Divergence and Curl of the Magnetic Field

The *static* electric field $\mathbf{E}(x,y,z)$ — such as the field of static charges — obeys equations

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\,\rho, \tag{1}$$

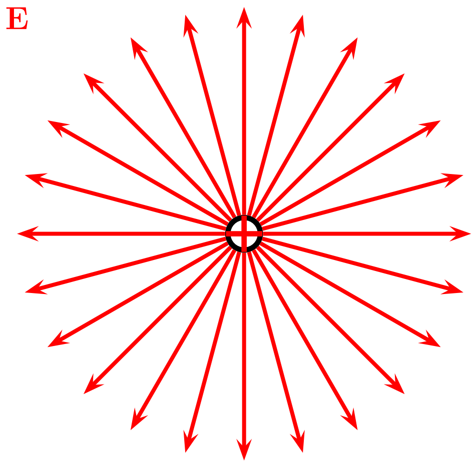
$$\nabla \times \mathbf{E} = 0. \tag{2}$$

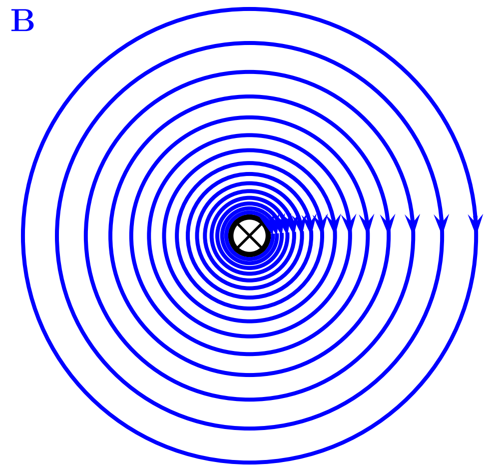
The *static* magnetic field $\mathbf{B}(x,y,z)$ — such as the field of steady currents — obeys different equations

$$\nabla \cdot \mathbf{B} = 0, \tag{3}$$

$$\nabla \times \mathbf{B} = \mu_0\,\mathbf{J}. \tag{4}$$

Due to this difference, the magnetic field of long straight wire looks quite different from the electric field of a point charge or a linear charge:





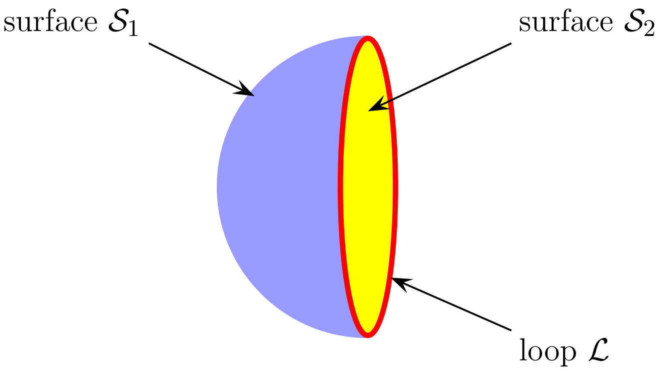
Later in these notes I shall derive eqs. (3) and (4) from the Biot–Savart–Laplace Law. But first, let me explore some of their consequences.

The zero-divergence equation (3) is valid for any magnetic field, even if it is time-dependent rather than static. Physically, it means that there are no magnetic charges — otherwise we would have $\nabla \cdot \mathbf{B} \propto \rho_{\mathrm{mag}}$ instead of $\nabla \cdot \mathbf{B} = 0$. Consequently, the magnetic field lines never begin or end anywhere in space; instead they form closed loops or run from infinity to infinity.

The integral form of eq. (3) follows by the Gauss theorem: *the magnetic flux through any closed surface is zero*,

$$\oiint_{\mathcal{S}} \mathbf{B}\cdot\mathbf{d^2A} \;=\; 0 \quad \text{for any closed surface } \mathcal{S}. \tag{5}$$

Consequently, *any open surfaces $\mathcal{S}_1, \mathcal{S}_2, \ldots$ spanning the same loop $\mathcal{L}$ have the same magnetic flux through them*, for example



$$\iint_{\mathcal{S}_1} \mathbf{B}\cdot\mathbf{d^2A} \;=\; \iint_{\mathcal{S}_2} \mathbf{B}\cdot\mathbf{d^2A} \;=\; \Phi_B[\text{through loop } \mathcal{L}]. \tag{6}$$

Later in class, we shall find this identity very useful for stating the Faraday's Law of magnetic induction.

