Multipole Expansion of the Electrostatic Potential

Mathematical Background

Let me start with a bit of mathematical **theorem:** Consider two points with respective radius vectors $\mathbf{R}$ and $\mathbf{r}$. Suppose the second point is closer to the origin than the first point, $r < R$. Then the inverse distance between the two points

$$\frac{1}{|\mathbf{R} - \mathbf{r}|} = \frac{1}{\sqrt{R^2 + r^2 - 2Rr\cos\theta}} \tag{1}$$

(where θ is the angle between the vectors $\mathbf{R}$ and $\mathbf{r}$) can be expanded in powers of the ratio r/R as

$$\frac{1}{|\mathbf{R} - \mathbf{r}|} = \sum_{\ell=0}^{\infty} \frac{r^\ell}{R^{\ell+1}} \times P_\ell(\cos\theta) \tag{2}$$

where $P_\ell(x)$ is the Legendre polynomial of degree ℓ. Spelling out the first few Legendre polynomials explicitly, the expansion (2) becomes

$$\frac{1}{|\mathbf{R} - \mathbf{r}|} = \frac{1}{R} \times 1 + \frac{r}{R^2} \times \cos\theta + \frac{r^2}{R^3} \times \frac{3\cos^2\theta - 1}{2} + \frac{r^3}{R^4} \times \frac{5\cos^3\theta - 3\cos\theta}{2} + \cdots. \tag{3}$$

The proof of the theorem (2) involves complex contour integration — a technique some students have not yet learned, — so I present it at as *optional* reading material at the end of this set of notes.

Meanwhile, let me simply verify the first few terms in the expansion (3) for $r \ll R$. For the sake of compactness, let's denote

$$\alpha = \frac{r}{R} \ll 1, \quad x = \cos\theta, \quad \beta = 2\alpha x - \alpha^2 \ll 1.$$

In these notations,

$$\frac{1}{\sqrt{R^2 + r^2 - 2Rr\cos\theta}} = \frac{1}{\sqrt{R^2(1 + \alpha^2 - 2\alpha x)}} = \frac{1}{R} \times \frac{1}{\sqrt{1 - \beta}}. \tag{4}$$

Next, let's expand the $1/\sqrt{1 - \beta}$ into powers of β:

$$S = \frac{1}{\sqrt{1 - \beta}} = 1 + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{2^n\, n!} \times \beta^n = 1 + \tfrac{1}{2}\beta + \tfrac{3}{8}\beta^2 + \tfrac{5}{16}\beta^3 + \tfrac{35}{128}\beta^4 + \cdots. \tag{5}$$

Now, remember that $\beta = 2\alpha x - \alpha^2$, plug that into the above expansion, open up the $(\cdots)$, and rearrange the terms according to the powers of α. For simplicity, let's stop with terms $\sim \alpha^4$

and truncate the higher powers of α, thus

$$
\begin{aligned}
S \;=\; & 1 \;+\; \tfrac{1}{2}(2\alpha x - \alpha^2) \;+\; \tfrac{3}{8}(2\alpha x - \alpha^2)^2 \;+\; \tfrac{5}{16}(2\alpha x - \alpha^2)^3 \;+\; \tfrac{35}{128}(2\alpha x - \alpha^2)^4 \qquad\qquad + \cdots \\[4pt]
=\; & 1 \;+\; \alpha x \qquad\qquad\quad -\; \tfrac{1}{2}\alpha^2 \\
& \qquad\qquad\qquad +\; \tfrac{3}{2}\alpha^2 x^2 \qquad\qquad -\; \tfrac{3}{2}\alpha^3 x \qquad\qquad +\; \tfrac{3}{8}\alpha^4 \\
+\;\cdots \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad +\; \tfrac{5}{2}\alpha^3 x^3 \qquad\qquad\quad -\; \tfrac{15}{4}\alpha^4 x^2 \qquad\qquad + \cdots \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +\; \tfrac{35}{8}\alpha^4 x^4 \qquad\qquad - \cdots \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +\; \cdots \\[4pt]
=\; & 1 \;+\; \alpha \times x \qquad\quad +\; \alpha^2 \times \frac{3x^2 - 1}{2} \;+\; \alpha^3 \times \frac{5x^3 - 3x}{2} \;+\; \alpha^4 \times \frac{35x^4 - 30x^2 + 3}{8} \;+\; \cdots \\[4pt]
=\; & 1 \;+\; \alpha \times P_1(x) \quad +\; \alpha^2 \times P_2(X) \quad +\; \alpha^3 \times P_3(x) \qquad +\; \alpha^4 \times P_4(x) \qquad\qquad\quad + \cdots
\end{aligned}
\tag{6}
$$

where $P_1(x)$, $P_2(x)$, and $P_3(x)$ are the Legendre polynomials of respective degrees 1, 2, 3. Plugging this result back into eq. (4), we obtain

$$
\begin{aligned}
\frac{1}{\sqrt{R^2 + r^2 - 2Rr\cos\theta}} \;=\; & \frac{1}{R} \;+\; \frac{r}{R^2} \times P_1(\cos\theta) \;+\; \frac{r^2}{R^3} \times P_2(\cos\theta) \\[4pt]
& +\; \frac{r^3}{R^4} \times P_3(\cos\theta) \;+\; \frac{r^4}{R^5} \times P_4(\cos\theta) \;+\; \cdots,
\end{aligned}
\tag{7}
$$

in perfect agreement with eq. (3).

