{\text{reg}} \frac{d^4k}{(2\pi)^4}\,\frac{\mathcal{N}^\mu}{\mathcal{D}} \end{aligned} \tag{5}$$

where

$$\mathcal{N}^\mu = \gamma^\nu(\not k + \not p' + m)\gamma^\mu(\not k + \not p + m)\gamma_\nu \tag{6}$$

and

$$\mathcal{D} = \big[k^2 + i0\big] \times \big[(p+k)^2 - m^2 + i0\big] \times \big[(p'+k)^2 - m^2 + i0\big]. \tag{7}$$

The purpose of this section of the notes is to simplify these numerator and denominator. Using the Feynman parameter trick, we may combine the 3 denominator factors as

$$\frac{1}{\mathcal{D}} = \iiint\limits_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{2}{\Big[x((p+k)^2-m^2) \,+\, y((p'+k)^2-m^2) \,+\, z(k^2) \,+\, i0\Big]^3}\,. \tag{8}$$

Inside the big square brackets here we have

$$\begin{aligned}
[\cdots] \;&=\; x\times\bigl((p+k)^2-m^2\bigr) \;+\; y\times\bigl((p'+k)^2-m^2\bigr) \;+\; z\times k^2\\
&=\; k^2\times(x+y+z=1) \;+\; 2k_\mu(xp+yp')^\mu \;+\; x(p^2-m^2) \;+\; y(p'^2-m^2)\\
&=\; (k+xp+yp')^2 \;-\; \Delta
\end{aligned} \tag{9}$$

where

$$\begin{aligned}
\Delta \;&=\; (xp+yp')^2 \;-\; xp^2 \;-\; yp'^2 \;+\; (x+y)m^2\\
&=\; xy\times\bigl(2p\cdot p' \;=\; p^2+p'^2-(p'-p)^2\bigr) \;+\; (x^2-x)\times p^2 \;+\; (y^2-y)\times p'^2 \;+\; (x+y)\times m^2\\
&=\; -xy\times q^2 \;-\; x(1-x-y)\times p^2 \;-\; y(1-x-y)\times p'^2 \;+\; (x+y)\times m^2\\
&=\; -xy\times q^2 \;-\; xz\times p^2 \;-\; yz\times p'^2 \;+\; (1-z)\times m^2
\end{aligned} \tag{10}$$

For the on-shell electron momenta, $p^2=p'^2=m^2$, we may further simplify

$$(1-z)\times m^2 \;-\; xz\times p^2 \;-\; yz\times p'^2 \;=\; m^2\times\Bigl((1-z)\;-\;(x+y)z \;=\; (1-z)^2\Bigr) \tag{11}$$

which gives

$$\Delta \;=\; (1-z)^2\times m^2 \;-\; xy\times q^2. \tag{12}$$

Let us also define the shifted loop momentum

$$\ell \;=\; k\;+\;xp\;+\;yp', \tag{13}$$

then we can rewrite the denominator as

$$\frac{1}{\mathcal{D}} \;=\; \iiint_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{2}{[\ell^2-\Delta+i0]^3}\,. \tag{14}$$

As usual, we plug this denominator into the loop integral (5), then change the order of integration — $\int$ over the loop momentum before $\int$ over the Feynman parameters, — and then shift the momentum integration variable from k to ℓ, thus

$$\Gamma^{\mu}_{1\,\mathrm{loop}}(p',p) \;=\; -2ie^2 \iiint\limits_0^1 dx\,dy\,dz\,\delta(x+y+z-1) \int\limits_{\mathrm{reg}} \frac{d^4\ell}{(2\pi)^4}\, \frac{\mathcal{N}^{\mu}}{\left[\ell^2-\Delta+i0\right]^3}\,. \tag{15}$$

But to make full use of the momentum shift, we need to re-express the numerator $\mathcal{N}^{\mu}$ in terms of the shifted momentum ℓ. It would also help to simplify the numerator (6) *in the context of this monstrous integral.*

The first step towards simplifying the $\mathcal{N}^{\mu}$ is obvious: Let us get rid of the γ^{ν} and γ_{ν} factors using the γ matrix algebra, *eg.*, $\gamma^{\nu}\not{a}\gamma_{\nu}=-2\not{a}$, *etc.*. However, in order to allow for the dimensional regularization, we need to re-work the algebra for an arbitrary spacetime dimension D where $\gamma^{\nu}\gamma_{\nu}=D\neq 4$. Consequently,

$$\begin{aligned}
\gamma^{\nu}\not{a}\gamma_{\nu} \;&=\; -2\not{a} \;+\; (4-D)\not{a},\\
\gamma^{\nu}\not{a}\not{b}\gamma_{\nu} \;&=\; 4(ab) \;-\; (4-D)\not{a}\not{b},\\
\gamma^{\nu}\not{a}\not{b}\not{c}\gamma_{\nu} \;&=\; -2\not{c}\not{b}\not{a} \;+\; (4-D)\not{a}\not{b}\not{c},
\end{aligned} \tag{16}$$

and therefore

$$\begin{aligned}
\mathcal{N}^{\mu} \;\stackrel{\mathrm{def}}{=}\; &\gamma^{\nu}(\not{k}+\not{p}'+m)\gamma^{\mu}(\not{k}+\not{p}+m)\gamma_{\nu}\\
=\; &-2m^2\gamma^{\mu} \;+\; 4m(p'+p+2k)^{\mu} \;-\; 2(\not{p}+\not{k})\gamma^{\mu}(\not{p}'+\not{k})\\
&+\; (4-D)(\not{p}'+\not{k}-m)\gamma^{\mu}(\not{p}+\not{k}-m).
\end{aligned} \tag{17}$$

The second step is to re-express this numerator in terms of the loop momentum ℓ rather than k using eq. (13). Expanding the result in powers of ℓ, we get quadratic, linear and ℓ–independent terms, but the linear terms do not contribute to the $\int d^D\ell$ integral because they are odd with respect to $\ell\to-\ell$ while everything else in that integral is even. Consequently, *in the context of* *eq.* (15) we may neglect the linear terms, thus

$$\begin{aligned}
\mathcal{N}^\mu \;&=\; -2m^2\gamma^\mu \;+\; 4m(p' + p + 2\ell - 2xp - 2yp')^\mu \\
&\quad -\; 2(\not{p} + \not{\ell} - x\,\not{p} - y\,\not{p}')\gamma^\mu(\not{p}' + \not{\ell} - x\,\not{p} - y\,\not{p}') \\
&\quad +\; (4-D)(\not{p}' + \not{\ell} - x\,\not{p} - y\,\not{p}' - m)\gamma^\mu(\not{p} + \not{\ell} - x\,\not{p} - y\,\not{p}' - m) \\
&\langle\!\langle \text{skipping terms linear in } \ell \rangle\!\rangle \\
&\cong\; -2m^2\gamma^\mu \;+\; 4m(p + p' - 2xp - 2yp')^\mu \\
&\quad -\; 2\,\not{\ell}\gamma^\mu\not{\ell} \;-\; 2(\not{p} - x\,\not{p} - y\,\not{p}')\,\gamma^\mu\,(\not{p}' - x\,\not{p} - y\,\not{p}') \\
&\quad +\; (4-D)\,\not{\ell}\gamma^\mu\not{\ell} \;+\; (4-D)(\not{p}' - y\,\not{p}' - x\,\not{p} - m)\,\gamma^\mu\,(\not{p} - x\,\not{p} - y\,\not{p}' - m).
\end{aligned} \tag{18}$$

Next, we make use of $p' - p = q$ and $1 - x - y = z$ to rewrite

$$\begin{aligned}
2xp \;+\; 2yp' \;&=\; (x+y)\times(p+p') \;+\; (x-y)\times(p-p'), \\
p \;+\; p' \;-\; 2xp \;-\; 2yp' \;&=\; z\times(p'+p) \;+\; (x-y)\times q, \\
p \;-\; xp \;-\; yp' \;&=\; z\times p \;-\; y\times q \\
&=\; z\times p' \;-\; (1-x)\times q, \\
p' \;-\; xp \;-\; yp' \;&=\; z\times p' \;+\; x\times q \\
&=\; z\times p \;+\; (1-y)\times q,
\end{aligned} \tag{19}$$

and consequently

$$\begin{aligned}
\mathcal{N}^\mu \;\cong\; & -2m^2\gamma^\mu \;+\; 4mz(p'+p)^\mu \;+\; 4m(x-y)q^\mu \\
& +\; (-2 + 4 - D)\times\not{\ell}\gamma^\mu\not{\ell} \\
& -\; 2(z\,\not{p}' + (x-1)\,\not{q})\,\gamma^\mu\,(z\,\not{p} + (1-y)\,\not{q}) \\
& +\; (4-D)(z\,\not{p}' + x\,\not{q} - m)\,\gamma^\mu\,(z\,\not{p} - y\,\not{q} - m).
\end{aligned} \tag{20}$$