Now consider the curl equation $\nabla\times\mathbf{B} = \mu_0\mathbf{J}$, which is the differential form of the *Ampere's Law*. The integral form of the Ampere's Law obtains by the Stokes' theorem: For any closed loop $\mathcal{L}$ and any surface $\mathcal{S}$ spanning that loop,

$$\oint_{\mathcal{L}} \mathbf{B}\cdot d\vec{\ell} \;=\; \iint_{\mathcal{S}} (\nabla\times\mathbf{B})\cdot\mathbf{d^2A} \;=\; \mu_0 \iint_{\mathcal{S}} \mathbf{J}\cdot\mathbf{d^2A} \;=\; \mu_0\times I_{\text{net}}[\text{through } \mathcal{L}] \tag{7}$$

where $I_{\text{net}}[\text{through } \mathcal{L}]$ is the net electric current flowing through the loop $\mathcal{L}$. The integral form (7) of the Ampere's Law is particularly convenient when the current flows through a wire or several wires; in this case all we need is to check which wire goes through the loop $\mathcal{L}$ and which does not, then add up the currents in the wires that do go through $\mathcal{L}$ and mind their directions. But it is also convenient for the volume currents flowing through thick conductors or for current sheets flowing on surfaces. I shall give several examples of using the Ampere's Law in a separate set of notes.

As written in eqs. (4) or (7), the Ampere's Law applies only to the magnetic fields of steady currents. Otherwise, we need to use the more general Maxwell–Ampere Law, with an extra term for the time-dependent electric field. I shall discuss this more general law in a few weeks. For the moment, let me simply say that the the original Ampere's Law is simply mathematically inconsistent unless the electric current has zero divergence, $\nabla\cdot\mathbf{J}=0$. Indeed, the left hand side of the curl equation $\nabla\times\mathbf{B}=\mu_0\mathbf{J}$ always have zero divergence

$$\nabla\cdot(\nabla\times\mathbf{B}) \;=\; 0, \tag{8}$$

so we cannot have $\nabla\times\mathbf{B}=\mu_0\mathbf{J}$ unless the RHS has zero divergence as well, $\nabla\cdot\mathbf{J}=0$. As to the integral form (7) of the Ampere's Law, we need a divergence-less current density $\mathbf{J}$ to make sure that the net current through the loop $\mathcal{L}$ is the same for any surface $\mathcal{S}$ spanning the loop, otherwise we simply cannot define the net current through $\mathcal{L}$.

Proving the Curl and Divergence Equations from the Biot–Savart–Laplace Law

The magnetic field of a steady current density $\mathbf{J}$ is given by the Biot–Savart–Laplace equation

$$\mathbf{B}(\mathbf{r}) \;=\; \frac{\mu_0}{4\pi}\iiint\mathbf{J}(\mathbf{r}')\times\mathbf{G}(\mathbf{r}-\mathbf{r}')\,d^3\mathrm{Vol} \tag{9}$$

where

$$\mathbf{G}(\mathbf{r}-\mathbf{r}') \;=\; \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \;=\; \frac{\text{unit vector from }\mathbf{r}'\text{ to }\mathbf{r}}{(\text{distance from }\mathbf{r}'\text{ to }\mathbf{r})^2}\,. \tag{10}$$

Let me show that the field (9) indeed obeys the divergence and the curl equations (3) and (4).

In eq. (9) I did not specify the volume over which the integral is taken. This volume must include everywhere the current flows through, but we may also use a larger integration volume, or even the whole 3D space. In any case, this volume should not depend on the point

$\mathbf{r}$ where I measure the magnetic field. Consequently, when I take a derivative of the integral — such as the divergence or the curl of the magnetic field — I can move the derivative inside the integral, thus

$$\begin{aligned} \nabla \cdot \mathbf{B}(\mathbf{r}) &= \frac{\mu_0}{4\pi} \iiint \nabla_{\mathbf{r}} \cdot \big(\mathbf{J}(\mathbf{r}') \times \mathbf{G}(\mathbf{r}-\mathbf{r}')\big) d^3\text{Vol}, & (11) \\ \nabla \times \mathbf{B}(\mathbf{r}) &= \frac{\mu_0}{4\pi} \iiint \nabla_{\mathbf{r}} \times \big(\mathbf{J}(\mathbf{r}') \times \mathbf{G}(\mathbf{r}-\mathbf{r}')\big) d^3\text{Vol}, & (12) \end{aligned}$$

where $\nabla_{\mathbf{r}}$ takes derivatives with respect to components (x, y, z) of $\mathbf{r}$ rather than (x', y', z') of $\mathbf{r}'$. Therefore, when acting on the prodict $\mathbf{J}(\mathbf{r}') \times \mathbf{G}(\mathbf{r}-\mathbf{r}')$, the derivative $\nabla_{\mathbf{r}}$ acts on the *kernel* $\mathbf{G}(\mathbf{r}-\mathbf{r}')$ but not on the current $\mathbf{J}(\mathbf{r}')$ since the latter depends on the $\mathbf{r}'$ rather than $\mathbf{r}$. Specifically, by the Leibniz product rules (rules (6) and (8) inside the textbook front cover),

