

Math of Multipole Expansion

In this note I explain how to expand

$$\frac{1}{|\mathbf{R} - \mathbf{r}|} = \frac{1}{\sqrt{R^2 + r^2 - 2Rr \cos \theta}} \quad (1)$$

into a power series in (r/R) for $r < R$, and then apply this expansion to the Coulomb potential.

Let's start with the few leading terms in this expansion for $r \ll R$. For the sake of compactness, let's denote

$$\alpha = \frac{r}{R} \ll 1, \quad x = \cos \theta, \quad \beta = 2\alpha x - \alpha^2 \ll 1.$$

In these notations,

$$\frac{1}{\sqrt{R^2 + r^2 - 2Rr \cos \theta}} = \frac{1}{\sqrt{R^2(1 + \alpha^2 - 2\alpha x)}} = \frac{1}{R} \times \frac{1}{\sqrt{1 - \beta}}. \quad (2)$$

Next, let's expand the $1/\sqrt{1 - \beta}$ into powers of β :

$$S = \frac{1}{\sqrt{1 - \beta}} = 1 + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{2^n n!} \times \beta^n = 1 + \frac{1}{2}\beta + \frac{3}{8}\beta^2 + \frac{5}{16}\beta^3 + \dots \quad (3)$$

Now, remember that $\beta = 2\alpha x - \alpha^2$, plug that into the above expansion, then truncate it to powers of α no larger than 3:

$$\begin{aligned} S &= 1 + \frac{1}{2}(2\alpha x - \alpha^2) + \frac{3}{8}(2\alpha x - \alpha^2)^2 + \frac{5}{16}(2\alpha x - \alpha^2)^3 + \dots \\ &= 1 + \alpha x && - \frac{1}{2}\alpha^2 \\ &&& + \frac{3}{2}\alpha^2 x^2 && - \frac{3}{2}\alpha^3 x && + \dots \\ &&&&& + \frac{5}{2}\alpha^3 x^3 && + \dots \\ &&&&&&& + \dots \\ &= 1 + \alpha \times x && + \alpha^2 \times \frac{3x^2 - 1}{2} && + \alpha^3 \times \frac{5x^3 - 3x}{2} && + \dots \\ &= 1 + \alpha \times P_1(x) && + \alpha^2 \times P_2(x) && + \alpha^3 \times P_3(x) && + \dots \end{aligned} \quad (4)$$

where $P_1(x)$, $P_2(x)$, and $P_3(x)$ are the Legendre polynomials of respective degrees 1, 2, 3.

Plugging this result back into eq. (2), we obtain

$$\frac{1}{\sqrt{R^2 + r^2 - 2Rr \cos \theta}} = \frac{1}{R} + \frac{r}{R^2} \times P_1(\cos \theta) + \frac{r^2}{R^3} \times P_2(\cos \theta) + \frac{r^3}{R^4} \times P_3(\cos \theta) + \dots \quad (5)$$

The expansion (5) clearly suggests similar terms for the higher powers of r/R , and indeed there is a **theorem**:

$$\text{For } r < R, \quad \frac{1}{\sqrt{R^2 + r^2 - 2Rr \cos \theta}} = \sum_{\ell=0}^{\infty} \frac{r^\ell}{R^{\ell+1}} \times P_\ell(\cos \theta). \quad (6)$$

Proof: Let's start with the Rodrigues formula for the Legendre polynomials,

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} (x^2 - 1)^\ell, \quad (7)$$

and the residue method for taking contour integrals in the complex plane,

$$\oint_{\Gamma} \frac{dz}{2\pi i} \frac{f(z)}{(z-x)^{n+1}} = \text{Residue} \left[\frac{f(z)}{(z-x)^{n+1}} \right]_{@z=x} = \frac{1}{n!} \left. \frac{d^n f(z)}{dz^n} \right|_{@z=x} \quad (8)$$

provided the contour Γ circles x and that the function $f(x)$ is analytic and has no singularities inside the contour Γ . Applying this method *in reverse* — *i.e.*, turning n^{th} into a contour integral — to the Rodrigues formula, we obtain

$$P_\ell(x) = \frac{1}{2^\ell} \oint_{\Gamma} \frac{dz}{2\pi i} \frac{(z^2 - 1)^\ell}{(z-x)^{\ell+1}} \quad (9)$$

