Applications of Gauss Law

The *Gauss Law* of electrostatics relates the net electric field flux through a complete surface S of some volume V to the net electric charge inside that volume,

$$\Phi_E[S] \stackrel{\text{def}}{=} \iint\limits_S \mathbf{E} \cdot d\mathbf{A} \;=\; \frac{1}{\epsilon_0} \times Q_{\text{net}}[\text{inside}\, V]. \tag{1}$$

For a highly symmetric configuration of electric charges, the Gauss Law can be used to obtain the electric field $\mathbf{E}$ without taking any hard integrals. Instead, one uses a *Gaussian surface* whose symmetry alone assures that $\mathbf{E}$ is normal to the surface and has constant magnitude along the surface. Consequently, the flux through such a surface is simply

$$\Phi_E[S] \;=\; E \times A(S), \tag{2}$$

hence by Gauss Law (1),

$$E \;=\; \frac{1}{\epsilon_0} \times \frac{Q[\text{surrounded by } S]}{A(S)}\,. \tag{3}$$

In these notes, I shall illustrate this method for configurations of electric charges which have spherical, cylindrical, or planar symmetry.

Spherical Symmetry

The spherically symmetric charge configurations include a point charge, a uniform spherical shell of charges, a uniformly charged solid ball, several concentric spherical shells, as well as more exotic configurations where the 3D charge density ρ depends on the radius but not on the direction of the radius-vector. In spherical coordinates (r, θ, ϕ),

$$dQ \;=\; \rho(r) \times d\text{Volume} \;=\; \rho(r) \times dr \times r^2 d\Omega \tag{4}$$

where $d\Omega = d\theta \times \sin\theta d\phi$ is the infinitesimal solid angle (in steradians) and $\rho(r, \cancel{\theta}, \cancel{\phi})$ depends only on the radius but not on the angular coordinates θ or ϕ.

By symmetry, the electric field $\mathbf{E}$ of spherically symmetric charges always points in the radial direction — towards the origin, or away from the origin — while its magnitude depends only on the radius,

$$\mathbf{E}(r, \theta, \phi) \;=\; E(r\,\text{only})\,\hat{\mathbf{r}}\,. \tag{5}$$

To find the radial dependence of $E(r)$ we use the Gauss Law. Let S be a sphere of some radius r centered at the origin. By symmetry, the radial electric field is always perpendicular to S, while its magnitude $E(r)$ stays constant along S. This makes S a Gaussian surface, so the flux through S is simply

$$\Phi_E[S] \;=\; E(r) \times A(S) \;=\; E(r) \times 4\pi r^2. \tag{6}$$

Hence, by Gauss Law,

$$E(r) \times 4\pi r^2 \;=\; \frac{1}{\epsilon_0} \times Q_{\text{net}}[\text{inside}\,S] \tag{7}$$

and therefore

$$E(r) \;=\; \frac{1}{4\pi\epsilon_0\, r^2} \times Q_{\text{net}} \begin{bmatrix} \text{inside sphere} \\ \text{of radius } r \end{bmatrix}. \tag{8}$$

Example: Point Charge.

For an isolated point charge Q, any sphere surrounding the charge contains the same net charge $Q(r) = Q$, hence eq. (8) reproduces the Coulomb Law,

$$E(r) \;=\; \frac{Q}{4\pi\epsilon_0\, r^2}\,. \tag{9}$$

Example: Thin Spherical Shell.

Now consider a thin spherical shell of radius R and uniform surface charge density

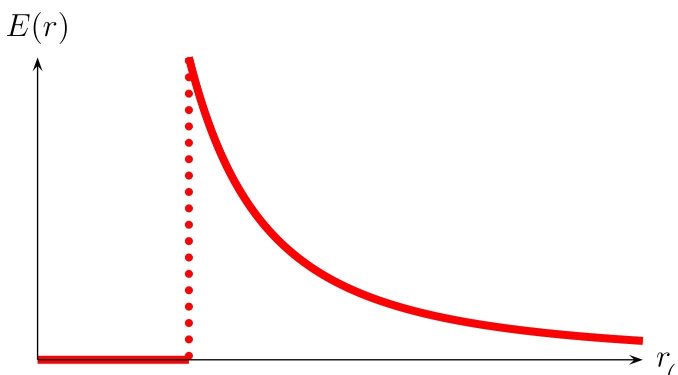
$$\sigma \;=\; \frac{dQ}{dA} \;=\; \frac{Q_{\rm net}}{4\pi R^2}\,. \tag{10}$$

