

ELECTRIC DIPOLES

In these notes, I write down the electric field of a dipole, and also the net force and the torque on a dipole in the electric field of other charges. For simplicity, I focus on ideal dipoles — also called pure dipoles — where the distance a between the positive and the negative charges is infinitesimal, but the charges are so large that the dipole moment \mathbf{p} is finite.

Electric Field of a Dipole

The potential due to an ideal electric dipole \mathbf{p} is

$$V(\mathbf{r}) = \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{4\pi\epsilon_0 r^2}, \quad (1)$$

or in terms of spherical coordinates where the North pole ($\theta = 0$) points in the direction of the dipole moment \mathbf{p} ,

$$V(r, \theta) = \frac{p \cos \theta}{4\pi\epsilon_0 r^2}. \quad (2)$$

Taking (minus) gradient of this potential, we obtain the dipole's electric field

$$\mathbf{E} = \frac{p}{4\pi\epsilon_0} \left(\frac{2 \cos \theta}{r^3} \nabla r + \frac{\sin \theta}{r^2} \nabla \theta \right) = \frac{p}{4\pi\epsilon_0} \frac{1}{r^3} (2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\boldsymbol{\theta}}). \quad (3)$$

In this formula, the unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ themselves depend on θ and ϕ . Translating them to Cartesian unit vectors, we have

$$\begin{aligned} \hat{\mathbf{r}} &= \sin \theta \cos \phi \hat{\mathbf{x}} + \sin \theta \sin \phi \hat{\mathbf{y}} + \cos \theta \hat{\mathbf{z}}, \\ \hat{\boldsymbol{\theta}} &= \cos \theta \cos \phi \hat{\mathbf{x}} + \cos \theta \sin \phi \hat{\mathbf{y}} - \sin \theta \hat{\mathbf{z}}, \end{aligned} \quad (4)$$

hence

$$2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\boldsymbol{\theta}} = 3 \sin \theta \cos \theta (\cos \phi \hat{\mathbf{x}} + \sin \phi \hat{\mathbf{y}}) + (2 \cos^2 \theta - \sin^2 \theta = 3 \cos^2 \theta - 1) \hat{\mathbf{z}}, \quad (5)$$

and therefore

$$\begin{aligned} E_x(r, \theta, \phi) &= \frac{p}{4\pi\epsilon_0} \frac{3 \sin \theta \cos \theta \cos \phi}{r^3}, \\ E_y(r, \theta, \phi) &= \frac{p}{4\pi\epsilon_0} \frac{3 \sin \theta \cos \theta \sin \phi}{r^3}, \\ E_z(r, \theta, \phi) &= \frac{p}{4\pi\epsilon_0} \frac{3 \cos^2 \theta - 1}{r^3}. \end{aligned}$$