where the contour Γ circles x . Now let's plug this formula into the series on the RHS of

eq. (6):

$$\begin{aligned}
\text{series} &= \sum_{\ell=0}^{\infty} \frac{r^\ell}{R^{\ell+1}} \times P_\ell(x) \\
&= \sum_{\ell=0}^{\infty} \frac{r^\ell}{R^{\ell+1}} \times \frac{1}{2^\ell} \oint_{\Gamma} \frac{dz}{2\pi i} \frac{(z^2 - 1)^\ell}{(z - x)^{\ell+1}} \\
&\quad \langle\langle \text{putting the sum inside the integral} \rangle\rangle \\
&= \oint_{\Gamma} \frac{dz}{2\pi i} \sum_{\ell=0}^{\infty} \frac{r^\ell}{R^{\ell+1}} \times \frac{(z^2 - 1)^\ell}{2^\ell (z - x)^{\ell+1}} \\
&= \oint_{\Gamma} \frac{dz}{2\pi i} \frac{1}{R(z - x)} \times \sum_{\ell=0}^{\infty} \left(\frac{r(z^2 - 1)}{2R(z - x)} \right)^\ell \\
&= \oint_{\Gamma} \frac{dz}{2\pi i} \frac{1}{R(z - x)} \times \frac{1}{1 - \frac{r(z^2 - 1)}{2R(z - x)}} \\
&= \oint_{\Gamma} \frac{dz}{2\pi i} \frac{-2}{rz^2 - 2Rz + 2Rx - r}.
\end{aligned} \tag{10}$$

Note: before the summation, each term on the third line has poles at $z = x$ and at $z = \infty$, but after the summation, both poles have moved to the roots of the quadratic equation

$$rz^2 - 2Rz + 2Rx - r = 0, \tag{11}$$

thus

$$z_{1,2} = \frac{R \pm \sqrt{R^2 - 2rRx + r^2}}{r}; \quad \text{for } r \ll R, \quad z_1 \approx \frac{2R}{r} \rightarrow \infty, \quad \text{while } z_2 \approx x. \tag{12}$$

This tells us how to choose the integration contour Γ : It should circle around x and have enough room to accommodate the shifting of the pole from x to z_2 , but it should not include the other pole at z_1 which have moved in from the infinity. Consequently, evaluating the integral on the bottom line of eq. (10) by the residue method, we have

$$\oint_{\Gamma} \frac{dz}{2\pi i} \frac{-2}{rz^2 - 2Rz + 2Rx - r} = \text{Residue} \left[\frac{-2}{rz^2 - 2Rz + 2Rx - r} \right]_{@z=z_2}. \tag{13}$$

Specifically,

$$\frac{-2}{rz^2 - 2Rz + 2Rx - r} = \frac{-2}{r} \times \frac{1}{(z - z_1)(z - z_2)}, \quad (14)$$

so the residue of this function at $z = z_2$ is simply

$$\text{Residue} = \frac{-2}{r} \times \frac{1}{z_2 - z_1} = \frac{-2}{r} \times \frac{r}{-2\sqrt{R^2 - 2rRx + r^2}} = +\frac{1}{\sqrt{R^2 - 2rRx + r^2}}. \quad (15)$$

Thus,

$$\text{the series} = \sum_{\ell=0}^{\infty} \frac{r^\ell}{R^{\ell+1}} \times P_\ell(x) = \frac{1}{\sqrt{R^2 - 2rRx + r^2}}, \quad (16)$$

which proves the theorem (6), *quod erat demonstrandum*.

ASIDE: A few words about the convergence of the expansion (6). For $x = \cos\theta$ ranging between -1 and $+1$, all the Legendre polynomials take values between -1 and $+1$. Consequently, the series (6) converges for any $r < R$. If we analytically continue it to the complex r , it would converge for $|r| < R$; in other words, it has *radius of convergence* $= R$. Indeed, as a function of complex r , the $1/\sqrt{\dots}$ on the LHS of (6) has singularities at

$$r_{1,2} = R \cos\theta \pm iR \sin\theta, \quad |r_{1,2}| = R,$$

and that's what sets the radius of convergence to $|r| < R$.

