Separation of Variables in Polar and Spherical Coordinates

POLAR COORDINATES

Suppose we are given the potential on the inside surface of an infinitely long cylindrical cavity, and we want to find the potential inside the cylinder. For simplicity, suppose the boundary potential depends only on the angular coordinate ϕ and not on the z coordinate along the cylinder, $V_b(\phi, z) = V_b(\phi$ only), so the potential inside the cylinder should also be independent on z, $V(s, \phi, z) = V(s, \phi$ only).

Mathematically, we have a two-dimensional problem: Find $V(s, \phi)$ such that:

[1] $\nabla^2 V(s, \phi) \equiv 0$ for $s \leq R$.

[2] V is periodic in ϕ, $V(s, \phi + 2\pi) = V(s, \phi)$.

[3] V is well-behaved at $s = 0$ (the axis).

[4] At $s = R$ (the surface), $V(s, \phi) = $ given $V_b(\phi)$.

In the separation-of-variables method, we start by looking at solutions to conditions [1,2,3] (but not [4]) in the form

$$V(s, \phi) \;=\; f(s) \times g(\phi). \tag{1}$$

Let's start with the 2D Laplacian, which in polar coordinates (s, ϕ) acts as

$$\triangle V(s.\phi) \;=\; \frac{\partial^2 V}{\partial s^2} \;+\; \frac{1}{s} \times \frac{\partial V}{\partial s} \;+\; \frac{1}{s^2} \times \frac{\partial^2 V}{\partial \phi^2}. \tag{2}$$

For the potential $V(s.\phi)$ of the form (1), this Laplacian becomes

$$\triangle V \;=\; f''(s) \times g(\phi) \;+\; \frac{f'(s)}{s} \times g(\phi) \;+\; \frac{f(s)}{s^2} \times g''(\phi), \tag{3}$$

hence

$$\frac{s^2}{V} \times \triangle V \;=\; \frac{s^2 f''(s)}{f(s)} \;+\; \frac{s f'(s)}{f(s)} \;+\; \frac{g''(\phi)}{g(\phi)}, \tag{4}$$

and therefore $\triangle V \equiv 0$ for all s and ϕ requires

$$\frac{s^2 f''(s)}{f(s)} \;+\; \frac{s f'(s)}{f(s)} \;=\; -\frac{g''(\phi)}{g(\phi)} \;=\; \text{const.} \tag{5}$$

Next, consider the g equation $g''(\phi) + Cg(\phi) = 0$ for a constant C. In general, the solutions

to this equation are

$$\text{for } C = +m^2 \geq 0, \quad g(\phi) = A\cos(m\phi) + B\sin(m\phi), \tag{6}$$

$$\text{for } C = -\mu^2 \leq 0, \quad g(\phi) = A\cosh(\mu\phi) + B\sinh(\mu\phi). \tag{7}$$

However, we want not just any solution but a *periodic solution* $g(\phi+2\pi) = g(\phi)$, which requires trigonometric rather than hyperbolic sine and cosine, hence $C = +m^2 > 0$. Moreover, *a period compatible with 2π requires integer $m = 0, 1, 2, 3, 4, \ldots$.* Thus,

$$C = +m^2 \text{ for } m = 0, 1, 2, 3, \ldots \quad \text{and} \quad g(\phi) = A\cos(m\phi) + B\sin(m\phi). \tag{8}$$

Now consider the f equation for $C = +m^2$,

$$s^2 \times f''(s) + s \times f'(s) - m^2 \times f(s) = 0. \tag{9}$$

This equation is linear in f and *homogeneous* in s, so let's look for solutions of the form $f(s) = s^\alpha$ for some power α. Indeed, plugging such f into the equation yields

$$0 = s^2 \times \alpha(\alpha-1)s^{\alpha-2} + s \times \alpha s^{\alpha-1} - m^2 \times s^\alpha = s^\alpha \times (\alpha^2 - m^2), \tag{10}$$

which is satisfies whenever

$$\alpha^2 - m^2 = 0 \quad \Longrightarrow \quad \alpha = \pm m. \tag{11}$$

For $m \neq 0$ we have two distinct roots, hence two independent solutions to eq. (9), so the general solution looks like