The third step is to make use of the external fermions being on-shell. This means more than just $p^2 = p'^2 = m^2$: We also sandwich the vertex $ie\Gamma^\mu$ between the Dirac spinors $\bar{u}(p')$ on the left and $u(p)$ on the right. The two spinors satisfy the appropriate Dirac equations $\not{p}u(p) = mu(p)$ and $\bar{u}(p')\,\not{p}' = \bar{u}(p')m$, so in the context of $\bar{u}(p')\Gamma^\mu u(p)$,

$$A\times\not{p} \;\cong\; A\times m \quad\text{and}\quad \not{p}'\times B \;\cong\; m\times B \tag{21}$$

for any terms in Γ^μ that look like $A\times\,\not{p}$ or $\not{p}'\times B$ for some A or B. Consequently, the terms on the last two lines of eq. (20) are equivalent to

$$
\begin{aligned}
(z\,\not{p}' + (x-1)\,\not{q})\,\gamma^\mu\,(z\,\not{p} + (1-y)\,\not{q}) \;&\cong\; (zm + (x-1)\,\not{q})\,\gamma^\mu\,(zm + (1-y)\,\not{q}) \\
&=\; z^2m^2\times\gamma^\mu \;-\; (1-x)(1-y)\times\not{q}\gamma^\mu\,\not{q} \\
&\qquad +\; z(x-y)m\times\left(\tfrac{1}{2}\{\gamma^\mu,\not{q}\} \;=\; q^\mu\right) \\
&\qquad +\; z(2-x-y)m\times\left(\tfrac{1}{2}[\gamma^\mu,\not{q}] \;=\; -i\sigma^{\mu\nu}q_\nu\right) \\
(z\,\not{p}' + x\,\not{q} - m)\,\gamma^\mu\,(z\,\not{p} - y\,\not{q} - m) \;&\cong\; ((z-1)m + x\,\not{q})\,\gamma^\mu\,((z-1)m - y\,\not{q}) \\
&=\; (1-z)^2m^2\times\gamma^\mu \;-\; xy\times\not{q}\gamma^\mu\,\not{q} \\
&\qquad -\; (1-z)(x-y)m\times\left(\tfrac{1}{2}\{\gamma^\mu,\not{q}\} \;=\; q^\mu\right) \\
&\qquad +\; (1-z)(x+y)m\times\left(\tfrac{1}{2}[\gamma^\mu,\not{q}] \;=\; -i\sigma^{\mu\nu}q_\nu\right).
\end{aligned}
\tag{22}
$$

Let's plug these expressions back into eq. (20), collect similar terms together, and make use of $(1-x)(1-y) = 1-x-y+xy = z+xy$. This gives us

$$
\begin{aligned}
\mathcal{N}^\mu \;\cong\; & -(D-2)\,\not{\ell}\gamma^\mu\not{\ell} \;+\; 4mz(p'+p)^\mu \\
& +\; m^2\gamma^\mu\times\Big(-2-2z^2 \;+\; (4-D)(1-z)^2\Big) \\
& +\; \not{q}\gamma^\mu\not{q}\times\Big(2(z+xy) \;-\; (4-D)xy\Big) \\
& +\; mq^\mu\times(x-y)\Big(4-2z \;-\; (4-D)(1-z)\Big) \\
& +\; im\sigma^{\mu\nu}q_\nu\times\Big(2z(1+z) \;-\; (4-D)(1-z)^2\Big).
\end{aligned}
\tag{23}
$$

Furthermore, in the context of the Dirac sandwich $\bar{u}(p')\Gamma^\mu u(p)$ we have

$$
\not{q}\gamma^\mu\not{q} \;=\; 2q^\mu\,\not{q} \;-\; q^2\gamma^\mu \;\cong\; -q^2\gamma^\mu \tag{24}
$$

because $\bar{u}(p')\,\not{q}u(p) = 0$, and also

$$
(p'+p)^\mu \;\cong\; 2m\gamma^\mu \;-\; i\sigma^{\mu\nu}q_\mu \tag{25}
$$

(the Gordon identity). Plugging these formulae into eq. (23), we arrive at

$$
\begin{aligned}
\mathcal{N}^\mu \;\cong\; & -(D-2)\,\not{\ell}\gamma^\mu\not{\ell} \;+\; m^2\gamma^\mu\times\Big(8z - 2(1+z^2) + (4-D)(1-z)^2\Big) \\
& -\; q^2\gamma^\mu\times\Big(2(z+xy) - (4-D)xy\Big) \;-\; im\sigma^{\mu\nu}q_\nu\times(1-z)\Big(2z + (4-D)(1-z)\Big) \\
& +\; mq^\mu\times(x-y)\Big(4-2z-(4-D)(1-z)\Big).
\end{aligned}
\tag{26}
$$

To further simplify this expression, let us go back to the symmetries of the integral (15). The integral over the Feynman parameters, the integral $\int d^D\ell$, and the denominator $[l^2-\Delta]^3$ are all invariant under the parameter exchange $x\leftrightarrow y$. In eq. (26) for the numerator, the first two lines are invariant under this symmetry, but the last line changes sign. Consequently, only the first two lines contribute to the integral (15) while the third line integrates to zero and may be disregarded, thus

$$\begin{aligned}\mathcal{N}^\mu \;\cong\; & -(D-2)\,\not\ell\gamma^\mu\not\ell \;+\; m^2\gamma^\mu\times\Big(8z-2(1+z^2)+(4-D)(1-z)^2\Big)\\ & -\; q^2\gamma^\mu\times\Big(2(z+xy)-(4-D)xy\Big) - im\sigma^{\mu\nu}q_\nu\times(1-z)\Big(2z+(4-D)(1-z)\Big).\end{aligned} \tag{27}$$

Finally, thanks to the Lorentz invariance of the $\int d^D\ell$ integral,

$$\ell_\lambda\ell_\nu \;\cong\; g_{\lambda\nu}\times\frac{\ell^2}{D}\,, \tag{28}$$

and hence

$$\not\ell\gamma^\mu\not\ell \;=\; \gamma^\lambda\gamma^\mu\gamma^\nu\times\ell_\lambda\ell_\nu \;\cong\; \gamma^\lambda\gamma^\mu\gamma^\nu\times g_{\lambda\nu}\frac{\ell^2}{D} \;=\; -(D-2)\gamma^\mu\times\frac{\ell^2}{D}\,. \tag{29}$$

Plugging this formula into eq. (26) and grouping terms according to their γ–matrix structure, we arrive at

$$\mathcal{N}^\mu \;=\; \mathcal{N}_1\times\gamma^\mu - \mathcal{N}_2\times\frac{i\sigma^{\mu\nu}q_\nu}{2m} \tag{30}$$

where

$$\begin{aligned}\mathcal{N}_1 \;\cong\; & \frac{(D-2)^2}{D}\times\ell^2 \;+\; \Big(8z-2(1+z^2)+(4-D)(1-z)^2\Big)\times m^2\\ & -\;\Big(2(z+xy)-(4-D)xy\Big)\times q^2\\ =\; & \frac{(D-2)^2}{D}\times\ell^2 \;-\; (D-2)\times\Delta \;+\; 2z\times(2m^2-q^2),\end{aligned} \tag{31}$$

$$\mathcal{N}_2 \;\cong\; (1-z)\Big(4z+2(4-D)(1-z)\Big)\times m^2. \tag{32}$$

Note that splitting the numerator according to eq. (30) is particularly convenient for calculating the electron's form factors:

$$\Gamma^\mu_{1\,\mathrm{loop}} \;=\; F_1^{1\,\mathrm{loop}}(q^2)\times\gamma^\mu + F_2^{1\,\mathrm{loop}}(q^2)\times\frac{i\sigma^{\mu\nu}q_\nu}{2m}\,, \tag{33}$$

$$F_1^{1\,\text{loop}}(q^2) \;=\; -2ie^2 \iiint_0^1 dx\,dy\,dz\,\delta(x+y+z-1) \int \frac{d^D\ell}{(2\pi)^D}\,\frac{\mathcal{N}_1}{\left[\ell^2-\Delta+i0\right]^3}\,, \tag{34}$$

$$F_2^{1\,\text{loop}}(q^2) \;=\; +2ie^2 \iiint_0^1 dx\,dy\,dz\,\delta(x+y+z-1) \int \frac{d^D\ell}{(2\pi)^D}\,\frac{\mathcal{N}_2}{\left[\ell^2-\Delta+i0\right]^3}\,. \tag{35}$$

Electron's Gyromagnetic Moment

As explained earlier in class, electron's spin couples to the static magnetic field as

$$\hat{H} \;\supset\; \frac{-eg}{2m_e}\,\mathbf{S}\cdot\mathbf{B} \qquad \text{where} \quad g \;=\; 2\left(F_{\text{mag}} = F_1 + F_2\right)\Big|_{q^2=0}\,. \tag{36}$$

The electric form factor $F_1 \equiv F_{el}$ for $q^2 = 1$ is constrained by the Ward identity,

$$F_1^{\text{tot}} \;=\; F_1^{\text{tree}} \,+\, F_1^{\text{loops}} \,+\, F_1^{\text{counter}-\text{terms}} \;\xrightarrow[q^2\to 0]{}\; 1. \tag{37}$$

Therefore, the gyromagnetic moment is

$$g \;=\; 2 \,+\, 2F_2(q^2=0) \tag{38}$$

where $F_2 = F_2^{\text{loops}}$ because the there are no tree-level or counter-term contributions to the F_2, only to the F_1. Thus, to calculate the $g-2$ at the one-loop level, all we need is to evaluate the integral (35) for $q^2 = 0$.