For $r > R$ we may longer expand the inverse distance into powers of r/R . Instead, we may expand it into powers of the inverse ratio R/r :

$$\text{For } r > R, \quad \frac{1}{\sqrt{R^2 + r^2 - 2Rr \cos\theta}} = \sum_{\ell=0}^{\infty} \frac{R^\ell}{r^{\ell+1}} \times P_\ell(\cos\theta), \quad (17)$$

which works exactly like eq. (6) once we exchange $r \leftrightarrow R$.

Physically, the expansion (6) is useful for potentials far outside complicated charged bodies, while the inverse expansion (17) is useful for potentials deep inside a cavity.

Multipole Expansion of the Electric Potential

Now consider the Coulomb potential of some continuous charge distribution $\rho(\vec{r})$,

$$V(\vec{R}) = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(\mathbf{r}) d^3\text{Vol}}{|\mathbf{R} - \mathbf{r}|}. \quad (18)$$

Suppose all the charges are limited to some compact volume, while we want to know the potential far away from that volume, so in the integral (18) we always have $r \ll R$. Consequently, we may expand the denominator in the Coulomb potential according to the Theorem (6), thus

$$\begin{aligned} V(\mathbf{R}) &= \frac{1}{4\pi\epsilon_0} \iiint d^3\text{Vol} \rho(\mathbf{r}) \times \sum_{\ell=0}^{\infty} \frac{r^\ell}{R^{\ell+1}} \times P_\ell(\cos \theta) \\ &= \sum_{\ell=0}^{\infty} \frac{1}{4\pi\epsilon_0 R^{\ell+1}} \times \iiint d^3\text{Vol} \rho(\mathbf{r}) \times r^\ell P_\ell(\cos \theta) \end{aligned} \quad (19)$$

where θ is the angle between the radius-vectors \mathbf{R} and \mathbf{r} . In terms of the unit vectors $\hat{\mathbf{R}}$ and $\hat{\mathbf{r}}$ in the directions of \mathbf{R} and \mathbf{r} ,

$$\cos \theta = \hat{\mathbf{R}} \cdot \hat{\mathbf{r}}. \quad (20)$$

Consequently, we may decompose the potential $V(\mathbf{R})$ of the charges $\rho(\mathbf{r})$ into a series of multipole potentials,

$$V(\mathbf{R}) = \sum_{\ell=0}^{\infty} \frac{\mathcal{M}_\ell(\hat{\mathbf{R}})}{4\pi\epsilon_0 R^{\ell+1}} \quad (21)$$

where

$$\mathcal{M}_\ell(\hat{\mathbf{R}}) = \iiint r^\ell P_\ell(\hat{\mathbf{r}} \cdot \hat{\mathbf{R}}) \times \rho(\mathbf{r}) d^3\text{Vol}. \quad (22)$$

is the 2^ℓ -pole moment of the charge distribution $\rho(\mathbf{r})$. Or rather, it's the *component of the 2^ℓ -pole moment in the direction $\hat{\mathbf{R}}$* .

Let's take a closer look at these components:

- $\ell = 0$ • The monopole moment \mathcal{M}_0 is simply the net charge of the distribution,

$$\mathcal{M}_0 = \iiint \rho(\mathbf{r}) d^3\text{Vol} = Q^{\text{net}}, \quad (23)$$

and it obviously does not depend on the direction $\widehat{\mathbf{R}}$, hence isotropic monopole potential,

$$V_{\text{monopole}} = \frac{Q^{\text{net}}}{4\pi\epsilon_0 R}. \quad (24)$$

- $\ell = 1$ • The dipole moment is a vector

$$\mathbf{p} = \iiint \mathbf{r} \rho(\mathbf{r}) d^3\text{Vol} \quad (25)$$

and the $\mathcal{M}_1(\widehat{\mathbf{R}})$ in the series (21) is simply

$$\mathcal{M}_1(\widehat{\mathbf{R}}) = \widehat{\mathbf{R}} \cdot \mathbf{p}, \quad (26)$$

hence the dipole's potential

$$V_{\text{dipole}} = \frac{\mathbf{p} \cdot \widehat{\mathbf{R}}}{4\pi\epsilon_0 R^2}. \quad (27)$$