For this shell, a Gaussian sphere of radius $r < R$ contains no charge at all, while a Gaussian sphere of radius $r > R$ contains the whole charge $Q_{\rm net}$ of the shell, thus

$$Q(r) \;=\; \begin{cases} 0 & \text{for } r < R, \\ Q_{\rm net} & \text{for } r > R, \end{cases} \tag{11}$$

and therefore

$$E(r) \;=\; \left\{ \begin{matrix} 0 & \text{for } r < R, \\ \dfrac{Q_{\rm net}}{4\pi\epsilon_0 r^2} & \text{for } r > R, \end{matrix} \right\} \tag{12}$$



In other words, *inside the shell there is no electric field, but outside the shell the electric field is the same as if the whole charge was at the center.*

EXAMPLE: SOLID BALL.

For our next example we take a solid ball of radius R and uniform 3D charge density

$$\rho\ =\ \frac{dQ}{dV}\ =\ \frac{Q_{\rm net}}{\frac{4\pi}{3}R^3}\,. \tag{13}$$

Again, a Gaussian sphere of radius $r>R$ contains the entire charge of the ball, hence

$$\text{for } r>R,\quad E(r)\ =\ \frac{Q_{\rm net}}{4\pi\epsilon_0\,r^2}\,, \tag{14}$$

outside the ball, its electric field is the same as if the whole charge was at the center of the ball. This is an example of a general rule: *outside any spherically symmetric charged body, its electric field is the same as if the whole charge was at the center.*

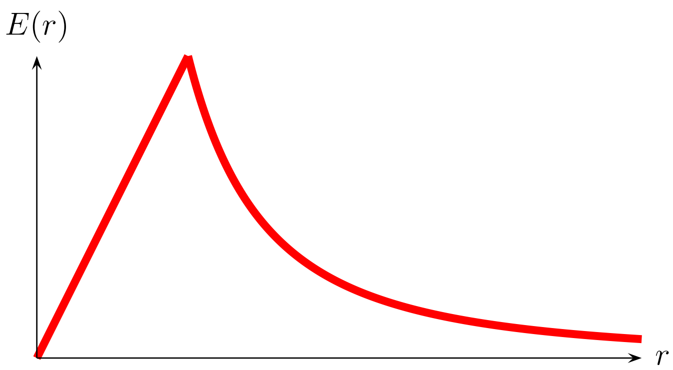
But inside the body, only the charges inside radius r contribute to the $E(r)$. For the solid ball, the Gaussian sphere of radius $r<R$ contains charge

$$Q(r)\ =\ \rho\times V\begin{bmatrix}\text{ball of}\\ \text{radius } r\end{bmatrix}\ =\ \rho\times\frac{4\pi}{3}r^3\ =\ Q_{\rm net}\times\frac{r^3}{R^3}\,. \tag{15}$$

Consequently, the electric field at radius $r<R$ is

$$E(r)\ =\ \frac{1}{4\pi\epsilon_0\,r^2}\times\frac{Q_{\rm net}\,r^3}{R^3}\ =\ \frac{Q_{\rm net}}{4\pi\epsilon_0\,R^2}\times\frac{r}{R}\,. \tag{16}$$

Thus, inside the ball, the electric field increases linearly with the radial coordinate, while outside the ball it decreases as $1/r^2$. Altogether,



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Example: Non-Uniform density.

Finally, consider a solid ball of radius R whose charge density depends on the radial coordinate as

$$\rho(r) \;=\; C\times r^{\alpha} \tag{18}$$

for some constants C and α. The net charge of such a ball is

$$\begin{aligned}
Q_{\rm net} \;&=\int\limits_0^R \rho(r)\times d\mathrm{Volume}(r)\;=\int\limits_0^R Cr^{\alpha}\times 4\pi r^2\,dr\\
&=\;4\pi C\times\int\limits_0^R r^{\alpha+2}\,dr\;=\;4\pi C\times\frac{R^{\alpha+3}}{\alpha+3}\\
&=\;\frac{4\pi}{\alpha+3}\,CR^{\alpha+3}.
\end{aligned}\tag{19}$$