Let's start with the momentum integral

$$\int \frac{d^D\ell}{(2\pi)^D}\,\frac{\mathcal{N}_2}{\left[\ell^2-\Delta+i0\right]^3} \tag{39}$$

where $\Delta = (1-z)^2m^2$ for $q^2 = 0$ and $\mathcal{N}_2$ is as in eq. (32). Because the numerator here does not depend on the loop momentum ℓ, this integral converges in $D = 4$ dimensions and there is no need for dimensional regularization. All we need is to rotate the momentum into Euclidean space,

$$\begin{aligned}
\int\frac{d^4\ell}{(2\pi)^4}\,\frac{\mathcal{N}_2}{\left[\ell^2-\Delta+i0\right]^3} \;&=\; \mathcal{N}_2\times\int\frac{i\,d^4\ell_E}{(2\pi)^4}\,\frac{1}{-(\ell_E^2+\Delta)^3}\\
&=\; \frac{-i\mathcal{N}_2}{16\pi^2}\times\int\limits_0^\infty d\ell_E^2\,\frac{\ell_E^2}{(\ell_E^2+\Delta)^3}\\
&=\; \frac{-i\mathcal{N}_2}{16\pi^2}\times\frac{1}{2\Delta}\\
&=\; \frac{-i}{32\pi^2}\times\frac{\mathcal{N}_2=4z(1-z)m^2\quad\langle\!\langle\text{for } D=4\rangle\!\rangle}{\Delta=(1-z)^2m^2\quad\langle\!\langle\text{for } q^2=0\rangle\!\rangle}\\
&=\; \frac{-i}{32\pi^2}\times\frac{4z}{1-z}\,.
\end{aligned}\tag{40}$$

Substituting this formula into eq. (35), we have

$$F_2^{1\,\text{loop}}(q^2=0) \;=\; \frac{e^2}{16\pi^2}\iiint\limits_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\times\frac{4z}{1-z}\,. \tag{41}$$

The integrand here depends on z but not on the other two Feynman parameters, so we can immediately integrate over x and y and obtain

$$\iint\limits_0^1 dx\,dy\,\delta(x+y+z-1) \;=\; \int\limits_0^{1-z} dx \;=\; 1-z. \tag{42}$$

Consequently,

$$F_2^{1\,\text{loop}}(q^2=0) \;=\; \frac{e^2}{16\pi^2}\times\int\limits_0^1 dz\,(1-z)\times\frac{4z}{1-z} \;=\; \frac{e^2}{16\pi^2}\times 2 \;=\; \frac{\alpha}{2\pi} \tag{43}$$

and the gyromagnetic moment is

$$g \;=\; 2\;+\;\frac{\alpha}{\pi}\;+\;O(\alpha^2). \tag{44}$$

The higher-loop corrections to this gyromagnetic moment are harder to calculate because the number of diagrams grows very rapidly with the number of loops; at the 4-loop order there are thousands of diagrams, and one needs a computer just to count them! Also, at higher orders one has to include the effects of strong and weak interactions because the photons interact with hadrons and $W^\pm$ particles, which in turn interact with other hadrons, Z^0, Higgs, *etc.*, *etc.* Nevertheless, people have calculated the electron's and the muon's *anomalous magnetic moments*

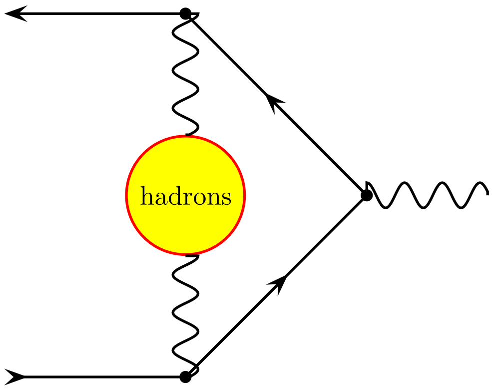
$$a_e = \frac{g_e - 2}{2} = F_2^{\text{electron}}(0) \quad \text{and} \quad a_\mu = \frac{g_\mu - 2}{2} = F_2^{\text{muon}}(0) \tag{45}$$

to the order α^4 back in the 1970s, and more recent calculations are good up to the order α^5. Meanwhile, the experimentalists have measured a_e to a comparable accuracy of 12 significant digits and a_μ to 9 significant digits

$$a_e^{\text{exp}} = 0.001\,159\,652\,180\,28\,(24), \qquad a_\mu^{\text{exp}} = 0.001\,165\,920\,89\,(63). \tag{46}$$

The theoretical value of the electron's anomalous magnetic moment is in good agreement with the experimental value, while for the muon there is a small discrepancy $a_\mu^{\text{exp}} - a_\mu^{\text{theory}} \approx (29\pm 8)\cdot 10^{-10}$. This discrepancy might stem from some physics beyond the Standard Model, maybe supersymmetry, maybe something else. Note that the effect of heavy particles on the a_μ is proportional to $(m_\mu/M_{\text{heavy}})^2$, that's why the muon's anomalous magnetic moment is much more sensitive to the new physics than the electron's.

However, the discrepancy between the a_μ^{exp} and the a_μ^{theory} might also stem from a small theoretical error in modeling the photon-hadron interactions, which affects the a_μ^{theory} via 2+ loop diagrams like



(47)

For a recent review of the muon's high-precision anomalous magnetic moment — both the experiments and the theory — see http://arxiv.org/abs/1311.2198 and the references cited therein.

I would like to complete this section of the notes by calculating the $F_2^{1\,\mathrm{loop}}(q^2)$ form factor for $q^2 \neq 0$. Proceeding as in eq. (40) but letting $\Delta = (1-z)^2 m^2 - xyq^2$, we have

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{\mathcal{N}_2}{\left[\ell^2 - \Delta + i0\right]^3} = \frac{-i}{32\pi^2} \times \frac{4z(1-z)m^2}{(1-z)^2 m^2 - xyq^2} \tag{48}$$

and hence

$$F_2^{1\,\mathrm{loop}}(q^2) = \frac{e^2}{16\pi^2} \iiint_0^1 dx\,dy\,dz\,\delta(x+y+z-1) \times \frac{4z(1-z)m^2}{(1-z)^2 m^2 - xyq^2}\,. \tag{49}$$

To evaluate this integral over Feynman parameters, we change variables from x, y, z to $w = 1-z$ and $\xi = x/(x+y)$,

$$x = w\xi, \quad y = w(1-\xi), \quad z = 1-w, \qquad dx\,dy\,dz\,\delta(x+y+z-1) = w\,dw\,d\xi. \tag{50}$$

Consequently,

$$\begin{aligned}
F_2^{1\,\mathrm{loop}}(q^2) &= \frac{e^2}{16\pi^2} \int_0^1 d\xi \int_0^1 dw\,w \times \frac{4(1-w)w \times m^2}{w^2 \times m^2 \,-\, w^2\xi(1-\xi) \times q^2} \\
&= \frac{e^2}{16\pi^2} \int_0^1 d\xi \frac{m^2}{m^2 \,-\, \xi(1-\xi)q^2} \times \int_0^1 dw\,w \times \frac{4w(1-w)}{w^2} \\
&= \frac{e^2}{8\pi^2} \times \int_0^1 d\xi \frac{m^2}{m^2 \,-\, \xi(1-\xi)q^2} \\
&= \frac{\alpha}{2\pi} \times \frac{4m^2}{\sqrt{q^2 \times (4m^2 - q^2)}} \times \arctan\sqrt{\frac{q^2}{4m^2 - q^2}} \\
&= \frac{\alpha}{2\pi} \times \frac{4m^2}{\sqrt{(-q^2) \times (4m^2 - q^2)}} \times \log \frac{\sqrt{4m^2 - q^2} + \sqrt{-q^2}}{2m}\,.
\end{aligned} \tag{51}$$

For $q^2 < 0$ and $-q^2 \gg m^2$,

$$F_2^{1\,\mathrm{loop}}(q^2) \approx \frac{\alpha}{2\pi} \times \frac{2m^2}{-q^2} \times \log \frac{-q^2}{m^2}\,. \tag{52}$$

The Electric Form Factor

Now consider the electric form factor $F_1(q^2)$. In the first section we have obtained

$$F_1^{1\,\text{loop}}(q^2) \;=\; -2ie^2 \iiint_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\int\frac{d^D\ell}{(2\pi)^D}\,\frac{\mathcal{N}_1}{\left[\ell^2-\Delta+i0\right]^3}\,, \tag{34}$$

for

$$\mathcal{N}_1 \;\cong\; \frac{(D-2)^2}{D}\times\ell^2 \;-\; (D-2)\times\Delta \;+\; 2z\times(2m^2-q^2) \tag{31}$$

and $\Delta=(1-z)^2m^2-xyq^2$.