To see how this works, note that $r^1 \times P_1(\widehat{\mathbf{R}} \cdot \widehat{\mathbf{r}}) = r(\widehat{\mathbf{R}} \cdot \widehat{\mathbf{r}}) = \widehat{\mathbf{R}} \cdot \mathbf{r}$, hence

$$\mathcal{M}_1(\widehat{\mathbf{R}}) = \iiint (\widehat{\mathbf{R}} \cdot \mathbf{r}) \rho(\mathbf{r}) d^3\text{Vol} = \widehat{\mathbf{R}} \cdot \iiint \mathbf{r} \rho(\mathbf{r}) d^3\text{Vol} = \widehat{\mathbf{R}} \cdot \mathbf{p}. \quad (28)$$

- $\ell = 2$ • The quadrupole moment is a 2-index symmetric tensor

$$\mathcal{Q}_{i,j} = \iiint \left(\frac{3}{2} r_i r_j - \frac{1}{2} \delta_{i,j} r^2 \right) \rho(\mathbf{r}) d^3\text{Vol} \quad (29)$$

where the indices i, j run over x, y, z , the r_i are the components of the vector \mathbf{r} , and $\delta_{i,j}$ is the Kronecker's delta (1 for $i = j$ and 0 for $i \neq j$).

To see the relation between this tensor and the $\mathcal{M}_2(\widehat{\mathbf{R}})$ in the series (21), let's expand the second Legendre polynomial $P_2(\cos \theta) = P_2(\widehat{\mathbf{R}} \cdot \widehat{\mathbf{r}})$:

$$P_2(\widehat{\mathbf{R}} \cdot \widehat{\mathbf{r}}) = \frac{3}{2}(\widehat{\mathbf{R}} \cdot \widehat{\mathbf{r}})^2 - \frac{1}{2}, \quad (30)$$

$$\begin{aligned} \frac{3}{2}(\widehat{\mathbf{R}} \cdot \widehat{\mathbf{r}})^2 &= \frac{3}{2} \left(\sum_i \widehat{R}_i \widehat{r}_i \right)^2 = \frac{3}{2} \left(\sum_i \widehat{R}_i \widehat{r}_i \right) \times \left(\sum_j \widehat{R}_j \widehat{r}_j \right) \\ &= \frac{3}{2} \sum_{i,j} \widehat{R}_i \widehat{R}_j \widehat{r}_i \widehat{r}_j, \end{aligned} \quad (31)$$

$$\begin{aligned} \frac{1}{2} &= \frac{1}{2} \widehat{\mathbf{R}} \cdot \widehat{\mathbf{R}} \quad \langle\langle \text{since } \widehat{\mathbf{R}} \text{ is a unit vector} \rangle\rangle \\ &= \frac{1}{2} \sum_i \widehat{R}_i \widehat{R}_i = \frac{1}{2} \sum_{i,j} \delta_{i,j} \times \widehat{R}_i \widehat{R}_j, \end{aligned} \quad (32)$$

$$\text{hence } P_2(\widehat{\mathbf{R}} \cdot \widehat{\mathbf{r}}) = \sum_{i,j} \widehat{R}_i \widehat{R}_j \left(\frac{3}{2} \widehat{r}_i \widehat{r}_j - \frac{1}{2} \delta_{ij} \right), \quad (33)$$

$$\text{and } r^2 \times P_2(\widehat{\mathbf{R}} \cdot \widehat{\mathbf{r}}) = \sum_{i,j} \widehat{R}_i \widehat{R}_j \left(\frac{3}{2} r_i r_j - \frac{1}{2} r^2 \delta_{ij} \right). \quad (34)$$

