

## Divergence and Curl of the Magnetic Field

The *static* electric field  $\mathbf{E}(x, y, z)$  — such as the field of static charges — obeys equations

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho, \quad (1)$$

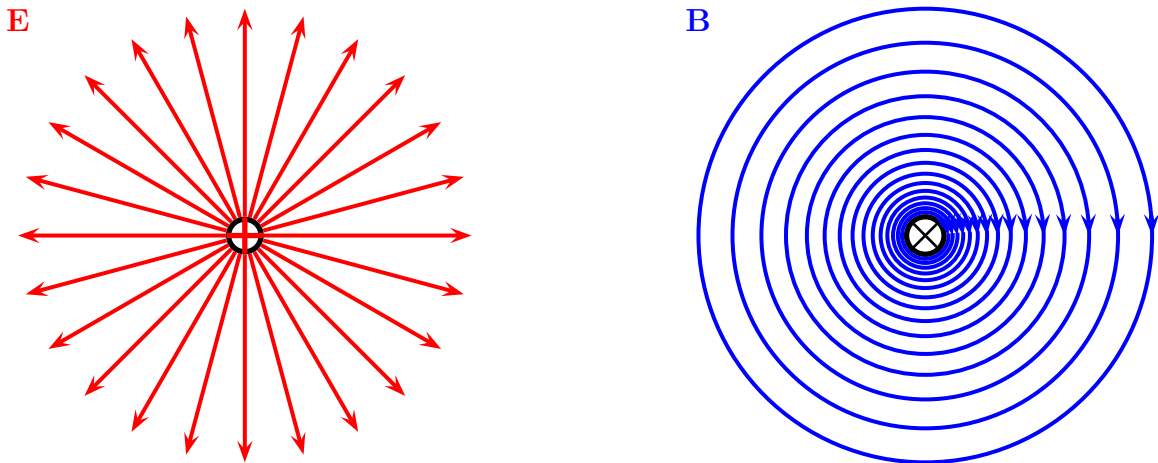
$$\nabla \times \mathbf{E} = 0. \quad (2)$$

The *static* magnetic field  $\mathbf{B}(x, y, z)$  — such as the field of steady currents — obeys different equations

$$\nabla \cdot \mathbf{B} = 0, \quad (3)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}. \quad (4)$$

Due to this difference, the magnetic field of long straight wire looks quite different from the electric field of a point charge or a linear charge:



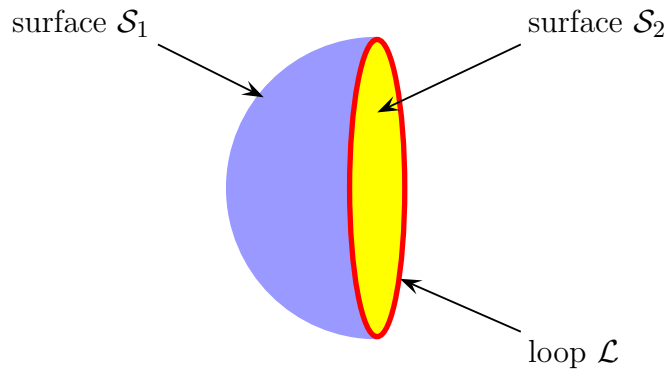
Later in these notes I shall derive eqs. (3) and (4) from the Biot–Savart–Laplace Law. But first, let me explore some of their consequences.

The zero-divergence equation (3) is valid for any magnetic field, even if it is time-dependent rather than static. Physically, it means that there are no magnetic charges — otherwise we would have  $\nabla \cdot \mathbf{B} \propto \rho_{\text{mag}}$  instead of  $\nabla \cdot \mathbf{B} = 0$ . Consequently, the magnetic field lines never begin or end anywhere in space; instead they form closed loops or run from infinity to infinity.

The integral form of eq. (3) follows by the Gauss theorem: *the magnetic flux through any closed surface is zero*,

$$\oiint_{\mathcal{S}} \mathbf{B} \cdot d^2\mathbf{A} = 0 \quad \text{for any closed surface } \mathcal{S}. \quad (5)$$

Consequently, *any open surfaces  $\mathcal{S}_1, \mathcal{S}_2, \dots$  spanning the same loop  $\mathcal{L}$  have the same magnetic flux through them*, for example



$$\iint_{\mathcal{S}_1} \mathbf{B} \cdot d^2\mathbf{A} = \iint_{\mathcal{S}_2} \mathbf{B} \cdot d^2\mathbf{A} = \Phi_B[\text{through loop } \mathcal{L}]. \quad (6)$$

Later in class, we shall find this identity very useful for stating the Faraday's Law of magnetic induction.