Let's start by calculating the momentum integral in eq. (34). The numerator $\mathcal{N}_1$ depends on ℓ as $a\ell^2+b$, so there is a logarithmic UV divergence for $\ell\to\infty$; to regularize this divergence, we work in $D=4-2\epsilon$ dimensions. Thus,

$$\begin{aligned}
-i\int_{\text{reg}}\frac{d^4\ell}{(2\pi)^4}\,\frac{a\ell^2+b}{[\ell^2-\Delta+i0]^3} \;&\equiv\; -i\mu^{4-D}\int\frac{d^D\ell}{(2\pi)^D}\,\frac{a\ell^2+b}{[\ell^2-\Delta+i0]^3} \;=\\
&=\; -i\mu^{4-D}\int\frac{id^D\ell_E}{(2\pi)^D}\,\frac{-a\ell_E^2+b}{-[\ell_E^2+\Delta]^3}\\
&=\; \mu^{4-D}\int\frac{d^D\ell_E}{(2\pi)^D}\times\left[\frac{a\ell_E^2-b}{(\ell_E^2+\Delta)^3} \;=\; \frac{a}{(\ell_E+\Delta)^2} \;-\; \frac{a\Delta+b}{(\ell_E^2+\Delta)^3}\right]\\
&=\; \mu^{4-D}\int\frac{d^D\ell_E}{(2\pi)^D}\int_0^\infty dt\Big(a\times t \;-\; (a\Delta+b)\times\tfrac{1}{2}t^2\Big)\times e^{-t(\Delta+\ell_E^2)}\\
&=\; \int_0^\infty dt\Big(a\times t \;-\; (a\Delta+b)\times\tfrac{1}{2}t^2\Big)e^{-t\Delta}\times\mu^{4-D}\int\frac{d^D\ell_E}{(2\pi)^D}\,e^{-t\ell_e^2}\\
&=\; \int_0^\infty dt\Big(a\times t \;-\; (a\Delta+b)\times\tfrac{1}{2}t^2\Big)e^{-t\Delta}\times\frac{\mu^{4-D}}{(4\pi t)^{D/2}}\\
&=\; \frac{\mu^{4-D}}{(4\pi)^{D/2}}\int_0^\infty dt\,e^{-t\Delta}\times\Big(a\times t^{1-(D/2)} \;-\; \tfrac{1}{2}(a\Delta+b)\times t^{2-(D/2)}\Big)\\
&=\; \frac{\mu^{4-D}}{(4\pi)^{D/2}}\left\{a\times\Gamma\left(2-\tfrac{D}{2}\right)\times\Delta^{\frac{D}{2}-2} \;-\; \tfrac{1}{2}(a\Delta+b)\times\Gamma\left(3-\tfrac{D}{2}\right)\times\Delta^{\frac{D}{2}-3}\right\}\\
&\to\; \frac{(4\pi\mu^2)^\epsilon}{16\pi^2}\times\frac{\Gamma(1+\epsilon)}{\Delta^\epsilon}\times\left\{\frac{a}{\epsilon} \;-\; \frac{a\Delta+b}{2\Delta}\right\}.
\end{aligned} \tag{53}$$

Going back to eq. (31), we identify a and b in eq. (52) as

$$\begin{aligned} a &= \frac{(D-2)^2}{D} = \frac{2(1-\epsilon)^2}{2-\epsilon}\,, \\ b &= 2z\times(2m^2-q^2) - (D-2)\times\Delta = \hat{b} - 2(1-\epsilon)\Delta\,, \\ \text{where } \hat{b} &= 2z(2m^2-q^2). \end{aligned} \tag{54}$$

Consequntly, on the last line of eq. (53) we have

$$\begin{aligned} \frac{a}{\epsilon} - \frac{a\Delta+b}{2\Delta} &= a\times\left(\frac{1}{\epsilon}-\frac{1}{2}\right) + \frac{2(1-\epsilon)\Delta}{2\Delta} - \frac{\hat{b}}{2\Delta} \\ &= \frac{2(1-\epsilon)^2}{2-\epsilon}\times\frac{2-\epsilon}{2\epsilon} + (1-\epsilon) - \frac{\hat{b}}{2\Delta} \\ &= \frac{1-\epsilon}{\epsilon}\times((1-\epsilon)+\epsilon=1) - \frac{\hat{b}}{2\Delta} \\ &= \frac{1-\epsilon}{\epsilon} - \frac{z(2m^2-q^2)}{\Delta}\,, \end{aligned} \tag{55}$$

so the momentum integral for the electric form factors evaluates to

$$\begin{aligned} -2ie^2\mu^{4-D}\int\frac{d^D\ell}{(2\pi)^D}\,\frac{\mathcal{N}_1}{[\ell^2-\Delta+i0]^3} &= \\ &= \frac{\alpha}{2\pi}\left(\frac{4\pi\mu^2}{\Delta}\right)^{\epsilon}\left\{\Gamma(\epsilon)\times(1-\epsilon) - \Gamma(1+\epsilon)\times\frac{z\times(2m^2-q^2)}{\Delta}\right\}. \end{aligned} \tag{56}$$

The next step in our claculation is to integrate the result in eq. (56) over the Feynman parameters. Changing the integration variables from x, y, z to w and ξ according to eq. (50), we have

$$F_1^{1\,\text{loop}}(q^2) = \frac{\alpha}{2\pi}\,(4\pi\mu^2)^{\epsilon}\int_0^1 d\xi\int_0^1 dw\,w\times\left\{\begin{aligned} &(1-\epsilon)\Gamma(\epsilon)\times\frac{1}{[\Delta(w,\xi)]^{\epsilon}} \\ &-\Gamma(1+\epsilon)\times\frac{(1-w)(2m^2-q^2)}{[\Delta(w,\xi)]^{1+\epsilon}} \end{aligned}\right\} \tag{57}$$

where

$$\Delta(w,\xi) = (1-z)^2m^2 - xyq^2 = w^2\times\Big(m^2 - \xi(1-\xi)q^2\Big), \tag{58}$$

or equivalently,

$$\Delta(w,\xi) = w^2\times H(\xi) \quad\text{where}\quad H(\xi) \stackrel{\text{def}}{=} m^2 - \xi(1-\xi)q^2. \tag{59}$$

The form (59) is particularly convenient for evaluating the $\int dw$ integral in eq. (57), which becomes

$$\int_0^1 dw\left\{\frac{2(1-\epsilon)\Gamma(\epsilon)}{H^\epsilon}\times\frac{w}{w^{2\epsilon}}\;-\;2\Gamma(1+\epsilon)\times\frac{2m^2-q^2}{H^{1+\epsilon}}\times\frac{w(1-w)}{w^{2+2\epsilon}}\right\}. \tag{60}$$

Near the lower limit $w\to 0$, the integrand is dominated by the second term, which is proportional to $w^{-1-2\epsilon}$. But for any $\epsilon\geq 0$ — *i.e.*, for any dimension $D\leq 4$ — the integral

$$\int_0^{\text{positive}}\frac{dw}{w^{1+2\epsilon}} \tag{61}$$

diverges: For $D=4$ the divergence is logarithmic while for $D<4$ it becomes power-like.

The Infrared Divergence

Physically, the divergence (61) is *infrared* rather than ultraviolet, that's why it gets worse as we lower the dimension D. Indeed, let's go back to the diagram (4) and look at the denominator $\mathcal{D}$ in eqs. (5) and (7). Taking the electron's momenta p and p' on-shell before introducing the Feynman parameters, we have

$$(p+k)^2-m^2\;=\;k^2\,+\,2kp\,+\,\cancel{p^2}\,-\,\cancel{m^2}\;=\;k^2\,+\,2kp\;=\;O(|k|)\quad\text{when }k\to 0, \tag{62}$$

and likewise

$$(p'+k)^2-m^2\;=\;k^2\,+\,2kp'\;=\;O(|k|)\quad\text{when }k\to 0. \tag{63}$$

Combining these two electron propagators with the $O(1/k^2)$ photon propagators, we see that the net denominator behaves as $\mathcal{D}\propto|k|^4$ for $k\to 0$ the numerator $\mathcal{N}^\mu$ remains finite, which makes the integral

$$\int d^Dk\,\frac{\mathcal{N}^\mu}{\mathcal{D}}\;\propto\;\int\frac{d^Dk}{|k|^4} \tag{64}$$

diverge for $k\to 0$. In $D=4$ dimensions, the infrared divergence here is logarithmic, while in lower dimensions $D<4$ it becomes power-like, *i.e.* $O\left((1/k_{\min})^{4-D}\right)$ — precisely as in eqs. (61) and (60).

We can regularize the infrared divergence (64) — and also (61) — by analytically continuing the spacetime dimension to $D > 4$. Such dimensional regularization of the IR divergences is used in many situations in both QFT and condensed matter. However, taking $D > 4$ makes the ultraviolet divergences worse, so if some amplitude has both UV and IR divergences, we cannot cure both of them at the same time by analytically continuing to $D \neq 4$. In particular, when calculating the electric form factor $F_1(q^2)$ of the electron, we need $D < 4$ to regulate the momentum integral $\int d^D\ell$, but then we need $D > 4$ to regulate the integral over the Feynman parameters.

A common *dirty trick* is to first continue to $D < 4$, shift the loop momentum from k^μ to $\ell^\mu = k^\mu + \text{shift}$, evaluate the $\int d^D\ell$ momentum integral in $D < 4$ dimension, then analytically continue the result to $D > 4$ and integrate over the Feynman parameters, and ultimately continue the final result to $D = 4$. However, in this kind of dimensional regularization it is hard to disentangle the $1/\epsilon$ poles coming from the UV divergence $\log(\Lambda^2/\mu^2)$ from the $1/\epsilon$ poles coming from the IR divergence $\log(\mu^2/k^2_{\text{min}})$, so we are not going to use it here.