$$f(s) = D \times s^{+m} + E \times s^{-m} \tag{12}$$

for some constants D and E. For $m = 0$. the roots (11) coincide, so we only get one solution, while the other solution involves the logarithm $\ln(s)$. thus in general

$$f(s) = D + E \times \ln(s). \tag{13}$$

In any case, we want more than a general solution to the equation (9), we want the solutions which obeys the condition [3], namely no singularity at the cylinder's axis, $s = 0$. This condition

does not allows negative powers for s for $m \neq 0$ or the logarithm for $m = 0$, thus in both cases we must have $E = 0$, which leaves us with

$$f(s) \;=\; \text{const} \times s^{+m} \;=\; \text{const}' \times \left(\frac{s}{R}\right)^m. \tag{14}$$

Altogether, we have an infinite series of solutions to conditions [1,2,3], namely

$$V(s, \phi) \;=\; A \cos(m\phi) \times (s/R)^m \;+\; B \sin(m\phi) \times (s/R)^m \quad \text{for integer } m = 0, 1, 2, 3, \ldots \tag{15}$$

Consequently, a general solution to [1,2,3] is

$$V(s, \phi) \;=\; \sum_{m=0}^{\infty} \Big(A_m \cos(m\phi) \;+\; B_m \sin(m\phi) \Big) \times \left(\frac{s}{R}\right)^m \tag{16}$$

for some constant coefficients A_m and B_m. Or in terms of complex exponentials $e^{\pm im\phi}$ with complex coefficients,

$$V(s, \phi) \;=\; A_0 \;+\; \sum_{m=1}^{\infty} \left(\tfrac{1}{2}(A_m + iB_m)e^{+im\phi} \;+\; \tfrac{1}{2}(A_m - iB_m)e^{-im\phi} \right) \times \left(\frac{s}{R}\right)^m$$

$$=\; \sum_{m=-\infty}^{+\infty} C_m \times e^{im\phi} \times \left(\frac{s}{R}\right)^{|m|}, \tag{17}$$

$$\text{where } C_0 \;=\; A_0, \quad C_{+m} \;=\; \tfrac{1}{2}(A_m + iB_m), \quad C_{-m} \;=\; \tfrac{1}{2}(A_m - iB_m) \;=\; C_{+m}^*. \tag{18}$$

Finally, the coefficients C_m follows from the boundary condition [4] on the surface of the cylinder:

$$@s = R, \quad V(R, \phi) \;=\; \sum_{m=-\infty}^{+\infty} C_m \times e^{im\phi} \;=\; V_b(\phi), \tag{19}$$

so the C_m obtain from expanding the periodic $V_b(\phi)$ into the Fourier series. Hence, the reverse Fourier transform gives

$$C_m \;=\; \frac{1}{2\pi} \int_0^{2\pi} V_b(\phi) \times e^{-im\phi} \, d\phi. \tag{20}$$

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As a specific example, suppose the cylinder's surface is split in two halves with potentials $\pm V_0$, for example

$$V_b(\phi) = \begin{cases} +V_0 & \text{for } 0 < \phi < \pi, \\ -V_0 & \text{for } \pi < \phi < 2\pi. \end{cases} \tag{21}$$

In this case,

$$C_m = \frac{V_0}{2\pi} \int_0^{\pi} e^{-im\phi} \, d\phi \;-\; \frac{V_0}{2\pi} \int_{\pi}^{2\pi} e^{-im\phi} \, d\phi$$