$$\begin{aligned} \nabla_{\mathbf{r}} \cdot \big(\mathbf{J}(\mathbf{r}') \times \mathbf{G}(\mathbf{r}-\mathbf{r}')\big) &= \mathbf{G}(\mathbf{r}-\mathbf{r}') \cdot \cancel{[\nabla_{\mathbf{r}} \times \mathbf{J}(\mathbf{r}')]} \; - \; \mathbf{J}(\mathbf{r}') \cdot [\nabla_{\mathbf{r}} \times \mathbf{G}(\mathbf{r}-\mathbf{r}')] \\ &= 0 \; - \; \mathbf{J}(\mathbf{r}') \cdot \big[\nabla_{\mathbf{r}} \times \mathbf{G}(\mathbf{r}-\mathbf{r}')\big], & (13) \\ \nabla_{\mathbf{r}} \times \big(\mathbf{J}(\mathbf{r}') \times \mathbf{G}(\mathbf{r}-\mathbf{r}')\big) &= \cancel{(\mathbf{G}(\mathbf{r}-\mathbf{r}') \cdot \nabla_{\mathbf{r}})\mathbf{J}(\mathbf{r}')} \; - \; \big(\mathbf{J}(\mathbf{r}') \cdot \nabla_{\mathbf{r}}\big)\mathbf{G}(\mathbf{r}-\mathbf{r}') \\ &\quad + \; \mathbf{J}(\mathbf{r}')\big[\nabla_{\mathbf{r}} \cdot \mathbf{G}(\mathbf{r}-\mathbf{r}')\big] \; - \; \mathbf{G}(\mathbf{r}-\mathbf{r}')\cancel{\big[\nabla_{\mathbf{r}} \cdot \mathbf{J}(\mathbf{r}')\big]} \\ &= \mathbf{J}(\mathbf{r}')\big[\nabla_{\mathbf{r}} \cdot \mathbf{G}(\mathbf{r}-\mathbf{r}')\big] \; - \; \big(\mathbf{J}(\mathbf{r}') \cdot \nabla_{\mathbf{r}}\big)\mathbf{G}(\mathbf{r}-\mathbf{r}'). & (14) \end{aligned}$$

Moreover, the kernel $\mathbf{G}(\mathbf{r}-\mathbf{r}')$ is a gradient,

$$\mathbf{G}(\mathbf{r}-\mathbf{r}') = \nabla_{\mathbf{r}} \left(\frac{-1}{|\mathbf{r}-\mathbf{r}'|} \right), \tag{15}$$

so it has zero curl,

$$\nabla_{\mathbf{r}'} \times \mathbf{G}(\mathbf{r}-\mathbf{r}') = 0. \tag{16}$$

Consequently, the RHS of eq. (13) vanishes altogether, so the LHS must also vanish, and this means identically zero integrand in eq. (11). Thus, the whole integral (11) vanishes, which proves *zero divergence of the magnetic field,*

$$\nabla\cdot\mathbf{B}(\mathbf{r}) \;=\; 0. \tag{17}$$

Next, consider the curl equation (12). Pluggin in the surviving terms in eq. (14) into the integral, we arrive at

$$\nabla\times\mathbf{B}(\mathbf{r}) \;=\; \frac{\mu_0}{4\pi}\iiint\mathbf{J}(\mathbf{r}')\big(\nabla_{\mathbf{r}}\cdot\mathbf{G}(\mathbf{r}-\mathbf{r}')\big)d^3\mathrm{Vol}' \;-\; \frac{\mu_0}{4\pi}\iiint\big(\mathbf{J}(\mathbf{r}')\cdot\nabla_{\mathbf{r}}\big)\mathbf{G}(\mathbf{r}-\mathbf{r}')d^3\mathrm{Vol}'. \tag{18}$$

Furthermore, the second term here happens to vanish. To see how this works, note that the kernel $\mathbf{G}(\mathbf{r}-\mathbf{r}')$ depends only on the difference $\mathbf{r}-\mathbf{r}'$, so its derivatives WRT to components of the $\mathbf{r}$ and the $\mathbf{r}'$ are related by sign reversal,

$$\frac{\partial G_i(\mathbf{r}-\mathbf{r}')}{\partial r_j} \;=\; -\frac{\partial G_i(\mathbf{r}-\mathbf{r}')}{\partial r'_j}, \quad i,j=x,y,z. \tag{19}$$