Now consider the curl equation  $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ , which is the differential form of the *Ampere's Law*. The integral form of the Ampere's Law obtains by the Stokes' theorem: For any closed loop  $\mathcal{L}$  and any surface  $\mathcal{S}$  spanning that loop,

$$\oint_{\mathcal{L}} \mathbf{B} \cdot d\vec{\ell} = \iint_{\mathcal{S}} (\nabla \times \mathbf{B}) \cdot d^2\mathbf{A} = \mu_0 \iint_{\mathcal{S}} \mathbf{J} \cdot d^2\mathbf{A} = \mu_0 \times I_{\text{net}}[\text{through } \mathcal{L}] \quad (7)$$

where  $I_{\text{net}}[\text{through } \mathcal{L}]$  is the net electric current flowing through the loop  $\mathcal{L}$ . The integral form (7) of the Ampere's Law is particularly convenient when the current flows through a wire or several wires; in this case all we need is to check which wire goes through the loop  $\mathcal{L}$  and which does not, then add up the currents in the wires that do go through  $\mathcal{L}$  and mind

their directions. But it is also convenient for the volume currents flowing through thick conductors or for current sheets flowing on surfaces. I shall give several examples of using the Ampere's Law in a [separate set of notes](#).

As written in eqs. (4) or (7), the Ampere's Law applies only to the magnetic fields of steady currents. Otherwise, we need to use the more general Maxwell–Ampere Law, with an extra term for the time-dependent electric field. I shall discuss this more general law in a few weeks. For the moment, let me simply say that the the original Ampere's Law is simply mathematically inconsistent unless the electric current has zero divergence,  $\nabla \cdot \mathbf{J} = 0$ . Indeed, the left hand side of the curl equation  $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$  always have zero divergence

$$\nabla \cdot (\nabla \times \mathbf{B}) = 0, \tag{8}$$

so we cannot have  $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$  unless the RHS has zero divergence as well,  $\nabla \cdot \mathbf{J} = 0$ . As to the integral form (7) of the Ampere's Law, we need a divergence-less current density  $\mathbf{J}$  to make sure that the net current through the loop  $\mathcal{L}$  is the same for any surface  $\mathcal{S}$  spanning the loop, otherwise we simply cannot define the net current through  $\mathcal{L}$ .

### Proving the Curl and Divergence Equations from the Biot–Savart–Laplace Law

The magnetic field of a steady current density  $\mathbf{J}$  is given by the Biot–Savart–Laplace equation

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \iiint \mathbf{J}(\mathbf{r}') \times \mathbf{G}(\mathbf{r} - \mathbf{r}') d^3\text{Vol} \tag{9}$$

where

$$\mathbf{G}(\mathbf{r} - \mathbf{r}') = \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} = \frac{\text{unit vector from } \mathbf{r}' \text{ to } \mathbf{r}}{(\text{distance from } \mathbf{r}' \text{ to } \mathbf{r})^2}. \tag{10}$$

Let me show that the field (9) indeed obeys the divergence and the curl equations (3) and (4).

In eq. (9) I did not specify the volume over which the integral is taken. This volume must include everywhere the current flows through, but we may also use a larger integration volume, or even the whole 3D space. In any case, this volume should not depend on the point

$\mathbf{r}$  where I measure the magnetic field. Consequently, when I take a derivative of the integral — such as the divergence or the curl of the magnetic field — I can move the derivative inside the integral, thus

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \iiint \nabla_{\mathbf{r}} \cdot (\mathbf{J}(\mathbf{r}') \times \mathbf{G}(\mathbf{r} - \mathbf{r}')) d^3\text{Vol}, \quad (11)$$

$$\nabla \times \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \iiint \nabla_{\mathbf{r}} \times (\mathbf{J}(\mathbf{r}') \times \mathbf{G}(\mathbf{r} - \mathbf{r}')) d^3\text{Vol}, \quad (12)$$

where  $\nabla_{\mathbf{r}}$  takes derivatives with respect to components  $(x, y, z)$  of  $\mathbf{r}$  rather than  $(x', y', z')$  of  $\mathbf{r}'$ .

