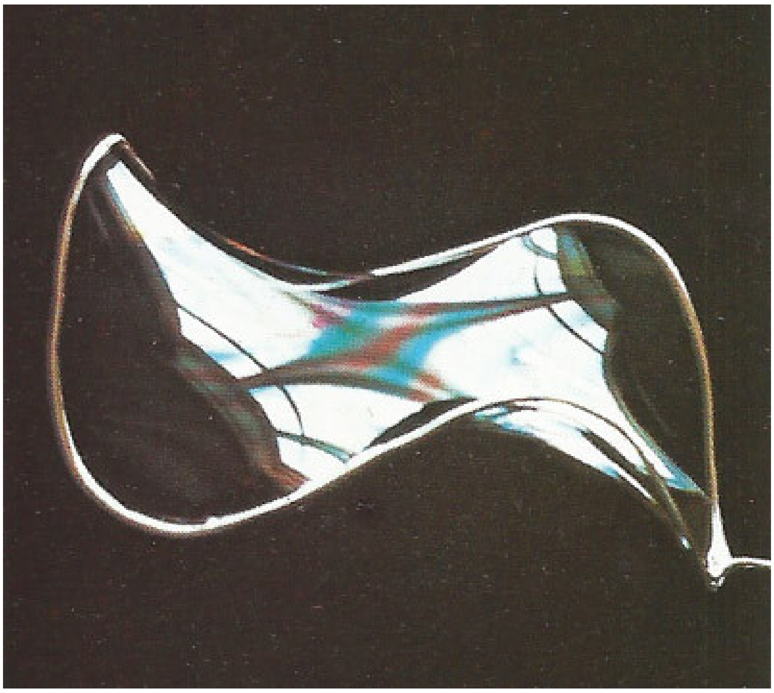
Soap Films and the Laplace Equation



Consider a soap film like on the above picture. Because of the surface tension, the geometry of the soap film is the minimal-area surface spanning the given loop. In Cartesian coordinates, the surface is specified by $z = f(x,y)$, and it obeys the Euler–Lagrange equation

$$\frac{\partial}{\partial x}\left(\frac{\frac{\partial z}{\partial x}}{\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}}\right) + \frac{\partial}{\partial y}\left(\frac{\frac{\partial z}{\partial y}}{\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}}\right) = 0. \tag{1}$$

This is a messy, non-linear partial differential equation! Fortunately, for the low-profile films with $\Delta z \ll \Delta x, \Delta y$ and hence small derivatives $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \ll 1$, the equation (1) simplifies and becomes the 2D Laplace equation:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0. \tag{2}$$

By shaping the wire loop spanned by the soap film, we specify the $z(s)$ along the boundary. For any such $s(s)$ at the boundary, the Laplace equation (2) has a *unique* solution.