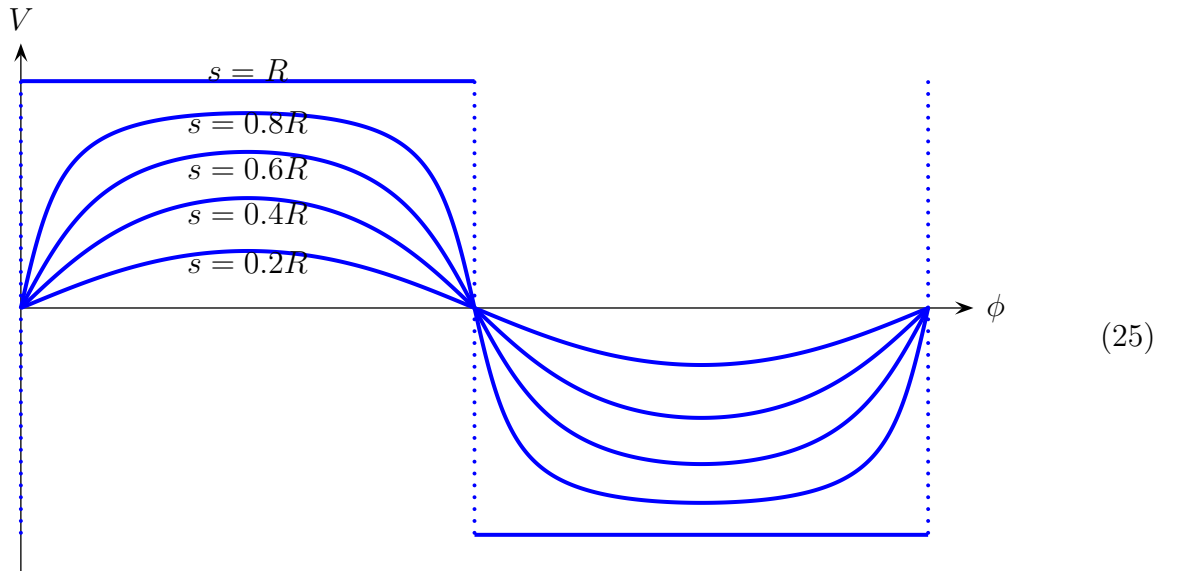$$= \begin{cases} \dfrac{4iV_0}{2\pi m} & \text{for odd } m, \\[2mm] 0 & \text{for even } m, \end{cases} \tag{22}$$

hence

$$V(s, \phi) = \frac{4V_0}{2\pi} \sum_{\substack{\text{odd } m \\ -\infty < m < +\infty}} \frac{i}{m} \times e^{im\phi} \times (s/R)^{|m|} \tag{23}$$

which evaluates to

$$V(r, s) = \frac{4V_0}{2\pi} \times \arctan\left(\frac{2Rs}{R^2 - s^2} \times \sin\phi \right). \tag{24}$$

To illustrate this potential graphically, let me plot it as a function of ϕ for $s = 0.2R$, $s = 0.4R$, $s = 0.6R$, $s = 0.8R$, and $s = R$:

$$\tag{25}$$

Note: the closer we are to the axis the smaller is the amplitude of the $V(\phi)$ curve the more the curve looks like the sine wave. The mathematical reason is that all non-constant terms in the

series (17) decrease with small s as positive powers of s/R, and the larger the $|m|$ the faster they decrease. Consequently, for small s/m the ϕ dependence of the potential is dominated by the modes with smallest $|m|$ with $C_m \neq 0$, — which for the problem at hand means $m = \pm 1$.

Outside the Cylinder

Now consider a slightly different problem: Given the potential $V_b(\phi)$ on a cylindrical surface, find the potential *outside the cylinder* rather than inside it. Proceeding similarly to the previous example, we ask for $V(s,\phi) = f(s) \times g(\phi)$ to obey the Laplace equation subject to periodicity requirement in ϕ, which leads to

$$C = +m^2 \text{ for } m = 0, 1, 2, 3, \ldots \quad \text{and} \quad g(\phi) = A\cos(m\phi) + B\sin(m\phi). \tag{8}$$

and hence

$$f(s) = \begin{cases} D + E \times \ln(s) & \text{for } m = 0, \\ D \times s^{+|m|} + E \times s^{-|m|} & \text{for } m \neq 0. \end{cases} \tag{26}$$

However, this time we are concerned with the asymptotic behavior for $s \to \infty$ rather than the axis of the cylinder at $s = 0$. Specifically, we want the potential to go to zero — or at least to stay finite — for $s \to \infty$, and this rules out the positive powers of s as well as $\ln(s)$. Consequently, outside of the cylinder

$$f(s) = \text{const} \times s^{-|m|} \tag{27}$$

instead of $f(s) \propto s^{+|m|}$ inside the cylinder.