Instead, we are going to use DR for the UV divergence only, while the IR divergence is regulated by a tiny but not-quite-zero photon mass $m_\gamma^2 \ll m_e^2$. Strictly speaking, a massive vector particle has three polarization states and its propagator is

$$= \frac{-i}{k^2 - m_\gamma^2 + i0} \times \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{m_\gamma^2}\right). \tag{65}$$

However, the longitudinal polarization of the massive but ultra-relativistic photon does not couple to a conserved current, so we are going to disregard the $k^\mu k^\nu$ terms in the propagator (65) and use

$$= \frac{-ig^{\mu\nu}}{k^2 - m_\gamma^2 + i0}. \tag{66}$$

In other words, we use the Feynman gauge in spite of the photon's mass; this is not completely consistent, but the inconsistencies go away in the $m_\gamma \to 0$ limit.

Using this infrared regulator for the internal photon line in the one-loop diagram (4), we get the vertex amplitude that looks exactly like eq. (5) except for one factor in the denominator,

$$\frac{1}{k^2 + i0} \quad \text{becomes} \quad \frac{1}{k^2 - m_\gamma^2 + i0}. \tag{67}$$

In terms of the integral (15), this change has no effect on the numerator $\mathcal{N}^\mu$ or the loop momentum

ℓ (which remains exactly as in eq. (13)), but the Δ in the denominator becomes

$$\Delta'(x,y,z) \;=\; \Delta(x,y,z) \;+\; z\times m_\gamma^2\,. \tag{68}$$

Consequently, the electric form factor is

$$F_1^{1\,\mathrm{loop}}(q^2) \;=\int d(FP)\int \mu^{4-D}\,\frac{d^D\ell}{(2\pi)^D}\,\frac{-2ie^2\times\mathcal{N}_1}{[\ell^2-\Delta'+i0]^3}\,, \tag{69}$$

exactly as in eq. (34), except for the Δ' instead of the Δ in the denominator. The momentum integral here converges for any $D<4$ and it evaluates exactly as in eq. (53). The only subtlety here is that in the numerator, the ℓ–independent term b involves the un-modified Δ instead of Δ' (*cf.* eq. (54)), but we can fix that by writing

$$b \;=\; 2z\times\left(2m_e^2-q^2+(1-\epsilon)m_\gamma^2\right) \;-\; 2(1-\epsilon)\times\Delta'. \tag{70}$$

Hence, instead of eq. (57) we get

$$F_1^{1\,\mathrm{loop}}(q^2) \;=\; \frac{\alpha}{2\pi}\,(4\pi\mu^2)^\epsilon\int_0^1\!d\xi\int_0^1\!dw\,w\times\left\{\begin{array}{l}(1-\epsilon)\Gamma(\epsilon)\times\dfrac{1}{[\Delta'(w,\xi)]^\epsilon}\\[2ex] -\,\Gamma(1+\epsilon)\times\dfrac{(1-w)(2m_e^2-q^2+(1-\epsilon)m_\gamma^2)}{[\Delta'(w,\xi)]^{1+\epsilon}}\end{array}\right\} \tag{71}$$

where

$$\Delta'(w,\xi) \;=\; (1-z)^2m_e^2 \,-\, xyq^2 \,+\, zm_\gamma^2 \;=\; w^2\times H(\xi) \;+\; (1-w)\times m_\gamma^2\,. \tag{72}$$

Note that the photon's mass is tiny, $m_\gamma^2\ll m_e^2,q^2$; were it not for the IR divergences, we would have used $m_\gamma^2=0$. This allows us to neglect various $O(m_\gamma^2)$ terms in eq. (71) except when it would cause a divergence for $w\to0$; in particular, we may neglect the $(1-\epsilon)m_\gamma^2$ term in the numerator of the second term in the integrand. As to the denominators, in eq. (72) the second term containing the photon's mass becomes important only in the $w\to0$ limit, and in that limit

$(1-w)m_\gamma^2 \to m_\gamma^2$. Thus, we approximate

$$\Delta'(w,\xi)\ \approx\ w^2\times H(\xi)\ +\ m_\gamma^2 \tag{73}$$

and the $\int dw$ integral in eq. (71) becomes

$$\begin{aligned}
&\int_0^1 dw\, w\times\left\{(1-\epsilon)\Gamma(\epsilon)\times\frac{1}{[w^2H(\xi)+m_\gamma^2]^\epsilon}\ -\ \Gamma(1+\epsilon)\times\frac{(1-w)(2m_e^2-q^2)}{[w^2H(\xi)+m_\gamma^2]^{1+\epsilon}}\right\}\\
&\qquad=\ \frac{(1-\epsilon)\Gamma(\epsilon)}{H^\epsilon}\times\int_0^1\frac{dw\,w}{[w^2+(m_\gamma^2/H)]^\epsilon}\\
&\qquad\quad+\ \Gamma(1+\epsilon)\,\frac{2m_e^2-q^2}{H^{1+\epsilon}}\times\int_0^1\frac{dw\,w^2}{[w^2+(m_\gamma^2/H)]^{1+\epsilon}}\\
&\qquad\quad-\Gamma(1+\epsilon)\,\frac{2m_e^2-q^2}{H^{1+\epsilon}}\times\int_0^1\frac{dw\,w}{[w^2+(m_\gamma^2/H)]^{1+\epsilon}}\,.
\end{aligned} \tag{74}$$

For $0<\epsilon<\frac{1}{2}$ — *i.e.*, for $3<D<4$ — the integrals on the second and third lines here converge even for $m_\gamma^2=0$,

$$\begin{aligned}
\int_0^1\frac{dw\,w}{[w^2]^\epsilon}\ &=\ \frac{1}{2-2\epsilon}\quad\text{for }\epsilon<1,\\
\int_0^1\frac{dw\,w^2}{[w^2]^{1+\epsilon}}\ &=\ \frac{1}{1-2\epsilon}\quad\text{for }\epsilon<\tfrac{1}{2},
\end{aligned} \tag{75}$$

so we may just as well evaluate them without the photon's mass. Only on the last line of eq. (74) we do need $m_\gamma^2\neq 0$ to make the integral converge for some $D\leq 4$:

$$\int_0^1\frac{dw\,w}{[w^2+(m_\gamma^2/H)]^{1+\epsilon}}\ =\ \frac{-1}{2\epsilon}\ \frac{1}{[w^2+(m_\gamma^2/H)]^\epsilon}\bigg|_0^1\ =\ \frac{1}{2\epsilon}\left[\left(\frac{H}{m_\gamma^2}\right)^\epsilon-1\right]. \tag{76}$$

Combining all these $\int dw$ integrals together, we get

$$\int\limits_0^1 dw\Bigl\{\cdots\Bigr\} \;=\; \frac{\Gamma(\epsilon)}{2H^\epsilon} \;+\; \frac{\Gamma(1+\epsilon)}{1-2\epsilon}\times\frac{2m_e^2-q^2}{H^{1+\epsilon}} \;-\; \frac{\Gamma(1+\epsilon)}{2\epsilon}\times\frac{2m_e^2-q^2}{H^{1+\epsilon}}\times\left[\left(\frac{H}{m_\gamma^2}\right)^\epsilon - 1\right]$$
$$= \frac{\Gamma(\epsilon)}{2H^\epsilon}\times\left\{1 \;+\; \frac{2m_e^2-q^2}{H(\xi)}\times\left(\frac{2\epsilon}{1-2\epsilon} \;+\; 1 \;=\; \frac{1}{1-2\epsilon}\right) \;-\; \frac{2m_e^2-q^2}{H(\xi)}\times\left(\frac{H(\xi)}{m_\gamma^2}\right)^\epsilon\right\} \tag{77}$$

and hence

$$F_1^{1\,\mathrm{loop}}(q^2) \;=\; \frac{\alpha}{4\pi}\int\limits_0^1 d\xi\,\Gamma(\epsilon)\left(\frac{4\pi\mu^2}{H(\xi)}\right)^\epsilon \times\left\{1 \;+\; \frac{2m_e^2-q^2}{H}\times\left[\frac{1}{1-2\epsilon} \;-\; \left(\frac{H}{m_\gamma^2}\right)^\epsilon\right]\right\} \tag{78}$$

where

$$H(\xi) \;=\; m_e^2 \;-\; \xi(1-\xi)q^2. \tag{59}$$

Before we even try to perform this last integral, let's remember that

$$\Gamma^\mu_{\mathrm{net}} \;=\; \gamma^\mu_{\mathrm{tree}} \;+\; \Gamma^\mu_{\mathrm{loops}} \;+\; \delta_1\times\gamma^\mu \tag{79}$$

and hence

$$F_1^{\mathrm{net}}(q^2) \;=\; 1^{\mathrm{tree}} \;+\; F_1^{\mathrm{loops}}(q^2) \;+\; \delta_1\,. \tag{80}$$

Also, the net electric charge does not renormalize, so we must have

$$F_1^{\mathrm{net}}(q^2) \;\to\; 1 \quad \text{for } q^2\to 0 \tag{81}$$

and hence

$$\delta_1 \;=\; -F_1^{\mathrm{loops}}(q^2=0). \tag{82}$$

In particular, the δ_1 counterterm to the order α follows from eq. (78) for $q^2=0$, in which case $H(\xi)\equiv m_e^2$ and the $\int d\xi$ integral becomes trivial (the integrand does not depend on ξ at all).