Consequently,

$$\begin{aligned}\big(\mathbf{J}(\mathbf{r}')\cdot\nabla_{\mathbf{r}}\big)G_i(\mathbf{r}-\mathbf{r}') \;&=\; -\big(\mathbf{J}(\mathbf{r}')\cdot\nabla_{\mathbf{r}'}\big)G_i(\mathbf{r}-\mathbf{r}')\\ &\qquad\langle\!\langle\text{ integrating by parts }\rangle\!\rangle\\ &=\; -\nabla_{\mathbf{r}'}\cdot\big(\mathbf{J}(\mathbf{r}')\,G_i(\mathbf{r}-\mathbf{r}')\big) \;+\; G_i(\mathbf{r}-\mathbf{r}')\big(\nabla_{\mathbf{r}'}\cdot\mathbf{J}(\mathbf{r}')\big).\end{aligned} \tag{20}$$

Moreover, the second term on the bottom line vanishes for the *steady* — and therefore divergence-less — current $\mathbf{J}(\mathbf{r}')$, which leaves us with

$$\big(\mathbf{J}(\mathbf{r}')\cdot\nabla_{\mathbf{r}}\big)G_i(\mathbf{r}-\mathbf{r}') \;=\; -\nabla_{\mathbf{r}'}\cdot\big(\mathbf{J}(\mathbf{r}')\,G_i(\mathbf{r}-\mathbf{r}')\big), \tag{21}$$

The integral of this total divergence obtains from the Gauss theorem:

$$\begin{aligned}\iiint\big(\mathbf{J}(\mathbf{r}')\cdot\nabla_{\mathbf{r}}\big)G_i(\mathbf{r}-\mathbf{r}')d^3\mathrm{Vol}' \;&=\; -\iiint\nabla_{\mathbf{r}'}\cdot\big(\mathbf{J}(\mathbf{r}')\,G_i(\mathbf{r}-\mathbf{r}')\big)d^3\mathrm{Vol}'\\ &=\; -\iint\limits_{\mathcal{S}} G_i(\mathbf{r}-\mathbf{r}')\,\mathbf{J}(\mathbf{r}')\cdot\mathbf{d^2A},\end{aligned} \tag{22}$$

where the surface $\mathcal{S}$ over which we integrate on the last line is the boundary of the volume $\mathcal{V}$ over which we integrate in the Biot–Savart–Laplace equation (9). This volume must include everywhere the current flows, but we may just as well use a bigger volume. Indeed, let's take a bigger volume $\mathcal{V}$, so no current flows on its surface $\mathcal{S}$ — which immediately kills the surface integral on the bottom line of eq. (22). Therefore,

$$\iiint \big(\mathbf{J}(\mathbf{r}') \cdot \nabla_{\mathbf{r}}\big) G_i(\mathbf{r}-\mathbf{r}') d^3\mathrm{Vol}' \;=\; 0, \tag{23}$$

or in vector notations

$$\iiint \big(\mathbf{J}(\mathbf{r}') \cdot \nabla_{\mathbf{r}}\big) \mathbf{G}(\mathbf{r}-\mathbf{r}')\, d^3\mathrm{Vol}' \;=\; 0. \tag{24}$$

In terms of eq. (18), this means that the entire curl of the magnetic field comes from the first term,

$$\nabla \times \mathbf{B}(\mathbf{r}) \;=\; \frac{\mu_0}{4\pi} \iiint \mathbf{J}(\mathbf{r}') \big(\nabla_{\mathbf{r}} \cdot \mathbf{G}(\mathbf{r}-\mathbf{r}')\big) d^3\mathrm{Vol}'. \tag{25}$$

Moreover, in the integrand here

$$\nabla_{\mathbf{r}} \cdot \mathbf{G}(\mathbf{r}-\mathbf{r}') \;=\; \nabla_{\mathbf{r}}^2 \left(\frac{-1}{|\mathbf{r}-\mathbf{r}'|}\right) \;=\; 4\pi\delta^{(3)}(\mathbf{r}-\mathbf{r}'), \tag{26}$$

so the integral becomes simply

$$\iiint \mathbf{J}(\mathbf{r}') \big(\nabla_{\mathbf{r}} \cdot \mathbf{G}(\mathbf{r}-\mathbf{r}')\big) d^3\mathrm{Vol}' = \; 4\pi \iiint \mathbf{J}(\mathbf{r}')\, \delta^{(3)}(\mathbf{r}-\mathbf{r}')\, d^3\mathrm{Vol}' \;=\; 4\pi \mathbf{J}(\mathbf{r}). \tag{27}$$

Therefore,

$$\nabla \times \mathbf{B}(\mathbf{r}) \;=\; \mu_0\, \mathbf{J}(\mathbf{r}), \tag{28}$$

which is indeed the Ampere's Law (3), *quod erat demonstrandum.*