Combining the s and ϕ dependence, we find

$$V(s,\phi) = \sum_{m=0}^{\infty} \Big(A_m \cos(m\phi) + B_m \sin(m\phi) \Big) \times \left(\frac{R}{s}\right)^m \tag{28}$$

for some constants A_m and B_m, or in terms of complex exponentials $e^{\pm im\phi}$,

$$V(s,\phi) = \sum_{m=-\infty}^{+\infty} C_m \times e^{im\phi} \times \left(\frac{R}{s}\right)^{|m|}. \tag{29}$$

Finally, the complex coefficients $C_m = C_{-m}^*$ here obtain from expanding the boundary potential

into the Fourier series,

$$V_b(\phi) \;=\; \sum_{m=-\infty}^{+\infty} C_m \times e^{+im\phi}, \tag{30}$$

$$C_m \;=\; \frac{1}{2\pi} \int_0^{2\pi} V_b(\phi) \times e^{-im\phi}\, d\phi. \tag{31}$$

Spherical Coordinates

Now consider a 3D problem: Find the potential $V(r, \theta, \phi)$ inside a spherical cavity — or outside a sphere — when we are given the potential $V_b(\theta, \phi)$ on the spherical surface. For simplicity, let's focus on potentials with axial symmetry:

$$V_b(\theta, \phi) \;=\; V_b(\theta \text{ only}) \quad \Longrightarrow \quad V(r, \theta, \phi) \;=\; V(r, \theta \text{ only}). \tag{32}$$

Mathematically, we seek the potential which:

[1] Obeys the 3D Laplace equation.

[2] Is single-valued, non-singular, and smooth as a function of θ.

[3] Is well behaved at the center $r \to 0$ if we work inside the sphere, or asymptotes to zero for $r \to \infty$ if we work outside the sphere.

[4] Has given boundary values at the sphere's surface, $V(r = R, \theta) = v_b(\theta)$.

Using the separation of variables method, we first seek to satisfy the conditions [1,2,3] for a potential of the form

$$V(r, \theta) \;=\; f(r) \times g(\theta), \tag{33}$$

find an infinite series of solutions, then look for a linear combination which satisfies the condition [4].

Let's start with the Laplace equation in the spherical coordinates:

$$
\begin{aligned}
\triangle V(r, \theta, \phi) \;=\; & \frac{\partial^2 V}{\partial r^2} \;+\; \frac{2}{r} \times \frac{\partial V}{\partial r} \\
& +\; \frac{1}{r^2} \times \frac{\partial^2 V}{\partial \theta^2} \;+\; \frac{1}{r^2 \tan\theta} \times \frac{\partial V}{\partial \theta} \\
& +\; \frac{1}{r^2 \sin^2\theta} \times \frac{\partial^2 V}{\partial \phi^2}\,.
\end{aligned}
\tag{34}
$$

For the potential of the form (33), the Laplacian becomes

$$\triangle V \;=\; \left(f''(r) + \frac{2f'(r)}{r} \right) \times g(\theta) \;+\; \frac{f(r)}{r^2} \times \left(g''(\theta) + \frac{g'(\theta)}{\tan\theta} \right), \tag{35}$$

hence

$$\frac{r^2}{V} \times \triangle V \;=\; \frac{r^2 f''}{f} \;+\; \frac{2rf'}{f} \;+\; \frac{g''}{g} \;+\; \frac{g}{g\tan\theta}, \tag{36}$$

and consequently the Laplace equation $\triangle V \equiv 0$ for all r, θ requires

$$r^2 \times \frac{f''(r)}{f(r)} \;+\; 2r \times \frac{f'(r)}{f(r)} \;=\; +C, \tag{37}$$

$$\frac{g''(\theta)}{g(\theta)} \;+\; \frac{1}{\tan\theta} \times \frac{g'(\theta)}{g(\theta)} \;=\; -C, \tag{38}$$

for the same constant C. (39)

Next, consider the g equation (38). Let's restate it in terms of the $x = \cos\theta$ variable instead of θ itself,

$$g(\theta) \;=\; P(\cos\theta), \tag{40}$$

$$\frac{dg}{d\theta} \;=\; -\sin\theta \times P'(\cos\theta), \tag{41}$$

$$\frac{d^2g}{d\theta^2} \;=\; -\cos\theta \times P'(\cos\theta) \;+\; \sin^2\theta \times P''(\cos\theta), \tag{42}$$

so that eq. (38) becomes the *Legendre equation* for the $P(x)$,

$$(1 - x^2) \times P''(x) \;-\; 2x \times P'(x) \;+\; C \times P(x) \;=\; 0. \tag{43}$$