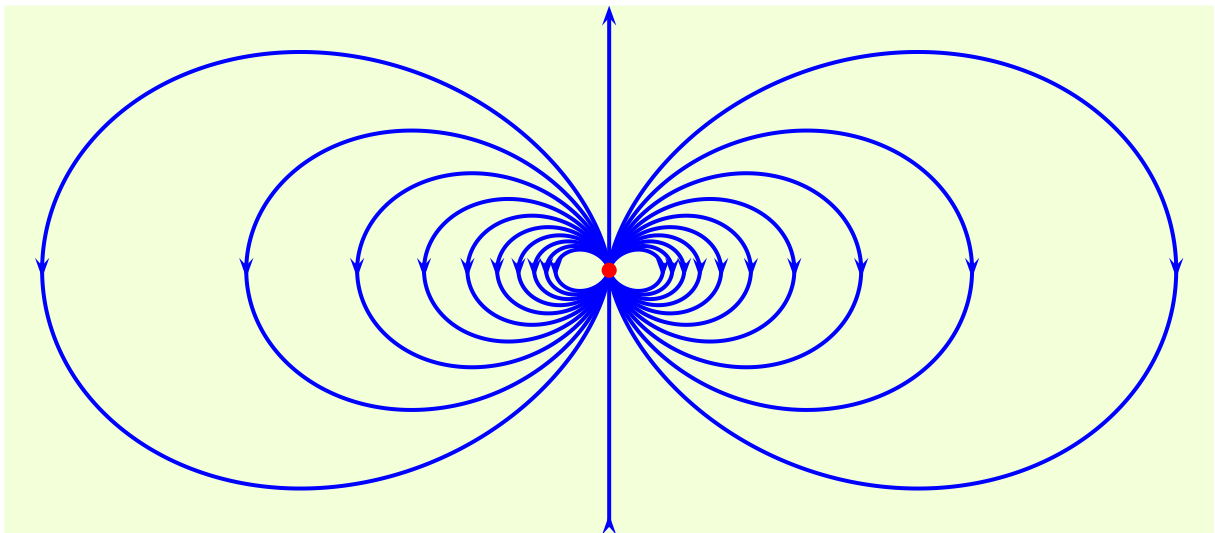
In terms of the (x, y, z) coordinates

$$\begin{aligned} E_x(x, y, z) &= \frac{p}{4\pi\epsilon_0} \frac{3xz}{(x^2 + y^2 + z^2)^{5/2}}, \\ E_y(x, y, z) &= \frac{p}{4\pi\epsilon_0} \frac{3yz}{(x^2 + y^2 + z^2)^{5/2}}, \\ E_z(x, y, z) &= \frac{p}{4\pi\epsilon_0} \frac{2z^2 - x^2 - y^2}{(x^2 + y^2 + z^2)^{5/2}}, \end{aligned} \tag{6}$$

or in vector notations,

$$\mathbf{E}(\mathbf{r}) = \frac{3(\mathbf{p} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}}{4\pi\epsilon_0 r^3}. \tag{7}$$

Here is the picture of the dipole's electric field lines (in the xz plane):



Force and Torque on a Dipole

Now consider an ideal dipole \mathbf{p} placed in an electric field $\mathbf{E}(x, y, z)$ due to some other sources. If this electric field is uniform, there is no net force on the dipole but there is a net torque. Indeed, the force $\mathbf{F}_+ = +q\mathbf{E}$ acting on the positive charge cancels the opposite force $\mathbf{F}_- = -q\mathbf{E} = -\mathbf{F}_+$ acting on the negative charge — so the net force is zero — but the two forces are acting at different points, which causes a torque. Specifically, the net torque of the two forces is

$$\vec{\tau} = \mathbf{r}_+ \times \mathbf{F}_+ + \mathbf{r}_- \times \mathbf{F}_- = (\mathbf{r}_+ - \mathbf{r}_-) \times q\mathbf{E} = q(\mathbf{r}_+ - \mathbf{r}_-) \times \mathbf{E}, \quad (8)$$

or in terms of the dipole moment $\mathbf{p} = q(\mathbf{r}_+ - \mathbf{r}_-)$,

$$\vec{\tau} = \mathbf{p} \times \mathbf{E}. \quad (9)$$

This torque vanishes when the dipole moment \mathbf{p} is parallel to the electric field \mathbf{E} . Otherwise, the torque twists the dipole trying to make it align with the field, $\mathbf{p} \rightarrow \mathbf{p}' \uparrow \mathbf{E}$.

When the electric field $\mathbf{E}(x, y, z)$ is not uniform, the two charges of the dipole feel slightly different electric fields, so the net force on the dipole does not quite vanish:

$$\mathbf{F}^{\text{net}} = q(\mathbf{E}(\mathbf{r}_+) - \mathbf{E}(\mathbf{r}_-)) \neq 0. \quad (10)$$

but for small displacements $\mathbf{a} = \mathbf{r}_+ - \mathbf{r}_-$ between the charges, we may expand the difference between the electric fields acting on them into a power series in \mathbf{a} . Let $\mathbf{r}_\pm = \mathbf{r}_m \pm \frac{1}{2}\mathbf{a}$ where \mathbf{r}_m is the middle of the dipole; then