Plugging the last line here into eq. (22) for $\ell = 2$, we obtain

$$\begin{aligned} \mathcal{M}_2(\widehat{\mathbf{R}}) &= \iiint \sum_{i,j} \widehat{R}_i \widehat{R}_j \left(\frac{3}{2} r_i r_j - \frac{1}{2} r^2 \delta_{ij} \right) \times \rho(\mathbf{r}) d^3 \text{Vol} \\ &= \sum_{i,j} \widehat{R}_i \widehat{R}_j \times \iiint \left(\frac{3}{2} r_i r_j - \frac{1}{2} r^2 \delta_{ij} \right) \times \rho(\mathbf{r}) d^3 \text{Vol} \\ &= \sum_{i,j} \widehat{R}_i \widehat{R}_j \times \mathcal{Q}_{i,j}, \end{aligned} \quad (35)$$

hence the quadrupole potential

$$V_{\text{quadrupole}}(\mathbf{R}) = \frac{\sum_{i,j} \mathcal{Q}_{i,j} \widehat{R}_i \widehat{R}_j}{4\pi\epsilon_0 R^3}. \quad (36)$$

- $\ell \geq 3$ • The higher multipoles are ℓ -index symmetric tensors. For example, the octupole moment is the 3-index tensor

$$\mathcal{O}_{i,j,k} = \iiint \left(\frac{5}{2} r_i r_j r_k - \frac{1}{2} \delta_{i,j} r_k - \frac{1}{2} \delta_{i,k} r_j - \frac{1}{2} \delta_{j,k} r_i \right) \times \rho(\mathbf{r}) d^3 \text{Vol} \quad (37)$$

whose potential is

$$V_{\text{octupole}}(\mathbf{R}) = \frac{\sum_{i,j,k} \mathcal{O}_{i,j,k} \widehat{R}_i \widehat{R}_j \widehat{R}_k}{4\pi\epsilon_0 R^4}. \quad (38)$$

Likewise, for higher ℓ the potential has form

$$V_{2^\ell\text{-pole}}(\mathbf{R}) = \frac{\sum_{i,j,\dots,n} \mathcal{M}_{i,j,\dots,n}^{(\ell)} \widehat{R}_i \widehat{R}_j \cdots \widehat{R}_n}{4\pi\epsilon_0 R^{\ell+1}} \quad (39)$$

where

$$\mathcal{M}_{i,j,\dots,n}^{(\ell)} = \iiint \left(\begin{array}{c} \text{homogeneous polynomial} \\ \text{of degree } \ell \text{ in } x, y, z \end{array} \right) \times \rho(\mathbf{r}) d^3 \text{Vol} \quad (40)$$

where the specific form of the degree- ℓ polynomial follows from the $P_\ell(\cos \theta)$.

AXIAL SYMMETRY

For the axially symmetric charge distributions $\rho(r, \theta, \phi) = \rho(r, \theta)$ only), we may re-express the angular dependence of the multipole expansion using the following

Lemma: Let (θ, ϕ) be the spherical angles of the direction $\widehat{\mathbf{R}}$ while (θ', ϕ') are spherical angles of the direction $\widehat{\mathbf{r}}$, then

$$\int_0^{2\pi} \frac{d\phi'}{2\pi} P_\ell(\widehat{\mathbf{R}} \cdot \widehat{\mathbf{r}}) = P_\ell(\cos \theta) \times P_\ell(\cos \theta'). \quad (41)$$

Consequently, for an axially symmetric charge distribution

$$\begin{aligned} \mathcal{M}_\ell(\widehat{\mathbf{R}}) &= \iiint r^\ell P_\ell(\widehat{\mathbf{R}} \cdot \widehat{\mathbf{r}}) \times \rho(r, \theta') \times r^2 \sin \theta' dr d\theta' d\phi' \\ &= P_\ell(\cos \theta) \times \mathcal{M}_{z\dots z}^{(\ell)} \end{aligned} \quad (42)$$

$$\text{where } \mathcal{M}_{z\dots z}^{(\ell)} = \iiint r^\ell P_\ell(\cos \theta') \times \rho(r, \theta') \times r^2 \sin \theta' dr d\theta' d\phi' \quad (43)$$

is the z, \dots, z component of the 2^ℓ -pole vector or tensor, for example p_z , $\mathcal{Q}_{z,z}$, or $\mathcal{O}_{z,z,z}$. For the axially symmetric charge distribution it's the only independent component, and it's also the only component we need for expanding the potential:

$$V(R, \theta) = \frac{\mathcal{M}_{z\dots z}^{(\ell)}}{4\pi\epsilon_0} \times \frac{P_\ell(\cos \theta)}{R^{\ell+1}}. \quad (44)$$

You should see examples of such expansion in your homework.