Without explaining how to solve this equation, let me briefly summarize its solutions. For generic C, all non-zero solutions to this equation have logarithmic singularities at $x = +1$

(which corresponds to $\theta = 0$) and/or at $x = -1$ (which corresponds to $theta = \pi$). The non-singular solutions obtain only for

$$C = \ell(\ell+1), \quad \text{integer } \ell = 0, 1, 2, 3, \ldots, \tag{44}$$

in which case the good solution is the *Legendre polynomial* of degree ℓ,

$$P_\ell(x) = \frac{1}{2^\ell\, \ell!} \frac{d^\ell}{dx^\ell} (x^2 - 1)^\ell. \tag{45}$$

The overall coefficient here is chosen such that at $x = +1$ all these polynomials become $P(1) = 1$. Here are a few explicit Legendre polynomials for small ℓ.

$$\begin{aligned}
P_0(x) &= 1, \\
P_1(x) &= x, \\
P_2(x) &= \tfrac{3}{2}x^2 - \tfrac{1}{2}, \\
P_3(x) &= \tfrac{5}{2}x^3 - \tfrac{3}{2}x, \\
P_4(x) &= \tfrac{35}{8}x^4 - \tfrac{15}{4}x^2 + \tfrac{3}{8}, \\
P_5(x) &= \tfrac{63}{8}x^5 - \tfrac{35}{4}x^3 + \tfrac{15}{8}x,
\end{aligned} \tag{46}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

The Legendre polynomial are 'orthogonal' to each other when we use $\int_{-1}^{+1} dx$ as a measure,

$$\int\limits_{-1}^{+1} P_\ell(x) \times P_{\ell'}(x) = \begin{cases} 0 & \text{for any } \ell' \neq \ell, \\[2mm] \dfrac{2}{2\ell+1} & \text{for } \ell' = \ell. \end{cases} \tag{47}$$

Consequently, any analytic function of x ranging from -1 to $+1$ may be expanded in a series of Legendre polynomials,

$$\text{any } H(x) = \sum_{\ell=0}^{\infty} H_\ell \times P_\ell(x) \quad \text{for} \quad H_\ell = \frac{2\ell+1}{2} \int\limits_{-1}^{+1} H(x) \times P_\ell(x)\, dx. \tag{48}$$

Anyhow, for $C = \ell(\ell+1)$ and $g(\theta) = P_\ell(\cos\theta)$, the f equation (37) becomes

$$r^2 \times f''(r) + 2r \times f'(r) - \ell(\ell+1) \times f(r) = 0. \tag{49}$$

This equation is linear in f and *homogeneous* in r, so let's look for the solutions of the form

$f(r) = r^\alpha$ for some constant power α. Indeed plugging such an f into the equation (49) yields

$$
\begin{aligned}
0 &= r^2 \times \alpha(\alpha - 1)r^{\alpha-2} + 2r \times \alpha r^{\alpha-1} - \ell(\ell + 1) \times r^\alpha \\
&= r^\alpha \times \big(\alpha(\alpha - 1) + 2\alpha - \ell(\ell + 1)\big)
\end{aligned}
\tag{50}
$$

so the differential equation is satisfied whenever

$$
\alpha(\alpha - 1) + 2\alpha = \alpha(\alpha + 1) = \ell(\ell + 1) \implies \alpha = \ell \text{ or } \alpha = -\ell - 1.
\tag{51}
$$

Thus, he general solution to eq. (37) has form

$$
f(r) = A \times r^\ell + \frac{B}{r^{\ell+1}}.
\tag{52}
$$

The specific solution we need depends on whether we are looking for the potential inside the sphere or outside the sphere.