Again, outside the ball the electric field is simply the Coulomb field of the net charge at the center of the ball,

$$\text{for } r>R,\quad E(r)\;=\;\frac{Q_{\rm net}}{4\pi\epsilon_0\,r^2}\,. \tag{20}$$

But inside the ball, we have a more complicated formula. Indeed, the charge inside a Gaussian sphere of radius $r<R$ is

$$\begin{aligned}
Q(r)\;&=\int\limits_0^r \rho(r')\times d\mathrm{Volume}(r')\;=\int\limits_0^r Cr'^{\alpha}\times 4\pi r'^2\,dr'\\
&=\;\frac{4\pi}{\alpha+3}\,Cr^{\alpha+3}\;=\;Q_{\rm net}\times\left(\frac{r}{R}\right)^{\alpha+3},
\end{aligned}\tag{21}$$

hence

$$E(r)\;=\;\frac{Q(r)}{4\pi\epsilon_0\,r^2}\;=\;Q_{\rm net}\times\left(\frac{r}{R}\right)^{\alpha+3}\times\frac{1}{4\pi\epsilon_0\,r^2}\;=\;\frac{Q_{\rm net}}{4\pi\epsilon_0\,R^2}\times\left(\frac{r}{R}\right)^{\alpha+1}. \tag{22}$$

Cylindrical Symmetry

Cylindrically symmetric configurations of electric charges include a long thin rod, a long hollow cylinder, several coaxial cylindrical shells (like a coaxial cable), as well as more general systems which are uniform and infinitely long in one dimension and axially symmetric in the other two dimensions. In cylindrical coordinates (r_c, ϕ, z),

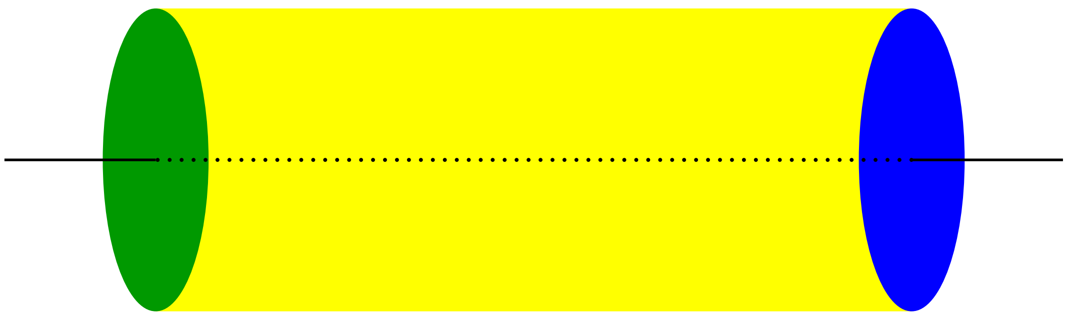
$$dQ \;=\; \rho(r_c)\times dV \;=\; \rho(r_c)\times dr_c\times r_c\,d\phi\times dz \tag{23}$$

where the charge density $\rho(r_c, \cancel{\phi}, \cancel{z})$ depends only on the cylindrical radius r_c but not on the angle ϕ or the lengthwise coordinate z.

By symmetry, the electric field of a cylindrically symmetric charge configuration always points directly away from the cylinder's axis or directly towards the axis, *i.e.* it has no z or ϕ components but only the r_c component. Also, the magnitude of the electric field depends only on the cylindrical radius r_c,

$$\mathbf{E}(r_c, \phi, z) \;=\; E(r_c\,\text{only})\,\hat{\mathbf{r}}_{\mathbf{c}}\,. \tag{24}$$

The Gaussian surfaces for such an electric field are cylinders C of generic radii r and lengths L but always coaxial with the charge distribution. For completeness sake, such a surface comprises both the outside cylinder at fixed $r_c = r$ and the circular endcaps at fixed $z = z_1$ or $z = z_2$,