Thus,

$$\delta_1 \;=\; -\frac{\alpha}{4\pi}\,\Gamma(\epsilon)\left(\frac{4\pi\mu^2}{m_e^2}\right)^{\epsilon}\times\left\{1\,+\,\frac{2}{1-2\epsilon}\;-\;2\left(\frac{m_e^2}{m_\gamma^2}\right)^{\epsilon}\right\}\;+\;O(\alpha^2). \tag{83}$$

This formula holds for any dimension D between 3 and 4 (*i.e.*, $0<\epsilon<\frac{1}{2}$). In the $D\to 4$ limit, it becomes

$$\delta_1 \;=\; -\frac{\alpha}{4\pi}\times\left\{\frac{1}{\epsilon}\;-\;\gamma_E\;+\;\log\frac{4\pi\mu^2}{m_e^2}\;+\;4\;-\;2\log\frac{m_e^2}{m_\gamma^2}\right\}+\;O(\alpha^2). \tag{84}$$

Now let's go back to the electric form factor $F_1^{\rm net}(q^2)$ for $q^2\neq 0$. According to eqs. (80) and (82), at the one-loop level

$$F_1^{\rm net}(q^2)\;-\;1\;=\;F_1^{\rm 1\,loop}(q^2)\;-\;F_1^{\rm 1\,loop}(0)\;+\;O(\alpha^2) \tag{85}$$

where $F_1^{\rm 1\,loop}(q^2)$ is given by eq. (78). Taking the $\epsilon\to 0$ limit of that formula, we arrive at

$$F_1^{\rm 1\,loop}(q^2)\;=\;\frac{\alpha}{4\pi}\int\limits_0^1\!d\xi\,\left\{\frac{1}{\epsilon}\;-\;\gamma_E\;+\;\log\frac{4\pi\mu^2}{H(\xi)}\;+\;\frac{2m_e^2-q^2}{H(\xi)}\times\left[2\;-\;\log\frac{H(\xi)}{m_\gamma^2}\right]\right\}, \tag{86}$$

and now we should subtract a similar a similar expression for $q^2=0$. This subtraction cancels the UV divergence and the associated $1/\epsilon$ pole but not the IR divergence. Moreover, not only the subtracted one-loop amplitude depends on the IR regulators, but the coefficient of the $\log m_\gamma^2$ has a non-trivial momentum dependence. Indeed,

$$\begin{aligned}
F_1^{\rm 1\,loop}(q^2)\;-\;F_1^{\rm 1\,loop}(0)\;&=\\
&=\;\frac{\alpha}{4\pi}\int\limits_0^1\!d\xi\,\left\{\log\frac{m_e^2}{H(\xi)}\;+\;\frac{2m_e^2-q^2}{H(\xi)}\times\left[2\;-\;\log\frac{H(\xi)}{m_\gamma^2}\right]\;-\;2\left[2\;-\;\log\frac{m_e^2}{m_\gamma^2}\right]\right\}\\
&=\;\frac{\alpha}{4\pi}\int\limits_0^1\!d\xi\,\left\{\log\frac{m_e^2}{H(\xi)}\;+\;\frac{2m_e^2-q^2}{H(\xi)}\times\left[2\;-\;\log\frac{m_e^2}{m_\gamma^2}\;-\;\log\frac{H(\xi)}{m_e^2}\right]\;-\;2\left[2\;-\;\log\frac{m_e^2}{m_\gamma^2}\right]\right\}\\
&=\;\frac{\alpha}{4\pi}\int\limits_0^1\!d\xi\,\left\{\log\frac{m_e^2}{H(\xi)}\times\left(1\;+\;\frac{2m_e^2-q^2}{H(\xi)}\right)\;+\;\left[2\;-\;\log\frac{m_e^2}{m_\gamma^2}\right]\times\left(\frac{2m_e^2-q^2}{H(\xi)}\;-\;2\right)\right\}\\
&=\;-\frac{\alpha}{2\pi}\times\left\{f_{\rm IR}(q^2/m_e^2)\times\log\frac{m_e^2}{m_\gamma^2}\;+\;h(q^2/m_e^2)\right\}
\end{aligned} \tag{87}$$

where

$$f_{\rm IR}(q^2/m_e^2) \;=\; \int\limits_0^1 d\xi \left(\frac{2m_e^2-q^2}{2H(\xi)}\;-\;1\right), \tag{88}$$

$$h(q^2/m_e^2) \;=\; -2f_{\rm IR}(q^2/m_e^2)\;-\;\frac{1}{2}\int\limits_0^1 d\xi \left(\frac{2m_e^2-q^2}{H(\xi)}\;+\;1\right)\times\log\frac{m_e^2}{H(\xi)}\,. \tag{89}$$

Note that both of these integrals are finite in the $m_\gamma\to0$ limit. Also, both $f_{\rm IF}$ and h vanish for $q=0$, while for non-zero but small $q^2\ll m_e^2$ both $f_{\rm IF}$ and h diminish as $O(q^2/m_e^2)$:

$$\text{For } q^2\ll m_e^2,\quad f_{\rm IR}(q^2/m_e^2)\;\approx\;-\frac{q^2}{3m_e^2}\,,\quad h(q^2/m_e^2)\;\approx\;-\frac{5q^2}{12m_e^2}\,. \tag{90}$$

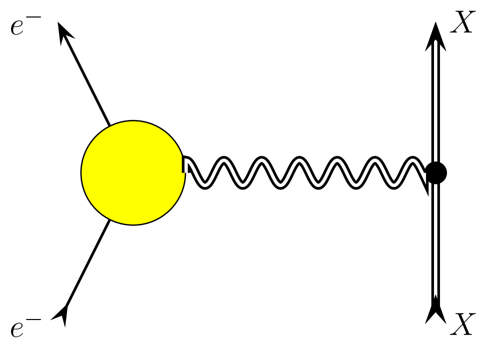
In the opposite limit of very large q^2, the integrals (88) and (89) grow as polynomials of $\log(q^2)$:

$$\text{For } q^2\gg m_e^2\quad \begin{cases} f_{\rm IR}(q^2/m_e^2)\;\approx\;\log\dfrac{-q^2}{m_e^2}\;-\;1, \\[2ex] h(q^2/m_e^2)\;\approx\;\dfrac{1}{2}\left(\log\dfrac{-q^2}{m_e^2}\;-\;2\right)\times f_{\rm IR}(q^2/m_e^2)\;-\;\dfrac{\pi^2}{6}\,. \end{cases} \tag{91}$$

Finite Cross-sections for IR–Divergent Amplitudes

Remarkably, the function $f_{\rm IR}(q^2/m_e^2)$ from eq. (88) which governs the infrared divergence of the electron's electric form factor $F_1(q^2)$ also govern the IR divergence of the the soft-photon bremsstrahlung. In terms of §6.1 of the Peskin & Schroeder textbook, $\mathcal{I}(\mathbf{v},\mathbf{v}')=2f_{\rm IR}(q^2/m_e^2)$. Let me postpone the proof of this identity towards the end of these notes (or you can find it in the textbook \$6.4, eqs. (6.69–70).) Meanwhile, let me explore the physical consequences of the related IR divergences of the F_1 form factor and of the bremsstrahlung.

Consider elastic Coulomb scattering of an electron off a very heavy point-like particle X:



(92)

For $q^2 \ll M_X^2$, we may approximate the X particle as a static (non-recoiling) source of electric field, and in its rest frame (the lab frame), the scattering amplitude evaluates to

$$\frac{\mathcal{M}(eX \to eX)}{2M_X} \;=\; \frac{4\pi\alpha_{\mathrm{eff}}(q^2)}{q^2} \times \bar{u}(p')\Gamma^0(p',p)u(p). \tag{93}$$

At the tree level $\Gamma^\mu(p',p) \equiv \gamma^\mu$, while at the one-loop level we get non-trivial form-factors. Focusing at their infrared divergences, we have

$$F_2^{1\,\mathrm{loop}}(q^2) \;=\; \frac{2\alpha}{2\pi} \times \mathrm{finite} \tag{94}$$

while

$$F_1^{\mathrm{tree}+1\,\mathrm{loop}}(q^2) \;=\; 1 \;-\; \frac{\alpha}{2\pi}\left(f_{\mathrm{IR}}(q^2) \times \log\frac{m_e^2}{m_\gamma^2} \;+\; \mathrm{finite}\right), \tag{95}$$

and therefore

$$\Gamma^\mu_{\mathrm{tree}+1\,\mathrm{loop}}(p',p) \;=\; \left(1 \;-\; \frac{\alpha}{2\pi} \times f_{\mathrm{IR}}(q^2) \times \log\frac{m_e^2}{m_\gamma^2}\right) \times \Gamma^\mu_{\mathrm{tree}} \;+\; \frac{\alpha}{2\pi} \times \mathrm{finite}. \tag{96}$$

