# ELECTRIC DIPOLES

In these notes, I write down the electric field of a dipole, and also the net force and the torque on a dipole in the electric field of other charges. For simplicity, I focus on ideal dipoles — also called pure dipoles — where the distance $a$ between the positive and the negative charges is infinitesimal, but the charges are so large that the dipole moment $\mathbf{p}$ is finite.

## Electric Field of a Dipole

The potential due to an ideal electric dipole $\mathbf{p}$ is

$$V(\mathbf{r}) \;=\; \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{4\pi\epsilon_0\, r^2}\,, \tag{1}$$

or in terms of spherical coordinates where the North pole ($\theta = 0$) points in the direction of the dipole moment $\mathbf{p}$,

$$V(r,\theta) \;=\; \frac{p}{4\pi\epsilon_0}\,\frac{\cos\theta}{r^2}\,. \tag{2}$$

Taking (minus) gradient of this potential, we obtain the dipole's electric field

$$\mathbf{E} \;=\; \frac{p}{4\pi\epsilon_0}\left(\frac{2\cos\theta}{r^3}\,\nabla r \;+\; \frac{\sin\theta}{r^2}\,\nabla\theta\right) \;=\; \frac{p}{4\pi\epsilon_0}\,\frac{1}{r^3}\Big(2\cos\theta\,\hat{\mathbf{r}} \;+\; \sin\theta\,\hat{\boldsymbol{\theta}}\Big). \tag{3}$$

In this formula, the unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ themselves depend on $\theta$ and $\phi$. Translating them to Cartesian unit vectors, we have

$$\begin{aligned}
\hat{\mathbf{r}} \;&=\; \sin\theta\cos\phi\,\hat{\mathbf{x}} \;+\; \sin\theta\sin\phi\,\hat{\mathbf{y}} \;+\; \cos\theta\,\hat{\mathbf{z}},\\
\hat{\boldsymbol{\theta}} \;&=\; \cos\theta\cos\phi\,\hat{\mathbf{x}} \;+\; \cos\theta\sin\phi\,\hat{\mathbf{y}} \;-\; \sin\theta\,\hat{\mathbf{z}},
\end{aligned} \tag{4}$$

hence

$$2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}} \;=\; 3\sin\theta\cos\theta(\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}) + (2\cos^2\theta - \sin^2\theta \;=\; 3\cos^2\theta - 1)\,\hat{\mathbf{z}}, \tag{5}$$

and therefore

$$\begin{aligned}
E_x(r,\theta,\phi) &= \frac{p}{4\pi\epsilon_0}\,\frac{3\sin\theta\cos\theta\cos\phi}{r^3}\,,\\
E_y(r,\theta,\phi) &= \frac{p}{4\pi\epsilon_0}\,\frac{3\sin\theta\cos\theta\sin\phi}{r^3}\,,\\
E_z(r,\theta,\phi) &= \frac{p}{4\pi\epsilon_0}\,\frac{3\cos^2\theta-1}{r^3}\,.
\end{aligned}$$