However, on the endcaps the electric field (24) lies parallel to the endcaps instead of crossing them, so the flux through the endcaps is zero. Instead, the entire flux comes through the outer cylinder where $\mathbf{E}$ stays normal to the surface and its magnitude stays constant $E(r)$. Altogether, the net flux through the cylinder C is

$$\Phi_E[C] \;=\; E(r_c=r)\times A(C) \;=\; E(r_c=r)\times 2\pi rL. \tag{25}$$

By Gauss Law, this implies

$$2\pi rL\times E(r_c=r) \;=\; \frac{1}{\epsilon_0}\times Q_{\text{net}}\begin{bmatrix}\text{inside cylinder of}\\ \text{length} = L \text{ and radius} = r\end{bmatrix} \tag{26}$$

and therefore

$$E(r_c) \;=\; \frac{2}{4\pi\epsilon_0\, rL}\times Q_{\text{net}}\begin{bmatrix}\text{inside cylinder of}\\ \text{length} = L \text{ and radius} = r_c\end{bmatrix} \tag{27}$$

By translational symmetry along the cylinder axis, the charge inside the cylinder is always proportional to its length L, so let us define the *linear charge density inside a given radius* r_c,

$$\lambda(r_c) \;\stackrel{\text{def}}{=}\; \frac{1}{L}\times Q_{\text{net}}\begin{bmatrix}\text{inside cylinder of}\\ \text{length} = L \text{ and radius} = r_c\end{bmatrix} \tag{28}$$

In terms of this linear density,

$$E(r_c) \;=\; \frac{\lambda(r_c)}{2\pi\epsilon_0\, r_c}\,. \tag{29}$$

EXAMPLE: THIN ROD.

Consider an infinitely long, infinitely thin rod of uniform linear charge density λ. Any Gaussian cylinder containing this rod has net charge $Q=\lambda\times L$ regardless of the cylinder's radius. In terms of eq. (28), this means $\lambda(r_c)\equiv\lambda$ for any $r_c>0$, hence by the Gauss Law equation (29)

$$E(r_c) \;=\; \frac{\lambda}{2\pi\epsilon_0\, r_c} \quad\Longrightarrow\quad \mathbf{E} \;=\; \frac{\lambda}{2\pi\epsilon_0\, r_c}\,\hat{\mathbf{r}}_{\mathbf{c}}\,. \tag{30}$$

Note: the same formula obtains from the direct integration of the Coulomb fields over the length of the rod:

$$
\begin{aligned}
E(r_c) &= \frac{1}{4\pi\epsilon_0}\int\limits_{-\infty}^{+\infty}\frac{\lambda r_c\,dz'}{(r_c^2+z'^2)^{3/2}} \\
&\qquad \langle\!\langle\, \text{changing variables } z' = r_c\times\tan\alpha \,\rangle\!\rangle \\
&= \frac{1}{4\pi\epsilon_0}\int\limits_{-\pi/2}^{+\pi/2}\lambda\times\frac{\cos^3\alpha}{r_c^3}\times\frac{r_c\,d\alpha}{\cos^2\alpha} \\
&= \frac{1}{4\pi\epsilon_0}\times\frac{\lambda}{r_c}\times\left(\int\limits_{-\pi/2}^{+\pi/2}\cos\alpha\,d\alpha \;=\; 2\right) \\
&= \frac{\lambda}{2\pi\epsilon_0\,r_c}\,.
\end{aligned}
\tag{31}
$$

But using the Gauss Law is easier than integrating.

Example: Thin Cylindrical Shell.

For our next example, consider an infinitely long thin cylindrical shell of radius R with a uniform surface charge density

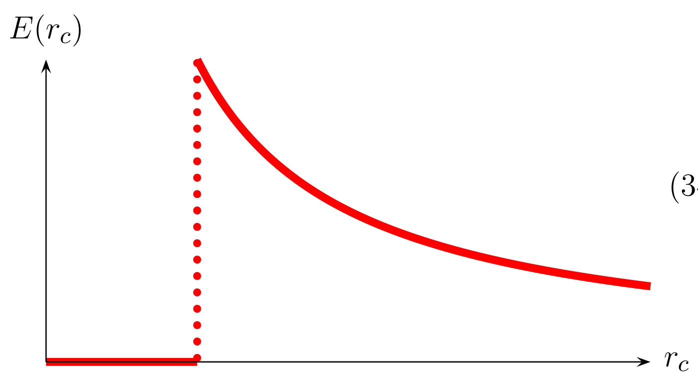
$$
\sigma \;\equiv\; \frac{dQ}{dA} \;=\; \frac{\lambda_{\text{net}}}{2\pi R}\,. \tag{32}
$$

For this shell, a Gaussian cylinder of radius $r_c < R$ contains no electric charge at all, while a cylinder of radius $r_c > R$ contains charge $Q = \lambda L$. In terms of $\lambda(r_c)$ this means

$$
\lambda(r_c) \;=\; \begin{cases} 0 & \text{for } r < R, \\ \lambda_{\text{net}} & \text{for } r > R, \end{cases} \tag{33}
$$

and therefore

$$
E(r_c) \;=\; \left\{\begin{matrix} 0 & \text{for } r_c < R, \\ \dfrac{\lambda_{\text{net}}}{2\pi\epsilon_0\,r_c} & \text{for } r_c > R, \end{matrix}\right\} \tag{34}
$$



In other words, *inside the shell there is no electric field, but outside the shell the electric field is the same as if the whole charge was on the axis.*

In general, *outside* any cylindrically symmetric charged body, the electric field is the same as outside a thin rod of the same net charge density λ_{net}.

Example: Solid Cylinder.

Now consider an infinitely long solid cylinder of radius R and uniform 3D charge density

$$\rho \;\equiv\; \frac{dQ}{dV} \;=\; \frac{\lambda_{\mathrm{net}}}{\pi R^2}\,. \tag{35}$$

This time, a Gaussian cylinder of radius $r_c > R$ contains charge $Q = L \times \lambda_{\mathrm{net}}$, while a Gaussian cylinder of radius $r_c < R$ contains non-zero but smaller charge

$$Q \;=\; \rho \times \pi r_c^2 L \;=\; L\lambda\mathrm{net} \times \frac{r_c^2}{R^2} \tag{36}$$

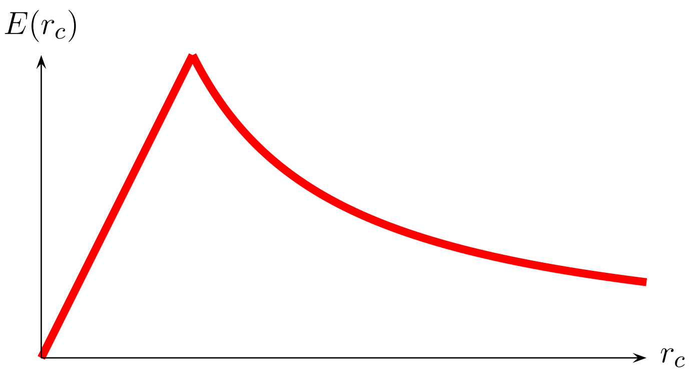
In terms of $\lambda(r_c)$, this means

$$\lambda(r_c) \;=\; \lambda_{\mathrm{net}} \times \begin{cases} \dfrac{r^2}{R^2} & \text{for } r_c < R, \\ 1 & \text{for } r_c > R, \end{cases} \tag{37}$$

and therefore

$$\begin{aligned} &\text{for } r_c < R, \quad E(r_c) \;=\; \lambda_{\mathrm{net}} \times \frac{r_c^2}{R^2} \times \frac{1}{2\pi\epsilon_0\, r_c} \;=\; \frac{\lambda_{\mathrm{net}}}{2\pi\epsilon_0\, R} \times \frac{r_c}{R}\,, \\ &\text{for } r_c > R, \quad E(r_c) \;=\; \frac{\lambda_{\mathrm{net}}}{2\pi\epsilon_0\, r_c}\,, \end{aligned} \tag{38}$$

inside the cylinder the electric field grows linearly with the radius as r_c while outside the cylinder it decreases as $1/r_c$,



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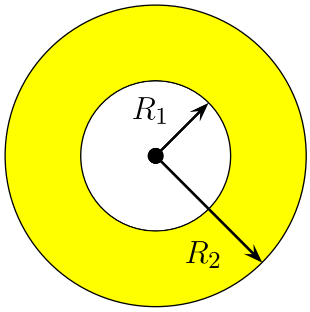
Example: Thick Cylindrical Shell.

This time, consider a thick cylindrical shell of inner radius R_1 and outer radius R_2,



and uniform 3D charge density ρ, hence net linear charge density

$$\lambda_{\rm net} \;=\; \rho\times\bigl(\pi R_2^2-\pi R_1^2\bigr). \tag{40}$$

This time, a Gaussian cylinder of radius smaller than the inner radius of the shell contains no electric charge at all, and there is no electric field in the hollow inside the cylinder,

$$\text{for } r_c<R_1,\quad \lambda(r_c)\;=\;0 \quad\Longrightarrow\quad E(r_c)\;=\;0, \tag{41}$$

On the other hand, the Gaussian cylinder of radius larger than the outer radius of the shell contains the entire linear charge density of the shell, thus

$$\text{for } r_c>R_2,\quad \lambda(r_c)\;=\;\lambda_{\rm net} \quad\Longrightarrow\quad E(r_c)\;=\;\frac{\lambda_{\rm net}}{2\pi\epsilon_0\,r_c}\,, \tag{42}$$

the electric field outside the shell is is similar to the field of a thin rod of the same $\lambda_{\rm net}$.

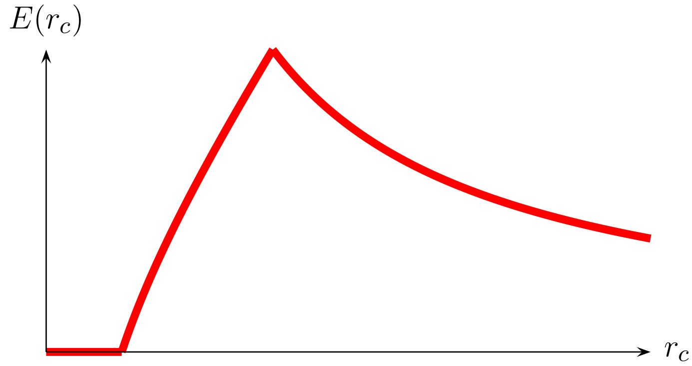
Finally, to find the electric field within the thickness of the shell, we use a Gaussian cylinder of radius $R_1<r_c<R_2$. The linear charge density inside this Gaussian cylinder is

$$\lambda(r_c)\;=\;\rho\times\bigl(\pi r_c^2-\pi R_1^2\bigr)\;=\;\lambda_{\rm net}\times\frac{r_c^2-R_1^2}{R_2^2-R_1^2}\,, \tag{43}$$

hence

$$E(r_c)\;=\;\lambda_{\rm net}\times\frac{r_c^2-R_1^2}{R_2^2-R_1^2}\times\frac{1}{2\pi\epsilon_0\,r_c}\,. \tag{44}$$

Altogether,



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Planar Symmetry

Charge configurations with planar symmetry include uniform 2D sheets, uniform slabs of finite thickness, as well as "sandwiches" of such sheets and slabs. In Cartesian coordinates (x, y, z), a general configuration with planar symmetry has 3D charge density $\rho(x, y, z)$ which depends only on z but not x or y. By symmetry, the electric field of such a configuration is parallel to the z axis and its magnitude depends only on the z coordinates but not on x or y,

$$\mathbf{E}(x, y, z) \;=\; E_z(z\,\text{only})\,\hat{\mathbf{z}}. \tag{46}$$

For simplicity, let me assume an additional upside-down symmetry $z \to -z$, thus

$$\rho(-z) \;=\; \rho(+z) \quad\Longrightarrow\quad E_z(-z) \;=\; -E_z(+z). \tag{47}$$

For charge configurations with planar and upside-down symmetries, the simplest Gaussian surfaces to use are brick-like parallelepipeds with top and bottom surfaces at $+z$ and $-z$ and horizontal area $A = \Delta x \times \Delta y$. Since the electric field is vertical everywhere, it has no flux through the four vertical sides of such a Gaussian brick, so the entire flux through the brick comes from the horizontal top and bottom surfaces. Thus,

$$\Phi_E[\text{brick}] \;=\; \iint\limits_{\text{top}} dx\,dy\,E_z(x, y, +z) \;-\; \iint\limits_{\text{bot}} dx\,dy\,E_z(x, y, -z) \tag{48}$$

where the $-$ sign between the two integrals comes from opposite directions of the inside-to-outside normals to the top and bottom surfaces, $\mathbf{n}_{\text{top}} = (0, 0, +1)$, $\mathbf{n}_{\text{bot}} = (0, 0, -1)$.