- For the inside of the sphere, we want the potential to be non-singular at the center, which rules out negative powers of the radius r. Consequently, in eq. (52) $B = 0$, which leaves us with

$$
f(r) = \text{const} \times r^\ell = \text{const}' \times \left(\frac{r}{R}\right)^\ell.
\tag{53}
$$

- For the outside of the sphere, we want the potential to asymptote to zero for $r \to \infty$, which rules out the positive powers of r, In terms of eq. (52), this means $A = 0$ and hence

$$
f(r) = \frac{\text{const}}{r^{\ell+1}} = \text{const}' \times \left(\frac{R}{r}\right)^{\ell+1}.
\tag{54}
$$

Altogether, the general solution to the conditions [1,2,3] is given by the series:

Inside the sphere,

$$
V(r, \theta) = \sum_{\ell=0}^{\infty} C_\ell \times P_\ell(\cos\theta) \times \left(\frac{r}{R}\right)^\ell.
\tag{55}
$$

Outside the sphere,

$$
V(r, \theta) = \sum_{\ell=0}^{\infty} C_\ell \times P_\ell(\cos\theta) \times \left(\frac{R}{r}\right)^{\ell+1}.
\tag{56}
$$

In both cases the coefficients C_ℓ follows from the boundary potential on the sphere's surface,

$$V(r = R, \theta) = \sum_{\ell=0}^{\infty} C_\ell \times P_\ell(\cos\theta) = V_b(\theta). \tag{57}$$

In other words, we must expand the axi-symmetric boundary potential into a series in Legendre polynomials of $\cos\theta$. But thanks to the orthogonality of the Legendre polynomials, the coefficients of such expansion obtain from eq. (48),

$$C_\ell = \frac{2\ell+1}{2} \int_{-1}^{+1} V_b\big(\theta = \arccos(x)\big) \times P_\ell(x)\, dx = \frac{2\ell+1}{2} \int_0^\pi V_b(\theta) \times P_\ell(\cos\theta) \times \sin\theta\, d\theta. \tag{58}$$

Charges on the spherical surface

Consider a thin spherical shell with some surface charge density $\sigma(\theta, \phi)$. For simplicity, assume axial symmetry, thus $\sigma(\theta$ only). Let's find out the potential both inside and outside the spherical shell due to this charge density.

Surface charge densities make for discontinuous electric fields, but the potential V is continuous across the charged surface. Thus, while in the present situation we do not know the boundary potential $V_b(\theta)$ on the spherical surface, we do know its the same potential both immediately inside and immediately outside the surface. Consequently, the potential $V(r, \theta)$ inside and outside the sphere are given by the equations (55) and (56) for the same coefficients C_ℓ, whatever they are. In other words,

$$\forall\, r, \theta: \quad V(r, \theta) = \sum_{\ell=0}^{\infty} C_\ell \times P_\ell(\cos\theta) \times \begin{cases} \left(\dfrac{r}{R}\right)^\ell & \text{for } r < R, \\[3mm] \left(\dfrac{R}{r}\right)^{\ell+1} & \text{for } r > R. \end{cases} \tag{59}$$

Next, consider the radial component of the electric field:

$$E_r = -\frac{\partial V(r\theta)}{\partial r} = \sum_{\ell=0}^{\infty} C_\ell \times P_\ell(\cos\theta) \times \begin{cases} -\ell\, \dfrac{r^{\ell-1}}{R^\ell} & \text{for } r < R, \\[3mm] +(\ell+1)\, \dfrac{R^{\ell+1}}{r^{\ell+2}} & \text{for } r > R. \end{cases} \tag{60}$$

Unlike the potential, this radial electric field is discontinuous across the sphere. Indeed, near

the sphere

$$E_r(r \approx R) \;=\; \sum_{\ell=0}^{\infty} C_\ell \times P_\ell(\cos\theta) \times \begin{cases} \dfrac{-\ell}{R} & \text{just inside the sphere,} \\[2mm] \dfrac{+(\ell+1)}{R} & \text{just outside the sphere,} \end{cases} \tag{61}$$

with discontinuity

$$\text{disc}(E_r) \;=\; E_r(r = R + \epsilon) \,-\, E_r(r = R - \epsilon) \;=\; \sum_{\ell=0}^{\infty} C_\ell \times P_\ell(\cos\theta) \times \frac{2\ell+1}{R}\,. \tag{62}$$

Physically, this discontinuity is caused by the surface charge density on the sphere,

$$\text{disc}(E_r) \;=\; \frac{\sigma}{\epsilon_0}\,. \tag{63}$$

Consequently, the charge density as a function of θ is related to the coefficients C_ℓ of the potential (59) according to

$$\sigma(\theta) \;=\; \epsilon_0\,\text{disc}(E_r(\theta)) \;=\; \frac{\epsilon_0}{R} \times \sum_{\ell=0}^{\infty} (2\ell+1) \times C_\ell \times P_\ell(\cos\theta). \tag{64}$$