Now consider the divergence  $\nabla_{\mathbf{r}}$  of the vector product  $\mathbf{J} \times \mathbf{G}$  in eq. (11). By the product rule,

$$\nabla_{\mathbf{r}} \cdot (\mathbf{J}(\mathbf{r}') \times \mathbf{G}(\mathbf{r} - \mathbf{r}')) = \mathbf{G}(\mathbf{r} - \mathbf{r}') \cdot (\nabla_{\mathbf{r}} \times \mathbf{J}(\mathbf{r}')) - \mathbf{J}(\mathbf{r}') \cdot (\nabla_{\mathbf{r}} \times \mathbf{G}(\mathbf{r} - \mathbf{r}')), \quad (13)$$

but in the first term on the RHS

$$\nabla_{\mathbf{r}} \times \mathbf{J}(\mathbf{r}') = 0 \quad (14)$$

because  $\mathbf{J}(\mathbf{r}')$  is a function of  $\mathbf{r}'$  rather than  $\mathbf{r}$ . As to the second term,  $\mathbf{G}(\mathbf{r} - \mathbf{r}')$  is a gradient with respect to  $\mathbf{r}$ ,

$$\mathbf{G}(\mathbf{r} - \mathbf{r}') = \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} = \nabla_{\mathbf{r}} \left( \frac{-1}{|\mathbf{r} - \mathbf{r}'|} \right), \quad (15)$$

so it has zero curl,

$$\nabla_{\mathbf{r}} \times \mathbf{G}(\mathbf{r} - \mathbf{r}') = 0. \quad (16)$$

We see that both terms on the RHS of eq. (13) are zeros, hence

$$\nabla_{\mathbf{r}} \cdot (\mathbf{J}(\mathbf{r}') \times \mathbf{G}(\mathbf{r} - \mathbf{r}')) = 0, \quad (17)$$

and therefore eq. (11) yields zero divergence of the magnetic field,

$$\nabla \cdot \mathbf{B} = 0. \quad (18)$$

Now consider the curl equation for the magnetic field. Applying the product rule to the

curls of a vector product in eq. (12), we have

$$\nabla_{\mathbf{r}} \times (\mathbf{J}(\mathbf{r}') \times \mathbf{G}(\mathbf{r} - \mathbf{r}')) = (\mathbf{G} \cdot \nabla_{\mathbf{r}})\mathbf{J} - (\mathbf{J} \cdot \nabla_{\mathbf{r}})\mathbf{G} + \mathbf{J}(\nabla_{\mathbf{r}} \cdot \mathbf{G}) - \mathbf{G}(\nabla_{\mathbf{r}} \cdot \mathbf{J}), \quad (19)$$

*cf.* textbook §1.2.6, rule (vi). However, two of the terms on the RHS here vanish since  $\mathbf{J}(\mathbf{r}')$  depends on the  $\mathbf{r}'$  rather than  $\mathbf{r}$ , hence

$$\nabla_{\mathbf{r}} \cdot \mathbf{J}(\mathbf{r}') = 0 \quad \text{and} \quad (\mathbf{G} \cdot \nabla_{\mathbf{r}})\mathbf{J}(\mathbf{r}') = 0, \quad (20)$$

therefore

$$\nabla_{\mathbf{r}} \times (\mathbf{J}(\mathbf{r}') \times \mathbf{G}(\mathbf{r} - \mathbf{r}')) = \mathbf{J}(\mathbf{r}')(\nabla_{\mathbf{r}} \cdot \mathbf{G}(\mathbf{r} - \mathbf{r}')) - (\mathbf{J}(\mathbf{r}') \cdot \nabla_{\mathbf{r}})\mathbf{G}(\mathbf{r} - \mathbf{r}'). \quad (21)$$

Plugging this formula into the integral (12) for the curl of the magnetic field, we arrive at

$$\nabla \times \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \iiint \mathbf{J}(\mathbf{r}')(\nabla_{\mathbf{r}} \cdot \mathbf{G}(\mathbf{r} - \mathbf{r}')) d^3 \text{Vol}' - \frac{\mu_0}{4\pi} \iiint (\mathbf{J}(\mathbf{r}') \cdot \nabla_{\mathbf{r}})\mathbf{G}(\mathbf{r} - \mathbf{r}') d^3 \text{Vol}'. \quad (22)$$