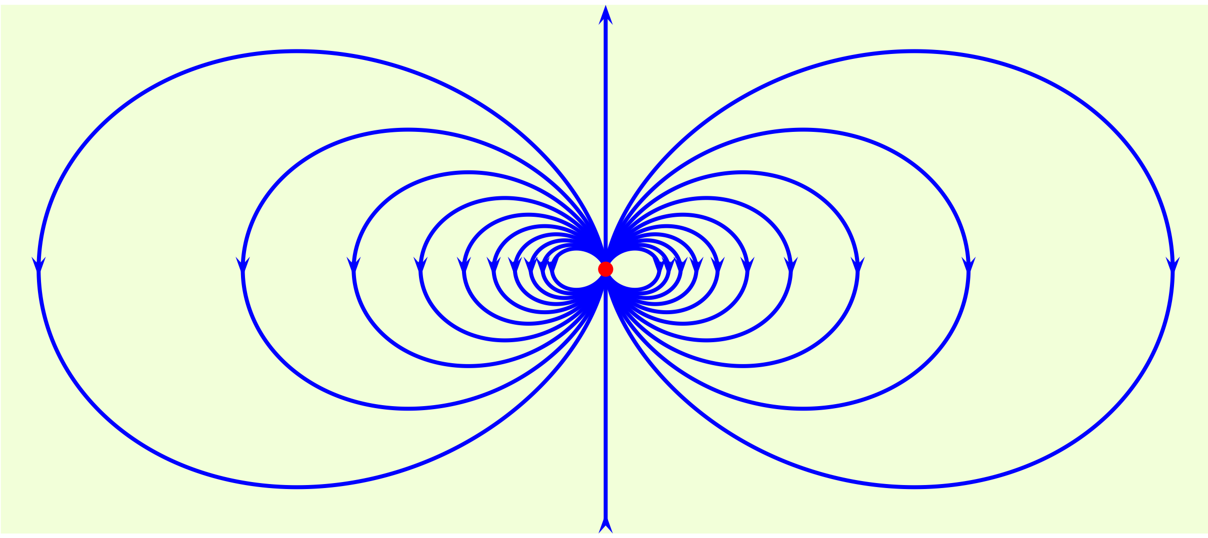
In terms of the $(x, y, z)$ coordinates

$$\begin{aligned}
E_x(x,y,z) &= \frac{p}{4\pi\epsilon_0}\,\frac{3xz}{(x^2+y^2+z^2)^{5/2}}\,,\\
E_y(x,y,z) &= \frac{p}{4\pi\epsilon_0}\,\frac{3yz}{(x^2+y^2+z^2)^{5/2}}\,,\\
E_z(x,y,z) &= \frac{p}{4\pi\epsilon_0}\,\frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}\,,
\end{aligned} \tag{6}$$

or in vector notations,

$$\mathbf{E}(\mathbf{r}) = \frac{3(\mathbf{p}\cdot\widehat{\mathbf{r}})\widehat{\mathbf{r}} - \mathbf{p}}{4\pi\epsilon_0\, r^3}\,. \tag{7}$$

Here is the picture of the dipole's electric field lines (in the $xz$ plane):



## Force and Torque on a Dipole

Now consider an ideal dipole $\mathbf{p}$ placed in an electric field $\mathbf{E}(x,y,z)$ due to some other sources. If this electric field is uniform, there is no net force on the dipole but there is a net torque. Indeed, the force $\mathbf{F}_+ = +q\mathbf{E}$ acting on the positive charge cancels the opposite force $\mathbf{F}_- = -q\mathbf{E} = -\mathbf{F}_+$ acting on the negative charge — so the net force is zero — but the two forces are acting at different points, which causes a torque. Specifically, the net torque of the two forces is

$$\vec{\tau} \;=\; \mathbf{r}_+ \times \mathbf{F}_+ \;+\; \mathbf{r}_- \times \mathbf{F}_- \;=\; (\mathbf{r}_+ - \mathbf{r}_-) \times q\mathbf{E} \;=\; q(\mathbf{r}_+ - \mathbf{r}_-) \times \mathbf{E}, \tag{8}$$

or in terms of the dipole moment $\mathbf{p} = q(\mathbf{r}_+ - \mathbf{r}_-)$,

$$\vec{\tau} \;=\; \mathbf{p} \times \mathbf{E}. \tag{9}$$

This torque vanishes when the dipole moment $\mathbf{p}$ is parallel to the electric field $\mathbf{E}$. Otherwise, the torque twists the dipole trying to make it align with the field, $\mathbf{p} \to \mathbf{p}' \upuparrows \mathbf{E}$.

When the electric field $\mathbf{E}(x,y,z)$ is not uniform, the two charges of the dipole feel slightly different electric fields, so the net force on the dipole does not quite vanish:

$$\mathbf{F}^{\text{net}} \;=\; q\big(\mathbf{E}(\mathbf{r}_+) \;-\; \mathbf{E}(\mathbf{r}_-)\big) \;\neq\; 0. \tag{10}$$

but for small displacements $\mathbf{a} = \mathbf{r}_+ - \mathbf{r}_-$ between the charges, we may expand the difference between the electric fields acting on them into a power series in $\mathbf{a}$. Let $\mathbf{r}_\pm = \mathbf{r}_m \pm \frac{1}{2}\mathbf{a}$ where $\mathbf{r}_m$ is the middle of the dipole; then

$$\mathbf{E}(\mathbf{r}_\pm) \;=\; \mathbf{E}(\mathbf{r}_m) \;\pm\; (\tfrac{1}{2}\mathbf{a}\cdot\nabla)\mathbf{E}\Big|_{@\mathbf{r}_m} \;+\; \tfrac{1}{2}(\tfrac{1}{2}\mathbf{a}\cdot\nabla)^2\mathbf{E}\Big|_{@\mathbf{r}_m} \;+\; \tfrac{1}{6}(\tfrac{1}{2}\mathbf{a}\cdot\nabla)^3\mathbf{E}\Big|_{@\mathbf{r}_m} \;+\; \cdots, \tag{11}$$

and hence the difference

$$\mathbf{E}(\mathbf{r}_+) \;-\; \mathbf{E}(\mathbf{r}_-) \;=\; (\mathbf{a}\cdot\nabla)\mathbf{E}\Big|_{@\mathbf{r}_m} \;+\; \tfrac{1}{24}(\mathbf{a}\cdot\nabla)^3\mathbf{E}\Big|_{@\mathbf{r}_m} \;+\; \cdots. \tag{12}$$

Consequently, the net force on the dipole is

$$\mathbf{F}^{\text{net}} \;=\; q(\mathbf{a}\cdot\nabla)\mathbf{E}\Big|_{@\mathbf{r}_m} \;+\; \frac{q}{24}(\mathbf{a}\cdot\nabla)^3\mathbf{E}\Big|_{@\mathbf{r}_m} \;+\; \cdots. \tag{13}$$

For a physical dipole with a finite distance $a$ between the two charges, we must generally take into account all the subleading terms in this expansion. But for an ideal dipole we take the limit $a \to 0$ while $q\times = p$ stays finite, so for any $n > 1$ $q \times a^n \to 0$. This makes all the subleading terms in eq. (13) negligible compared to the leading term, therefore the net force on an ideal dipole is simply

$$\mathbf{F}^{\rm net} \;=\; (\mathbf{p}\cdot\nabla)\mathbf{E}\Big|_{@\mathbf{r}_m}. \tag{14}$$

The force (14) is conservative and stems from the potential energy

$$U(\mathbf{r}_m, \widehat{\mathbf{p}}) \;=\; -\mathbf{p}\cdot\mathbf{E}(\mathbf{r}_m). \tag{15}$$

Indeed, (minus) the gradient of this $U$ WRT the dipole's location $\mathbf{r}_m$ taken for a fixed dipole orientation $\widehat{\mathbf{p}}$ produces the force (14),

$$-\nabla U\Big|_{\widehat{\mathbf{p}}}^{@\,\text{fixed}} \;=\; (\mathbf{p}\cdot\nabla)\mathbf{E}\Big|_{@\mathbf{r}_m} \;=\; \mathbf{F}^{\rm net}. \tag{16}$$

Also, variation of the potential energy (15) under infinitesimal rotations of the dipole moment $\mathbf{p}$ accounts for the torque

$$\vec{\tau} \;=\; \mathbf{p}\times\mathbf{E}. \tag{9}$$

To be precise, this is the *torque relative to the dipole center* $\mathbf{r}_m$. In a non-uniform electric field, the torque relative to some other point $\mathbf{r}_0$ has an extra term due to the net force (14) on the dipole, thus

$$\vec{\tau}^{\rm net} \;=\; (\mathbf{r}_m - \mathbf{r}_0)\times\mathbf{F}^{\rm net} \;+\; \mathbf{p}\times\mathbf{E}(\mathbf{r}_m) \;=\; (\mathbf{r}_m - \mathbf{r}_0)\times(\mathbf{p}\cdot\nabla)\mathbf{E}\Big|_{@\mathbf{r}_m} \;+\; \mathbf{p}\times\mathbf{E}(\mathbf{r}_m). \tag{17}$$

This net torque may also be obtained from the potential energy U — or rather its infinitesimal variation under simultaneous rotations of the dipole moment vector $\mathbf{p}$ and of the displacement $\mathbf{r}_m - \mathbf{r}_0$ of the dipole from the reference point — but I am not going to work it out in these notes.