Plugging this one-loop vertex into the Coulomb scattering amplitude (93), we obtain

$$\mathcal{M}^{\mathrm{tree}+1\,\mathrm{loop}}(eX \to eX) \;=\; \mathcal{M}^{\mathrm{tree}}(eX \to eX) \times \left(1 \;-\; \frac{\alpha}{2\pi} \times f_{\mathrm{IR}}(q^2) \times \log\frac{m_e^2}{m_\gamma^2} \;+\; \frac{\alpha}{2\pi} \times \mathrm{finite}\right) \tag{97}$$

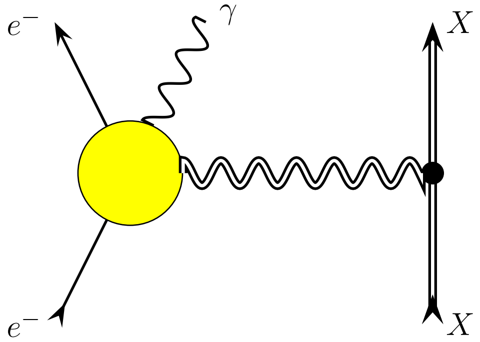
and hence the partial cross-section

$$\begin{aligned} \frac{d\sigma^{\mathrm{tree}+1\,\mathrm{loop}}(eX \to eX)}{d\Omega} \;&=\; \frac{d\sigma^{\mathrm{tree}}(eX \to eX)}{d\Omega} \times \\ &\times \left(1 \;-\; \frac{\alpha}{\pi} \times f_{\mathrm{IR}}(q^2) \times \log\frac{m_e^2}{m_\gamma^2} \;+\; \frac{\alpha}{\pi} \times \mathrm{finite} \;+\; O(\alpha^2)\right). \end{aligned} \tag{98}$$

Note the IR divergence of the one-loop term in this cross-section. For future reference, I would like to rephrase it in terms of $\log(E_e/m_\gamma)$ rather than $\log(m_e/m_\gamma)$, thus

$$\frac{d\sigma^{\text{tree+1 loop}}(eX \to eX)}{d\Omega} = \frac{d\sigma^{\text{tree}}(eX \to eX)}{d\Omega} \times \\ \times \left(1 - \frac{\alpha}{\pi} \times 2 f_{\text{IR}}(q^2) \log\frac{E_e}{m_\gamma} + \frac{\alpha}{\pi} \times \text{finite} + O(\alpha^2)\right). \tag{99}$$

Now consider the inelastic scattering in which a photon is emitted, $eX \to eX\gamma$,



(100)

In general, an extra incoming or outgoing photon costs a factor e in the amplitude and hence a factor α in the cross-section, thus at similar loop levels $\sigma(eX \to eX\gamma) = O(\alpha) \times \sigma(eX \to eX)$. Specifically, as explained in detail in §6.1 of the Peskin & Schroeder textbook, *for a soft photon whose energy is much smaller than the electron's,* $\omega_\gamma \ll E_e$,

$$\mathcal{M}^{\text{tree}}(eX \to eX\gamma) = \mathcal{M}^{\text{tree}}(eX \to eX) \times e\left(\frac{p' \cdot \epsilon^*_\gamma}{p' \cdot k_\gamma} - \frac{p \cdot \epsilon^*_\gamma}{p \cdot k_\gamma}\right) \tag{101}$$ (6.22)

and therefore

$$\frac{d\sigma^{\text{tree}}(eX \to eX\gamma)}{d\Omega_e\, d\omega_\gamma} = \frac{d\sigma^{\text{tree}}(eX \to eX)}{d\Omega} \times \frac{\alpha}{\pi} \times \frac{\mathcal{I}}{\omega_\gamma} \tag{102}$$ (6.25)

where

$$\mathcal{I}(p', p) \stackrel{\text{def}}{=} \int \frac{d^2\Omega_{\mathbf{n}}}{4\pi} \left[-\left(\frac{p'^\mu}{(np')} - \frac{p^\mu}{(np)}\right)^2\right]^{n^0 = |\mathbf{n}| = 1}. \tag{103}$$ (6.13)

Integrating the partial cross-section (102) over the photon's frequencies, we immediately run into the infrared divergence:

$$\int d\omega_\gamma \frac{d\sigma}{d\omega_\gamma} \propto \int\limits_0^{\omega_{\max} \sim E_e} \frac{d\omega_\gamma}{\omega_\gamma} = \infty. \tag{104}$$

To regulate this divergence, we need to impose a minimal energy requirement on the emitted photon, and the simplest way to do this is to assume a tiny but non-zero photon mass m_γ.

Consequently

$$\int\limits_{\rm reg}\frac{d\omega_\gamma}{\omega_\gamma}\ =\ \log\frac{\omega_{\rm max}}{m_\gamma}\ +\ \text{finite}\ =\ \log\frac{E_e}{m_\gamma}\ +\ \text{finite},\tag{105}$$

and therefore

$$\frac{d\sigma^{\rm tree}(eX\to eX\gamma)}{d\Omega}\ =\ \frac{d\sigma^{\rm tree}(eX\to eX)}{d\Omega}\times\frac{\alpha}{\pi}\left(\mathcal{I}(p',p)\times\log\frac{E_e}{m_\gamma}\ +\ \text{finite}\right).\tag{106}$$

We shall see momentarily that $\mathcal{I}(p',p)\ =\ 2f_{\rm IR}(q^2)$. Consequently, the one-loop-level cross-section (99) and the tree-level cross-section (106) have exactly the same infrared divergence, except for opposite signs,

$$\begin{aligned}\frac{d\sigma^{\rm tree+1\,loop}(Xe\to Xe)}{d\Omega}\ &=\ \frac{d\sigma^{\rm tree}(eX\to eX)}{d\Omega}\times\\ &\qquad\times\left(1-\frac{\alpha}{\pi}\times 2f_{\rm IR}(q^2)\log\frac{E_e}{m_\gamma}\ +\ \frac{\alpha}{\pi}\times\text{finite}\ +\ O(\alpha^2)\right),\end{aligned}\tag{99}$$

$$\begin{aligned}\frac{d\sigma^{\rm tree}(Xe\to Xe\gamma)}{d\Omega}\ &=\ \frac{d\sigma^{\rm tree}(eX\to eX)}{d\Omega}\times\\ &\qquad\times\left(0+\frac{\alpha}{\pi}\times 2f_{\rm IR}(q^2)\log\frac{E_e}{m_\gamma}\ +\ \frac{\alpha}{\pi}\times\text{finite}\right),\end{aligned}\tag{106}$$

and therefore the combined cross-section has no infrared divergence:

$$\frac{d\sigma^{\rm tree+1\,loop}(Xe\to Xe)\ +\ d\sigma^{\rm tree}(Xe\to Xe\gamma)}{d\Omega_e}\ =\ \frac{d\sigma^{\rm tree}(Xe\to Xe)}{d\Omega_e}\times\left(1\ +\ \frac{\alpha}{\pi}\times\text{finite}\ +\ O(\alpha)^2\right).\tag{107}$$

But what do we do with the IR divergences of the partial cross-sections (99) and (106)? While it is OK to UV-regulate or IR-regulate the intermediate stages of a calculation, but a finite result for a measurable quantity like a partial cross-section must be finite and it cannot depend on an IR regulator like m_γ. Nevertheless, eqs. (106) and (99) seem to contradict this rule, so what gives?

To resolve this paradox, consider a real-life scattering experiment. No photon detector can detect a photon with an arbitrarily low energy ω_γ, there is always a threshold $\omega_{\rm thr}>0$ below which the detector is blind. Thus, a final state $|Xe\gamma\rangle$ where the photon's energy is below the threshold — $\omega_\gamma<\omega_{\rm thr}$ — will be seen by the detector as simply $|Xe\rangle$ since the photon would not be detected. In other words, *observationally* we should not classify the final states by the net number of photons with any energy at all but rather by the number pf *detectable* photons with $\omega_\gamma > \omega_{\rm thr}$. As to the soft photons with $\omega_\gamma < \omega_{\rm thr}$, the detector would not tell us if they are there or not, so the physically measureable cross-sections should include all possible numbers of undetectably low-energy photons. In particular,

$$\begin{aligned}
d\sigma(X+e\to\text{observed }X+e) &= d\sigma(X+e\to X+e)\ +\ d\sigma(X+e\to X+e+\gamma(\omega<\omega_{\rm thr}))\\
&\quad +\ d\sigma(X+e\to X+e+\gamma(\omega<\omega_{\rm thr})+\gamma(\omega<\omega_{\rm thr}))\ +\ \cdots,\\
d\sigma(X+e\to\text{observed }X+e+\gamma) &= d\sigma(X+e\to X+e+\gamma(\omega>\omega_{\rm thr}))\\
&\quad +\ d\sigma(X+e\to X+e+\gamma(\omega>\omega_{\rm thr})+\gamma(\omega<\omega_{\rm thr}))\ +\ \cdots,
\end{aligned}$$

etc., etc.