I will show momentarily that the second term here evaluates to zero,

$$\iiint (\mathbf{J}(\mathbf{r}') \cdot \nabla_{\mathbf{r}})\mathbf{G}(\mathbf{r} - \mathbf{r}') d^3 \text{Vol}' = 0, \quad (23)$$

so the entire  $\nabla \times \mathbf{B}$  comes from the first term. In that first term,

$$\nabla_{\mathbf{r}} \cdot \mathbf{G}(\mathbf{r} - \mathbf{r}') = \nabla_{\mathbf{r}}^2 \left( \frac{-1}{|\mathbf{r} - \mathbf{r}'|} \right) = 4\pi \delta^{(3)}(\mathbf{r} - \mathbf{r}'), \quad (24)$$

so the integral becomes simply

$$\iiint \mathbf{J}(\mathbf{r}')(\nabla_{\mathbf{r}} \cdot \mathbf{G}(\mathbf{r} - \mathbf{r}')) d^3 \text{Vol}' = 4\pi \iiint \mathbf{J}(\mathbf{r}') \delta^{(3)}(\mathbf{r} - \mathbf{r}') d^3 \text{Vol}' = 4\pi \mathbf{J}(\mathbf{r}). \quad (25)$$

Consequently,

$$\nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{J}(\mathbf{r}) + 0, \quad (26)$$

which is indeed the Ampere's Law (3).

Finally, let me prove that the integral (23) indeed evaluates to zero. Note that  $\mathbf{G}(\mathbf{r} - \mathbf{r}')$  depends only on the difference  $\mathbf{r} - \mathbf{r}'$ , hence its derivatives WRT to components of the  $\mathbf{r}$  and the  $\mathbf{r}'$  are related by sign reversal,

$$\frac{\partial G_i(\mathbf{r} - \mathbf{r}')}{\partial r_j} = -\frac{\partial G_i(\mathbf{r} - \mathbf{r}')}{\partial r'_j}, \quad i, j = x, y, z. \quad (27)$$

Consequently,

$$\begin{aligned} (\mathbf{J}(\mathbf{r}') \cdot \nabla_{\mathbf{r}}) G_i(\mathbf{r} - \mathbf{r}') &= -(\mathbf{J}(\mathbf{r}') \cdot \nabla_{\mathbf{r}'}) G_i(\mathbf{r} - \mathbf{r}') \\ &\ll \text{integrating by parts} \gg \\ &= -\nabla_{\mathbf{r}'} \cdot (\mathbf{J}(\mathbf{r}') G_i(\mathbf{r} - \mathbf{r}')) + G_i(\mathbf{r} - \mathbf{r}') (\nabla_{\mathbf{r}'} \cdot \mathbf{J}(\mathbf{r}')). \end{aligned} \quad (28)$$

Moreover, the second term on the bottom line vanishes for the *steady* — and therefore divergence-less — current  $\mathbf{J}(\mathbf{r}')$ , which leaves us with

$$(\mathbf{J}(\mathbf{r}') \cdot \nabla_{\mathbf{r}}) G_i(\mathbf{r} - \mathbf{r}') = -\nabla_{\mathbf{r}'} \cdot (\mathbf{J}(\mathbf{r}') G_i(\mathbf{r} - \mathbf{r}')), \quad (29)$$

The integral of this total divergence obtains from the Gauss theorem:

$$\begin{aligned} \iiint (\mathbf{J}(\mathbf{r}') \cdot \nabla_{\mathbf{r}}) G_i(\mathbf{r} - \mathbf{r}') d^3 \text{Vol}' &= -\iiint \nabla_{\mathbf{r}'} \cdot (\mathbf{J}(\mathbf{r}') G_i(\mathbf{r} - \mathbf{r}')) d^3 \text{Vol}' \\ &= -\iint_{\mathcal{S}} G_i(\mathbf{r} - \mathbf{r}') \mathbf{J}(\mathbf{r}') \cdot \mathbf{d}^2 \mathbf{A}, \end{aligned} \quad (30)$$

where the surface  $\mathcal{S}$  over which we integrate on the last line is the boundary of the volume  $\mathcal{V}$  over which we integrate in the Biot–Savart–Laplace equation (9). This volume must include everywhere the current flows, but we may just as well use a bigger volume. Indeed, let's take a bigger volume  $\mathcal{V}$ , so no current flows on its surface  $\mathcal{S}$  — which immediately kills the surface integral on the bottom line of eq. (30). Therefore,

$$\iiint (\mathbf{J}(\mathbf{r}') \cdot \nabla_{\mathbf{r}}) G_i(\mathbf{r} - \mathbf{r}') d^3 \text{Vol}' = 0, \quad (31)$$

or in vector notations

$$\iiint (\mathbf{J}(\mathbf{r}') \cdot \nabla_{\mathbf{r}}) \mathbf{G}(\mathbf{r} - \mathbf{r}') d^3 \text{Vol}' = 0, \quad (23)$$

*quod erat demonstrandum.*