We may also reverse this relation according to eq. (48) to get the coefficients C_ℓ from the $\sigma(\theta)$,

$$C_\ell \;=\; \frac{R}{2\epsilon_0} \times \int_0^\pi \sigma(\theta) \times P_\ell(\cos\theta) \times \sin\theta\,d\theta. \tag{65}$$

For example, suppose the sphere is neutral on the whole, but has a quadrupole charge density

$$\sigma(\theta) \;=\; \hat\sigma \times \frac{3\cos^2\theta - 1}{2} \;=\; \hat\sigma \times P_2(\cos\theta). \tag{66}$$

Comparing this angular dependence with eq. (64), we immediately see that the only non-zero

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coefficient C_ℓ is the C_2, specifically

$$C_2 = \frac{R\hat{\sigma}}{5\epsilon_0}.$$ (67)

Consequently, inside the sphere the potential is

$$V(r, \theta) = \frac{\hat{\sigma}}{5\epsilon_0} \times \frac{r^2}{R} \times P_2(\cos\theta),$$ (68)

while outside the sphere

$$V(r, \theta) = \frac{\hat{\sigma}}{5\epsilon_0} \times \frac{R^4}{r^3} \times P_2(\cos\theta).$$ (69)

Metal Sphere in External Electric Field

Now consider another example: a metal sphere in a external electric field. Far away from the sphere, the electric field becomes constant $\mathbf{E} = E\hat{\mathbf{z}}$, hence

$$\text{for } r \to \infty, \quad V \to -Ez = -Er \times \cos\theta = -Er \times P_1(\cos\theta).$$ (70)

The sphere itself is neutral, so without loss of generality we may assume it has zero potential.

Let's find the potential outside the sphere for these boundary conditions. Since we no longer have $V \to 0$ at infinity, the radial function $f_\ell(r)$ could be a general combination of two solutions,

$$f_\ell(r) = A_\ell \times r^\ell + \frac{B_\ell}{r^{\ell+1}}$$ (71)

with $A_\ell \neq 0$. On the other hand, asking for $V = 0$ all over the sphere requires $f_\ell(r = R) = 0$ and hence

$$B_\ell = -R^{2\ell+1} \times A_\ell.$$ (72)

Consequently, the general form of the potential outside the sphere looks like

$$V(r, \theta) = \sum_{\ell=0}^{\infty} A_\ell \times P_\ell(\cos\theta) \times \left(r^\ell - \frac{R^{2\ell+1}}{r^{\ell+1}} \right)$$ (73)

for some coefficients A_ℓ.

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To find these coefficients, we compare the asymptotic behavior of the potential (73) for large r,

$$V \longrightarrow \sum_{\ell=0}^{\infty} A_\ell \times P_\ell(\cos\theta) \times r^\ell \tag{74}$$

to the desired asymptotics (70). This comparison immediately tells us that

$$A_1 = -E, \quad \text{all other } A_\ell = 0, \tag{75}$$

hence

$$V(r,\theta) = -E\left(r - \frac{R^3}{r^2}\right) \times \cos\theta, \tag{76}$$

or in Cartesian coordinates

$$V(x,y,z) = -Ez + ER^3 \times \frac{z}{(x^2+y^2+z^2)^{3/2}} \tag{77}$$

The first term here is due to the external electric field, while the second term is due to induced charges on the sphere's surface.

Taking the gradient of the potential (77), we obtain the net electric field,

$$\mathbf{E}(x,y,z) = E\hat{\mathbf{z}} + ER^3\left(\frac{3z}{r^4}\hat{\mathbf{r}} - \frac{1}{r^3}\hat{\mathbf{z}}\right) = E\hat{\mathbf{z}} + \frac{ER^3}{r^3}\left(2\frac{z}{r}\hat{\mathbf{z}} - \frac{x}{r}\hat{\mathbf{x}} - \frac{y}{r}\hat{\mathbf{y}}\right). \tag{78}$$

Here is the picture of the field lines for this electric field: