Inductance and Magnetic Energy

Mutual Inductance

Consider two wire loops or coils. Their geometries can be completely general, and there might be some magnetic materials inside the coils or around them — for example, iron cores — but let's assume that all the magnetic materials involved are linear ($\mathbf{B} = \mu\mu_0\mathbf{H}$). Let's run a steady current I_1 through the coil#1, so it creates a magnetic field $\mathbf{B}_1(\mathbf{r})$, which in turn has flux Φ_2 through the coil#2. Regardless of the gory details of the two coils' geometries or any linear magnetic materials, which may be present, the field $\mathbf{B}_1(\mathbf{r})$ at any point $\mathbf{r}$ is proportional to the current I_1 which creates it, so the flux Φ_2 through the second coil is also proportional to that current,

$$\Phi_2 \;=\; M_{21} \times I_1 \tag{1}$$

for some current-independent coefficient M_{21}. This coefficient is called the *mutual inductance* of the two coils.

Now let's slowly vary the current I_1 through the first coil with the time t. As long as this change is slow enough, we may use the quasi-static approximation for the magnetic field $\mathbf{B}_1(\mathbf{r}, t)$ created by this current, hence the magnetic flux through the second coil varies with time according to

$$\Phi_2(t) \;=\; M_{21} \times I_1(t). \tag{2}$$

The time derivative of this flux produces the electromotive force (EMF) in the second coil,

$$\mathcal{E}_2 \;=\; -\frac{d\Phi_2}{dt} \;=\; -M_{21} \times \frac{dI_1}{dt}\,. \tag{3}$$

For example, the AC current $I_1(t) = I_1^{(0)} \times \cos(\omega t)$ through the first coil induces the AC voltage

$$\mathcal{E}_2 \;=\; M_{21}\omega \times \sin(\omega t) \tag{4}$$

in the second coil. These formulae are very important for the transformers.

The MKSA unit of mutual inductance — and also of self-inductance explained later in these notes — is called *henry* [after American scientist Joseph Henry (1797–1878)] and denoted H,

$$1\ \mathrm{H} \times 1\ \mathrm{A} \;=\; 1\ \mathrm{W}\ (\mathrm{Weber}) \;=\; 1\ \mathrm{T} \times 1\ \mathrm{m}^2 \;=\; 1\ \mathrm{V} \times 1\ \mathrm{s}. \tag{5}$$

In Gaussian units, the mutual inductance is defined with an extra factor of c in eq. (1),

$$\Phi_2 \;=\; c \times M_{21} \times I_1 \tag{6}$$

to compensate for the $1/c$ factor in the Induction Law so that eq. (3) looks similarly in both unit systems,

$$\mathcal{E}_2 \;=\; -\frac{1}{c}\frac{d\Phi_2}{dt} \;=\; -M_{21} \times \frac{dI_1}{dt}\,. \tag{7}$$

The Gaussian unit if mutual inductance or self-inductance does not have a proper name, but by dimensional analysis it's equivalent to s^2/cm:

$$1\ (\text{Gaussian unit of inductance}) \;=\; \frac{\mathrm{statV}}{(\mathrm{Fr/s})/\mathrm{s}} \;=\; \frac{\mathrm{s}^2}{\mathrm{Fr/statV}} \;=\; \frac{\mathrm{s}^2}{\mathrm{cm}}\,. \tag{8}$$

A very useful theorem for calculating the mutual inductances of coils is the **symmetry theorem:** *for any two wire loops or coils, of whatever geometry, in presence or absence of any magnetic materials of whatever shapes, as long as all such magnetic materials are linear,*

$$M_{21} \;=\; M_{21} \tag{9}$$

Let me prove this theorem for the coils without iron cores or any other magnetic materials involved. In this case, the magnetic field due to current in the coil#1 obtains from the Biot–Savart–Laplace formula, or in terms of the vector potential,

$$\mathbf{A}_1(\mathbf{r}) \;=\; \frac{\mu_0 I_1}{4\pi} \oint\limits_{\text{loop}\#1} \frac{d\mathbf{r}_1}{|\mathbf{r} - \mathbf{r}_1|}\,. \tag{10}$$

The magnetic flux Φ_2 through the coil#2 obtains from this vector potential as

$$\Phi_2 = \iint\limits_{\text{a surface spanning coil\#2}} \mathbf{B}_1 \cdot d^2\mathbf{a} = \oint\limits_{\text{coil\#2 itself}} \mathbf{A}_1(\mathbf{r}_2) \cdot d\mathbf{r}_2 \tag{11}$$

where the second equality follows by the Stokes theorem. Consequently,

$$\Phi_2 = \oint\limits_{\text{coil\#2}} \mathbf{r}_2 \cdot \left(\frac{\mu_0 I_1}{4\pi} \oint\limits_{\text{coil\#1}} \frac{d\mathbf{r}_1}{|\mathbf{r}_1 - \mathbf{r}_2|} \right) = \frac{\mu_0 I_1}{4\pi} \oint\oint\limits_{\substack{\mathbf{r}_1 \in \text{coil\#1} \\ \mathbf{r}_2 \in \text{coil\#2}}} \frac{d\mathbf{r}_1 \cdot d\mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|}, \tag{12}$$

which means that the mutual inductance M_{21} of the two coils is

$$M_{21} = \frac{\mu_0}{4\pi} \oint\oint\limits_{\substack{\mathbf{r}_1 \in \text{coil\#1} \\ \mathbf{r}_2 \in \text{coil\#2}}} \frac{d\mathbf{r}_1 \cdot d\mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|}. \tag{13}$$

This formula is manifestly symmetric between the two coils, thus $M_{21} = M_{12}$, *quod erat demonstrandum.*

EXAMPLE: TWO COAXIAL SOLENOIDAL COILS.

Specifically, let the first coil be both shorter and narrower than the second coil, and let's put the first coil in the middle of the hollow space inside the second coil. Let's also assume that the second coil's length is much larger than its diameter. In this case, calculating the mutual inductance M_{12} is rather easy: The current I_2 in the second coil creates a uniform magnetic field

$$\mathbf{B}_2 = \frac{\mu_0 N_2 I_2}{L_2}\,\hat{\mathbf{z}} \tag{14}$$

everywhere inside that coil, including the coil#1, so the flux of this field through the coil#1 is

$$\Phi_1 = B_2 \times \pi r_1^2 N_1 = \frac{\mu_0 N_2 I_2}{L_2} \times \pi r_1^2 N_1 = \mu_0 N_1 N_2 \times \frac{\pi r_1^2}{L_2} \times I_2\,. \tag{15}$$

In terms of the mutual inductance, this means

$$M_{12} = \mu_0 N_1 N_2 \times \frac{\pi r_1^2}{L_2}\,. \tag{16}$$

On the other hand, the direct calculation of the M_{21} mutual inductance is much harder. Indeed, the magnetic field of the current I_1 in the first coil is approximately uniform inside that coil, but become rather complicated near its poles; and since the poles of the first coil are inside the second coil, calculating the net magnetic flux through the second coil becomes quite a challenge. Fortunately, the symmetry theorem allows us to avoid this hard calculation and simply use

$$M_{21} = M_{12} = \mu_0 N_1 N_2 \times \frac{\pi r_1^2}{L_2}. \tag{17}$$

Example: two coils on a common toroidal iron core.

The current I_1 in the first coil creates magnetic field

$$\begin{aligned} \text{outside the toroid} \quad \mathbf{H} &= 0, \\ \text{inside the toroid} \quad \mathbf{H} &= \frac{N_1 I_1}{2\pi s}\,\hat{\phi}, \end{aligned} \tag{18}$$

hence

$$\begin{aligned} \text{outside the toroid} \quad \mathbf{B} &= 0, \\ \text{inside the toroid} \quad \mathbf{B} &= \mu\mu_0\,\frac{N_1 I_1}{2\pi s}\,\hat{\phi}, \end{aligned} \tag{19}$$

where μ is the permeability of the iron in the toroid. For simplicity, let's assume the toroid is shaped like a bicycle tire rather than like a donut, so inside the toroid $s \approx \text{const} = \text{large}$ radius R. This makes for approximately uniform magnetic field inside the toroid, so if its cross-sectional area is A then the flux through the toroid is

$$\Phi \approx \mu\mu_0\,\frac{N_1 I_1}{2\pi R}\,A. \tag{20}$$

The second coil has this flux going through each of its loops, hence

$$\Phi_2 = N_2 \times \Phi = \mu\mu_0 N_1 N_2\,\frac{A}{2\pi R} \times I_2\,, \tag{21}$$

which means the mutual induction of the two coils is

$$M_{21} = M_{12} = \mu\mu_0 N_1 N_2\,\frac{A}{2\pi R}\,. \tag{22}$$

Note manifest symmetry of this formula between the 2 coils.

Self-Inductance and RL Circuits

Now consider a single coil, with or without an iron core. The current I through the coil creates magnetic field $\mathbf{B}(\mathbf{r})$, which has some flux Φ through the coil itself. By linearity,

$$\Phi \;=\; L\times I \tag{23}$$

for some coefficient L called the *self-inductance* of the coil; a less common name for the same coefficient is the *inductivity*. For example, consider a solenoidal coil with an iron core. A current I flowing through the coil creates approximately uniform field

$$\mathbf{B} \;=\; \mu\mu_0\,\frac{NI}{\ell}\,\hat{\mathbf{z}} \tag{24}$$

inside the core, and negligible $\mathbf{B}\approx 0$ outside the core. The flux of this field through the coil itself is

$$\Phi \;=\; N\times A\times B \;=\; \mu\mu_0 N^2\frac{A}{\ell}\times I \tag{25}$$

where A is the cross-sectional area of the core, which means the self-inductance of the coil is

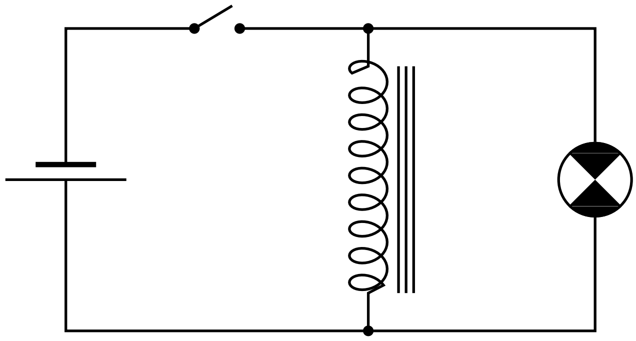
$$L \;=\; \mu\mu_0 N^2\,\frac{A}{\ell}\,. \tag{26}$$

When the current I changes with time, but slowly enough to use the quasi-static approximation, the flux changes with time as $\Phi(t)=L\times I(t)$, which induces EMF in the coil according to

$$\mathcal{E}(t) \;=\; -\frac{d\Phi}{dt} \;=\; -L\times\frac{dI}{dt}\,. \tag{27}$$

The minis sign here stems from the Lenz rule: the induced EMF resists changing the current flowing through the coil.

As an example of this Lenz rule in action, consider the following circuit



When the switch is closed, the light bulb and the coil receive the same voltage from the battery, but since the Ohmic resistance of the coil is much less than the resistance of the bulb, the current through the coil is much stronger than the current through the bulb. In fact, the current through the bulb is rather weak, so the bulb barely light up and stays rather dim. But when the switch is suddenly thrown open, the current which used to flow through the coil cannot stop right away — the coil's self-inductance prevents this according to eq. (27). Instead, this strong current has to flow through the bulb — which makes it flush bright. However, this flash lasts only a short time, as the current through the coil and the bulb decays rather fast.

Let's calculate the time scale and the manner of this decay. For simplicity let's treat the light bulb as a resistor of a constant resistance R_b. The current through the resulting RL circuit follows from the EMF in the coil by the Ohm's Law,

$$\mathcal{E} \;=\; IR_c \;+\; IR_b \;=\; IR \tag{28}$$

where R_c is the Ohmic resistance of the coil and $R = R_b + R_c$ is the net resistance of the RL circuit. At the same time, the EMF follows from the time derivative of the current according to eq. (27), hence

$$\frac{dI}{dt} \;=\; -\frac{\mathcal{E}}{L} \;=\; -\frac{RI}{L}\,. \tag{29}$$

Solving this differential equation with the initial condition $I(t = 0) = I_0$ gives us exponential decay

$$I(t) \;=\; I_0 \times \exp(-t/\tau) \tag{30}$$

with the time constant

$$\tau \;=\; \frac{L}{R} \tag{31}$$

For example, in the demo shown in the freshmen E&M class, the big coil has self-inductance about $L \approx 2$ H while the light bulb has resistance about $R \approx 100\ \Omega$, hence a rather short the time constant $\tau \approx\ 0.02$ seconds.

Magnetic Energy

Consider what happens when one tries to increase the current $I(t)$ flowing through an inductor coil. The coil's self-inductance F leads to EMF

$$\mathcal{E}_{\text{coil}} = -L\,\frac{dI}{dt} \tag{32}$$

which resists changing the current and performs negative work

$$dW_{\text{coil}} \;=\; \mathcal{E}_{\text{coil}} \times dQ \;=\; -L\,\frac{dI}{dt} \times I\,dt = \; -L \times I \times dI.$$

Note that this negative work is independent of the time it takes to change the current! This negative work has to be overcome by the positive work of the battery,

$$dW_{\text{battery}} \;=\; -dW_{\text{coil}} \;=\; +LI\,dI, \tag{33}$$

hence for a finite change of the current,

$$W_{\text{net}} \;=\int\limits_{I_1}^{I_2} LI\,dI \;=\; \frac{L}{2}\left(I_2^2 - I_1^2\right). \tag{34}$$

This work is *stored as the magnetic energy* of the inductor coil,

$$U_{\text{mag}} \;=\; \frac{LI^2}{2}\,, \tag{35}$$

which may be later used up to power some circuit for a short time, for example the light bulb in the example on the previous page.

Indeed, let's show that the net energy dissipated by the Ohmic resistance R in the RL circuit while the current is exponentially decaying is precisely the magnetic energy (35) stored in the inductor: The dissipated power is

$$P(t) \;=\; R\times I^2(t) \;=\; R\times I_0^2\times\exp(-2t/\tau), \tag{36}$$

hence net dissipated energy

$$W_{\rm net} \;=\; \int\limits_0^\infty P(t)\,dt \;=\; \int\limits_0^\infty RI_0^2\,\exp(-2t/\tau)\,dt \;=\; RI_0^2\times\frac{\tau}{2} \tag{37}$$

where $R\times\tau=L$ according to eq. (31), thus

$$W_{\rm net} \;=\; \frac{LI_0^2}{2}\,, \tag{38}$$

— which is precisely the initial energy stored in the inductor according to eq. (35).

Now let's relate the inductor coil energy (35) to the magnetic field in the inductor. For a coil of most general geometry,

$$U \;=\; \frac{LI^2}{2} \;=\; \frac{I}{2}\times\Phi_{\rm coil} \;=\; \frac{I}{2}\times\oint\limits_{\rm coil}\mathbf{A}\cdot d\vec{\ell}. \tag{39}$$

We may generalize this formula from a coil made form a thin wire to a thick conductor with some free current $\mathbf{J}_f$ flowing through its volume by simply changing $Id\vec{\ell}$ to $\mathbf{J}d^3{\rm Vol}$, thus

$$U \;=\; \frac{1}{2}\iiint\mathbf{A}\cdot\mathbf{J}_f\,d^3{\rm Vol}. \tag{40}$$

Moreover, by Ampere's Law $\mathbf{J}_f=\nabla\times\mathbf{H}$, hence

$$\mathbf{A}\cdot\mathbf{J} \;=\; \mathbf{A}\cdot(\nabla\times\mathbf{H}) \;=\; \nabla\cdot(\mathbf{H}\times\mathbf{A}) \;+\; \mathbf{H}\cdot(\nabla\times\mathbf{A}) =\; \nabla\cdot(\mathbf{H}\times\mathbf{A}) \;+\; \mathbf{H}\cdot\mathbf{B}. \tag{41}$$

Consequently, the magnetic energy becomes

$$\begin{aligned} U &= \frac{1}{2}\iiint\limits_{\mathcal{V}} \mathbf{H}\cdot\mathbf{B}\,d^3\mathrm{Vol} \;+\; \frac{1}{2}\iiint\limits_{\mathcal{V}} \nabla\cdot(\mathbf{H}\times\mathbf{A})\,d^3\mathrm{Vol} \\ &= \frac{1}{2}\iiint\limits_{\mathcal{V}} \mathbf{H}\cdot\mathbf{B}\,d^3\mathrm{Vol} \;+\; \frac{1}{2}\iint\limits_{\mathcal{S}} (\mathbf{H}\times\mathbf{A})\cdot d^2\mathbf{a}, \end{aligned} \tag{42}$$

where $\mathcal{V}$ is some volume which includes all the current-carrying conductors, and $\mathcal{S}$ is the complete surface of that volume, whatever it is. We can take the volume $\mathcal{V}$ to be as large as we want, so let's make it a ball of very large radius R. In the limit $R\to\infty$, the surface integral in eq. (42) vanishes; indeed, very far from all the currents,

$$\mathbf{A} \;\propto\; \frac{1}{R^2}\,, \quad \mathbf{H} \;\propto\; \frac{1}{R^3}\,, \quad \mathrm{Area}(\mathcal{S}) \;=\; 4\pi R^2, \tag{43}$$

hence

$$\iint\limits_{\mathcal{S}} (\mathbf{H}\times\mathbf{A})\cdot d^2\mathbf{a} \;\propto\; \frac{1}{R^3} \;\underset{R\to 0}{\longrightarrow}\; 0. \tag{44}$$

In the same $R\to\infty$ limit, the volume integral over $\mathcal{V}$ becomes the integral over the whole space, thus

$$U_{\mathrm{magnetic}} \;=\; \frac{1}{2}\iiint\limits_{\substack{\text{whole}\\\text{space}}} \mathbf{H}\cdot\mathbf{B}\,d^3\mathrm{Vol}. \tag{45}$$

Example: toroidal coil.

Earlier in these notes we have calculated the self-inductance of a toroidal coil with an iron core as

$$L \;=\; \frac{\mu\mu_0 N^2 A}{2\pi R}\,, \tag{46}$$

so the net magnetic energy stored in this coil is

$$U \;=\; \frac{LI^2}{2} \;=\; \frac{\mu\mu_0 N^2 A}{2\pi R} \times \frac{I^2}{2} \tag{47}$$

where I is the current through the coil. The magnetic fields $\mathbf{H}$ and $\mathbf{B}$ created by this coil are negligibly small outside the iron toroid, while inside the toroid

$$\mathbf{H} \approx \frac{NI}{2\pi R}\,\hat{\phi}, \quad \mathbf{B} = \mu\mu_0\mathbf{H}, \quad \Longrightarrow \quad \mathbf{H}\cdot\mathbf{B} \approx \mu\mu_0\left(\frac{NI}{2\pi R}\right)^2. \tag{48}$$

The energy (45) of these magnetic fields is therefore

$$\begin{aligned} U_{\text{magnetic}} &= \frac{1}{2}\iiint\limits_{\substack{\text{whole}\\\text{space}}} \mathbf{H}\cdot\mathbf{B}\,d^3\text{Vol} \approx \frac{1}{2}\iiint\limits_{\text{toroid}} \mathbf{H}\cdot\mathbf{B}\,d^3\text{Vol} \\ &\approx \frac{1}{2}\big(\text{approx. constant }\mathbf{H}\cdot\mathbf{B}\big)\times\big(\text{volume of the toroid}\big) \\ &= \frac{1}{2}\times\mu\mu_0\left(\frac{NI}{2\pi R}\right)^2\times 2\pi RA \\ &= \frac{I^2}{2}\times\frac{\mu\nu_0 N^2 A}{2\pi R}, \end{aligned} \tag{49}$$

in perfect agreement with eq. (47) for the magnetic energy of the coil.

This example is rather similar to the electric energy stored in a capacitor: we can calculate it as simply

$$U_{\text{capacitor}} = \frac{CV^2}{2} = \frac{Q^2}{2C}, \tag{50}$$

or we may calculate the electric tension and displacement fields $\mathbf{E}$ and $\mathbf{D}$ inside the capacitor, and then obtain their energy as

$$U_{\text{electric}} = \frac{1}{2}\iiint\limits_{\substack{\text{whole}\\\text{space}}} \mathbf{E}\cdot\mathbf{D}\,d^3\text{Vol}, \tag{51}$$

we would get the same net energy either way.

Note the remarkable similarity between the electric energy (51) and the magnetic energy (45). Microscopically — or in vacuum — these energies become

$$U_{\text{electric}} = \frac{\epsilon}{2}\iiint\limits_{\substack{\text{whole}\\\text{space}}} \mathbf{E}^2\,d^3\text{Vol}, \quad U_{\text{magnetic}} = \frac{1}{2\mu_0}\iiint\limits_{\substack{\text{whole}\\\text{space}}} \mathbf{B}^2\,d^3\text{Vol} \tag{52}$$

in MKSA units, or

$$U_{\text{electric}} \;=\; \frac{1}{8\pi} \iiint\limits_{\substack{\text{whole}\\\text{space}}} \mathbf{E}^2\, d^3\text{Vol}, \quad U_{\text{magnetic}} \;=\; \frac{1}{8\pi} \iiint\limits_{\substack{\text{whole}\\\text{space}}} \mathbf{B}^2\, d^3\text{Vol} \tag{53}$$

in Gaussian units. The similarity between these energies reflect similar behavior of the electric and magnetic fields in vacuum, the only difference being the way the E/B fields couple to the electric charges and currents. Indeed, had the Nature provided us with both electric and magnetic charges and currents, the similarity between the electric and the magnetic fields would be complete!