$$\mathbf{E}(\mathbf{r}_\pm) = \mathbf{E}(\mathbf{r}_m) \pm \left(\frac{1}{2}\mathbf{a} \cdot \nabla\right)\mathbf{E}\Big|_{\text{@}\mathbf{r}_m} + \frac{1}{2}\left(\frac{1}{2}\mathbf{a} \cdot \nabla\right)^2\mathbf{E}\Big|_{\text{@}\mathbf{r}_m} + \frac{1}{6}\left(\frac{1}{2}\mathbf{a} \cdot \nabla\right)^3\mathbf{E}\Big|_{\text{@}\mathbf{r}_m} + \dots, \quad (11)$$

and hence the difference

$$\mathbf{E}(\mathbf{r}_+) - \mathbf{E}(\mathbf{r}_-) = (\mathbf{a} \cdot \nabla)\mathbf{E}\Big|_{\text{@}\mathbf{r}_m} + \frac{1}{24}(\mathbf{a} \cdot \nabla)^3\mathbf{E}\Big|_{\text{@}\mathbf{r}_m} + \dots. \quad (12)$$

Consequently, the net force on the dipole is

$$\mathbf{F}^{\text{net}} = q(\mathbf{a} \cdot \nabla)\mathbf{E}\Big|_{\text{@}\mathbf{r}_m} + \frac{q}{24}(\mathbf{a} \cdot \nabla)^3\mathbf{E}\Big|_{\text{@}\mathbf{r}_m} + \dots. \quad (13)$$

For a physical dipole with a finite distance a between the two charges, we must generally

take into account all the subleading terms in this expansion. But for an ideal dipole we take the limit $a \rightarrow 0$ while $q \times = p$ stays finite, so for any $n > 1$ $q \times a^n \rightarrow 0$. This makes all the subleading terms in eq. (13) negligible compared to the leading term, therefore **the net force on an ideal dipole is simply**

$$\mathbf{F}^{\text{net}} = (\mathbf{p} \cdot \nabla) \mathbf{E} \Big|_{@ \mathbf{r}_m}. \quad (14)$$

The force (14) is conservative and stems from the potential energy

$$U(\mathbf{r}_m, \hat{\mathbf{p}}) = -\mathbf{p} \cdot \mathbf{E}(\mathbf{r}_m). \quad (15)$$

Indeed, (minus) the gradient of this U WRT the dipole's location \mathbf{r}_m taken for a fixed dipole orientation $\hat{\mathbf{p}}$ produces the force (14),

$$-\nabla U \Big|_{\hat{\mathbf{p}} @ \text{fixed}} = (\mathbf{p} \cdot \nabla) \mathbf{E} \Big|_{@ \mathbf{r}_m} = \mathbf{F}^{\text{net}}. \quad (16)$$

Also, variation of the potential energy (15) under infinitesimal rotations of the dipole moment \mathbf{p} accounts for the torque

$$\vec{\tau} = \mathbf{p} \times \mathbf{E}. \quad (9)$$

To be precise, this is the *torque relative to the dipole center* \mathbf{r}_m . In a non-uniform electric field, the torque relative to some other point \mathbf{r}_0 has an extra term due to the net force (14) on the dipole, thus

$$\vec{\tau}_m^{\text{net}} = (\mathbf{r}_m - \mathbf{r}_0) \times \mathbf{F}^{\text{net}} + \mathbf{p} \times \mathbf{E}(\mathbf{r}_m) = (\mathbf{r}_m - \mathbf{r}_0) \times (\mathbf{p} \cdot \nabla) \mathbf{E} \Big|_{@ \mathbf{r}_m} + \mathbf{p} \times \mathbf{E}(\mathbf{r}_m). \quad (17)$$

This net torque may also be obtained from the potential energy U — or rather its infinitesimal variation under simultaneous rotations of the dipole moment vector \mathbf{p} and of the displacement $\mathbf{r}_m - \mathbf{r}_0$ of the dipole from the reference point — but I am not going to work it out in these notes.