(108)

To the order $O(\alpha\times\sigma^{\rm tree}(Xe\to Xe)=O(\alpha^3)$, we should stop at one final-state photon, detectable or not, and calculate the $d\sigma(Xe\to Xe)$ to the one-loop level while the $d\sigma(Xe\to Xe\gamma)$ just to the tree level. Thus,

$$\begin{aligned}
\frac{d\sigma(X+e\to\text{observed }X+e+\gamma)}{d\Omega_e} &\approx \frac{d\sigma^{\rm tree}(X+e\to X+e+\gamma(\omega>\omega_{\rm thr}))}{d\Omega_e}\\
&= \int\limits_{\omega_{\rm thr}}^{\omega_{\max}=O(E_e)} d\omega_\gamma\,\frac{d\sigma^{\rm tree}(X+e\to X+e+\gamma)}{d\Omega_e\,d\omega_\gamma}\\
&= \frac{d\sigma^{\rm tree}(X+e\to X+e)}{d\Omega_e}\times\frac{\alpha}{\pi}\left(2f_{\rm IR}(q^2)\log\frac{E_e}{\omega_{\rm thr}}\ +\ \text{finite}\right),
\end{aligned}\tag{109}$$

where the last equality comes from

$$\frac{d\sigma^{\rm tree}(eX\to eX\gamma)}{d\Omega_e\,d\omega_\gamma}\ =\ \frac{d\sigma^{\rm tree}(eX\to eX)}{d\Omega}\times\frac{\alpha}{\pi}\times\frac{2f_{\rm IR}(q^2)}{\omega_\gamma}\qquad\langle\!\langle\text{ for }\omega_\gamma\ll E_e\,\rangle\!\rangle\tag{102}$$ (6.25)

and

$$\int\limits_{\omega_{\rm thr}}^{\omega_{\max}=O(E_e)}\frac{d\omega}{\omega}\ =\ \log\frac{E_e}{\omega_{\rm thr}}\ +\ \text{finite}.\tag{110}$$

Note that *observed* the cross-section (109) is infrared finite and does not depend on the m_γ (as long as $m_\gamma\ll\omega_{\rm thr}$). Instead, it depends on the photon detector's low-energy threshold $\omega_{\rm thr}$.

Similarly, to the same order $O(\alpha\times\sigma^{\rm tree}(Xe\to Xe)=O(\alpha^3)$,

$$\begin{aligned}
\frac{d\sigma(X+e\to\text{observed }X+e)}{d\Omega_e}\ &\approx\\
\approx\ &\frac{d\sigma^{\text{tree+1 loop}}(X+e\to X+e)}{d\Omega_e}\ +\ \frac{d\sigma^{\rm tree}(X+e\to X+e+\gamma(\omega<\omega_{\rm thr}))}{d\Omega_e}\\
\approx\ &\frac{d\sigma^{\rm tree}(X+e\to X+e)}{d\Omega_e}\times\left(1-\frac{\alpha}{\pi}\times 2f_{\rm IR}(q^2)\log\frac{E_e}{\cancel{m_\gamma}}\ +\ \frac{\alpha}{\pi}\times\text{finite}\right)\\
&+\ \frac{d\sigma^{\rm tree}(X+e\to X+e)}{d\Omega_e}\times\left(0+\frac{\alpha}{\pi}\times 2f_{\rm IR}(q^2)\log\frac{\omega_{\rm thr}}{\cancel{m_\gamma}}\ +\ \frac{\alpha}{\pi}\times\text{finite}\right)\\
=\ &\frac{d\sigma^{\rm tree}(X+e\to X+e)}{d\Omega_e}\times\left(1\ -\ \frac{\alpha}{\pi}\times 2f_{\rm IR}(q^2)\log\frac{E_e}{\omega_{\rm thr}}\ +\ \frac{\alpha}{\pi}\times\text{finite}\right).
\end{aligned}\tag{111}$$

Note how the IR regulator m_γ cancels between the two contributions to the net *observed* cross-section. Again, the observed cross-section is IR-finite, but it depends on the photon detector's threshold $\omega_{\rm thr}$.

Similar cancellations of IR divergences from the observed cross-sections happen at the higher loop orders. In general, go get a finite cross-section to the order $O(\alpha^L\times\sigma^{\rm tree})$ we should combine an L-loop cross-section with no soft photons, an $L-1$ loop cross-section with one soft photon, an $L-2$ cross-section with 2 soft photons, *etc.*, *etc.*, ending with a tree-level cross-section with L soft photons. Individually, all these formal cross-sections are infrared-divergent, but once we combine them together into a complete observed cross-section, the IR divergence should cancel out. Please read (or at least skim) textbook §6.5 to see how this works.

⋆ ⋆ ⋆

Let me conclude these notes by proving that $\mathcal{I}(p',p)=2f_{\rm IR}(q^2/m^2)$. By definition of the soft bremsstrahlung factor $\mathcal{I}(p',p)$ — or as the textbook calls it $\mathcal{I}(\mathbf{v},\mathbf{v}')$, since it only depends on the velocities and not the electron's mass, —

$$\mathcal{I}(p',p)\ =\int\!\frac{d^2\Omega_{\mathbf{n}}}{4\pi}\left[-\left(\frac{p'^\mu}{(np')}\ -\ \frac{p^\mu}{(np)}\right)^2\right]^{n^0=|\mathbf{n}|=1}.\tag{103}$$ (6.13)

Inside the integral here, we have

$$-\left(\frac{p'^\mu}{(np')}\ -\ \frac{p^\mu}{(np)}\right)^2\ =\ -\frac{m^2}{(np)^2}\ -\ \frac{m^2}{(np')^2}\ +\ \frac{2(pp')=2m^2-q^2}{(np)(np')}\,.\tag{112}$$

Now let's integrate each of these tree terms over the directions of the unit 3-vector $\mathbf{n}$. For the first term, we have

$$
\begin{aligned}
\int \frac{d^2\Omega_{\mathbf{n}}}{4\pi}\,\frac{1}{(np)^2} \;&=\; \frac{2\pi}{4\pi}\int\limits_{-1}^{+1} d\cos\theta\,\frac{1}{(E-|\mathbf{p}|\times\cos\theta)^2} \;=\; \frac{1}{2}\times\frac{1}{|\mathbf{p}|}\int\limits_{E-|\mathbf{p}|}^{E+|\mathbf{p}|}\frac{d(E-|\mathbf{p}|\times\cos\theta)}{(E-|\mathbf{p}|\times\cos\theta)^2} \\
&=\; \frac{1}{2|\mathbf{p}|}\left(\frac{1}{E-|\mathbf{p}|}\;-\;\frac{1}{E+|\mathbf{p}|}\right) \;=\; \frac{1}{E^2-|\mathbf{p}|^2} \;=\; \frac{1}{m^2}\,.
\end{aligned}
\tag{113}
$$

Likewise, for the second term

$$
\int \frac{d^2\Omega_{\mathbf{n}}}{4\pi}\,\frac{1}{(np')^2} \;=\; \frac{1}{m^2}\,.
\tag{114}
$$

Finally, for the third term we use the Feynman parameter trick:

$$
\frac{1}{(np)\times(np')} \;=\; \int\limits_0^1 d\xi\,\frac{1}{[(1-\xi)\times(np)+\xi\times(np')]^2} \;=\; \int\limits_0^1 d\xi\,\frac{1}{(np_\xi)^2}
\tag{115}
$$

where

$$
p_\xi^\mu \;=\; (1-\xi)\times p^\mu \;+\; \xi\times p'^\mu.
\tag{116}
$$

Consequently,

$$
\int \frac{d^2\Omega_{\mathbf{n}}}{4\pi}\,\frac{1}{(np)\times(np')} \;=\; \int\limits_0^1 d\xi\int \frac{d^2\Omega_{\mathbf{n}}}{4\pi}\,\frac{1}{(np_\xi)^2} \;=\; \int\limits_0^1 \frac{d\xi}{p_\xi^2}\,,
\tag{117}
$$

where

$$
\begin{aligned}
p_\xi^2 \;&=\; \bigl[(1-\xi)p+\xi p'\bigr]^2 \\
&=\; (1-\xi)^2\times(p^2=m^2) \;+\; \xi^2\times(p'^2=m^2) \;+\; \xi(1-\xi)\times(2pp'=2m^2-q^2) \\
&=\; m^2 \;-\xi(1-\xi)q^2.
\end{aligned}
\tag{118}
$$

Plugging all these formulae back into eqs. (103) and (112), we arrive at

$$
\mathcal{I}(p',p) \;=\; -\frac{m^2}{m^2} \;-\; \frac{m^2}{m^2} \;+\; (2m^2-q^2)\times\int\limits_0^1\frac{d\xi}{m^2-\xi(1-\xi)q^2} \;=\; \int\limits_0^1 d\xi\left(\frac{2m^2-q^2}{m^2-\xi(1-\xi)q^2}\;-\;2\right).
\tag{119}
$$

Finally comparing the last formula with the definition of the $f_{\rm IR}$,

$$f_{\rm IR}(q^2/m_e^2) \;=\; \int\limits_0^1 d\xi \left(\frac{2m_e^2 - q^2}{2[m^2 - \xi(1-\xi)q^2]} \;-\; 1 \right), \tag{88}$$

we immediately see that $\mathcal{I}(p', p) = 2 f_{\rm IR}(q^2/m_e^2)$, *quod erat demonstrandum.*