Fortunately, the integrals in eq. (48) are trivial since the electric field does not depend on x and y, hence

$$\Phi_E[\text{brick}] \;=\; A \times \big(E_z(+z) \,-\, E_z(-z)\big). \tag{49}$$

Finally, in light of the upside-down symmetry (47),

$$\Phi_E[\text{brick}] \;=\; 2A \times E_z(+z). \tag{50}$$

By Gauss Law, this flux is related to the net electric charge of the Gaussian brick,

$$2AE_z(+z) \;=\; \frac{1}{\epsilon_0} \times Q_{\text{net}}\begin{bmatrix}\text{inside brick}\\ A \times (-z\,\text{to}\,+z)\end{bmatrix} \tag{51}$$

Since the net charge inside such a brick is always proportional to the bricks horizontal area A, let's define *the net surface density of charge between* $-z$ *and* $+z$*,*

$$\sigma(z) \;\stackrel{\text{def}}{=}\; \frac{1}{A} \times Q_{\text{net}}\begin{bmatrix}\text{inside brick}\\ A \times (-z\,\text{to}\,+z)\end{bmatrix} \;=\int\limits_{-z}^{+z} \rho(z')\,dz'. \tag{52}$$

In terms of this surface density,

$$E_z(+z) \;=\; +\frac{1}{2\epsilon_0} \times \sigma(z), \quad E_z(-z) \;=\; -\frac{1}{2\epsilon_0} \times \sigma(z). \tag{53}$$

Example: Thin Sheet.

As a first example, consider an infinitely thin charged sheet of uniform surface charge density σ_{net}. For this sheet, any Gaussian brick has net charge $Q = \sigma_{\text{net}} \times A$ and therefore $\sigma(z) \equiv \sigma_{\text{net}}$. Consequently, the electric field of this sheet is

$$E_z(+z) \;=\; +\frac{\sigma_{\text{net}}}{2\epsilon_0}\,, \quad E_z(-z) \;=\; -\frac{\sigma_{\text{net}}}{2\epsilon_0}\,, \quad \text{for any } z > 0, \tag{54}$$

or in other words,

$$\mathbf{E} \;=\; \frac{\sigma_{\text{net}}}{2\epsilon_0}\,\text{sign}(z)\hat{\mathbf{z}}. \tag{55}$$

Again, we could have derived the same formula by integrating the Coulomb field of infinitesimal point charges all over the infinite 2D sheet. But using the Gauss Law is much easier than integrating!

Example: Thick Slab.

For our second example, consider a slab of finite thickness D and uniform 3D charge density

$$\rho = \frac{\sigma_{\rm net}}{D}\,. \tag{56}$$

For this charge configuration, a Gaussian brick of area A and thickness $2z < D$ has net charge $Q = \rho \times A \times 2z$, hence

$$\sigma(z) = \rho \times 2z = \sigma_{\rm net} \times \frac{2z}{D} \tag{57}$$

and therefore

$$\text{for } 0 < z < \frac{D}{2}\,,\quad E_z(+z) = +\frac{\sigma_{\rm net}}{2\epsilon_0} \times \frac{2z}{D} = \frac{\sigma_{\rm net}}{\epsilon_0} \times \frac{+z}{D}\,,\quad E_z(-z) = \frac{\sigma_{\rm net}}{\epsilon_0} \times \frac{-z}{D}\,. \tag{58}$$

In other words, *the electric field inside the slab is*

$$\text{for } -\frac{D}{2} < z < +\frac{D}{2}\,,\quad \mathbf{E}(z) = \frac{\rho}{\epsilon_0}\,z\,\hat{\mathbf{z}} = \frac{\sigma_{\rm net}}{\epsilon_0}\,\frac{z}{D/2}\,\hat{\mathbf{z}}. \tag{59}$$

On the other hand, the Gaussian brick or thickness $2z > D$ has net charge $Q = \sigma_{\rm net} \times A$, hence $\sigma(z) = \sigma_{\rm net}$ and therefore

$$\text{for } z > \frac{D}{2}\,,\quad E_z(+z) = +\frac{\sigma_{\rm net}}{2\epsilon_0}\,,\quad E_z(-z) = -\frac{\sigma_{\rm net}}{2\epsilon_0}\,. \tag{60}$$

In other words, *the electric field outside the slab is*

$$\text{for } |z| > \frac{D}{2}\,,\quad \mathbf{E}(z) = \frac{\sigma_{\rm net}}{2\epsilon_0}\,\mathrm{sign}(z)\,\hat{\mathbf{z}}, \tag{61}$$

which is the same as the field of a thin sheet of the same net surface charge density $\sigma_{\rm net}$.

Altogether,