Multipole Expansion of the Coulomb Potential

Now consider the Coulomb potential of some continuous charge distribution $\rho(\vec{r})$,

$$
V(\mathbf{R}) \;=\; \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(\mathbf{r})\, d^3\mathrm{Vol}}{|\mathbf{R} - \mathbf{r}|} \,.
\tag{8}
$$

Suppose all the charges are limited to some compact volume, while we want to know the potential far away from that volume, so in the integral (8) we always have $r \ll R$. Consequently, we may expand the denominator in the Coulomb potential according to the Theorem (2),

thus

$$V(\mathbf{R}) = \frac{1}{4\pi\epsilon_0} \iiint d^3\mathrm{Vol}\, \rho(\mathbf{r}) \times \sum_{\ell=0}^{\infty} \frac{r^\ell}{R^{\ell+1}} \times P_\ell(\cos\theta)$$

$$= \sum_{\ell=0}^{\infty} \frac{1}{4\pi\epsilon_0\, R^{\ell+1}} \times \iiint d^3\mathrm{Vol}\, \rho(\mathbf{r}) \times r^\ell P_\ell(\cos\theta) \tag{9}$$

where θ is the angle between the radius-vectors $\mathbf{R}$ and $\mathbf{r}$. In terms of the unit vectors $\widehat{\mathbf{R}}$ and $\widehat{\mathbf{r}}$ in the directions of $\mathbf{R}$ and $\mathbf{r}$,

$$\cos\theta = \widehat{\mathbf{R}} \cdot \widehat{\mathbf{r}}. \tag{10}$$

Consequently, we may decompose the potential $V(\mathbf{R})$ of the charges $\rho(\mathbf{r})$ into a series of multipole potentials,

$$V(\mathbf{R}) = \sum_{\ell=0}^{\infty} \frac{\mathcal{M}_\ell(\widehat{\mathbf{R}})}{4\pi\epsilon_0\, R^{\ell+1}} \tag{11}$$

where

$$\mathcal{M}_\ell(\widehat{\mathbf{R}}) = \iiint r^\ell P_\ell(\widehat{\mathbf{r}} \cdot \widehat{\mathbf{R}}) \times \rho(\mathbf{r})\, d^3\mathrm{Vol}. \tag{12}$$

is the 2^ℓ–pole moment of the charge distribution $\rho(\mathbf{r})$. Or rather, it's the *component of the 2^ℓ–pole moment in the direction* $\widehat{\mathbf{R}}$.

LET'S TAKE A CLOSER LOOK AT THESE COMPONENTS:

0 : The monopole moment is simply the net charge of the distribution. Indeed, $P_0(x) = 1$, hence $r^0 \times P_0(\widehat{\mathbf{r}} \cdot \widehat{\mathbf{R}}) = 1 \times 1 = 1$ and therefore

$$\mathcal{M}_0 = \iiint \rho(\mathbf{r})\, d^3\mathrm{Vol} = Q^{\mathrm{net}} \tag{13}$$

regardless of the direction $\widehat{\mathbf{R}}$. Consequently, the monopole term in the potential is isotropic,

$$V_{\mathrm{monopole}} = \frac{Q^{\mathrm{net}}}{4\pi\epsilon_0\, R}. \tag{14}$$

1 : The dipole moment is the vector

$$\mathbf{p} = \iiint \mathbf{r}\, \rho(\mathbf{r})\, d^3\mathrm{Vol} \tag{15}$$

and its component in the $\widehat{\mathbf{R}}$ direction is simply $\mathcal{M}_1(\widehat{\mathbf{R}}) = \widehat{\mathbf{R}} \cdot \mathbf{p}$. Indeed, $P_1(x) = x$,

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hence

$$r \times P_1(\widehat{\mathbf{R}} \cdot \widehat{\mathbf{r}}) \; = \; r \times (\widehat{\mathbf{R}} \cdot \widehat{\mathbf{r}}) \; = \; \widehat{\mathbf{R}} \cdot \mathbf{r} \tag{16}$$

and therefore

$$\mathcal{M}_1(\widehat{\mathbf{R}}) \; = \; \iiint (\widehat{\mathbf{R}} \cdot \mathbf{r})\rho(\mathbf{r}) \, d^3\mathrm{Vol} \; = \; \widehat{\mathbf{R}} \cdot \iiint \mathbf{r}\, \rho(\mathbf{r}) \, d^3\mathrm{Vol} = \; \widehat{\mathbf{R}} \cdot \mathbf{p}. \tag{17}$$

Consequently, the dipole potential has form

$$V_{\mathrm{dipole}} \; = \; \frac{\mathbf{p} \cdot \widehat{\mathbf{R}}}{4\pi\epsilon_0\, R^2} \,. \tag{18}$$

2 : The quadrupole moment is a 2-index symmetric tensor

$$\mathcal{Q}_{i,j} \; = \; \iiint \left(\tfrac{3}{2} r_i r_j \; - \; \tfrac{1}{2}\delta_{i,j} r^2 \right)\rho(\mathbf{r}) \, d^3\mathrm{Vol} \tag{19}$$

where the indices i, j run over x, y, z, the r_i are the components of the vector $\mathbf{r}$, and $\delta_{i,j}$ is the Kronecker's delta (1 for $i = j$ and 0 for $i \neq j$). The tensor (19) is symmetric WRT permutation of its two indices, $\mathcal{Q}_{i,j} = \mathcal{Q}_{j,i}$, and the component of this tensor in the direction $\widehat{\mathbf{R}}$ is simply the tensor analogue of the dot product with the unit vector $\widehat{\mathbf{R}}$,

$$\mathcal{M}_2(\widehat{\mathbf{R}}) \; = \; \sum_{i,j=x,y,z} \mathcal{Q}_{i,j}\widehat{R}_i\widehat{R}_j \,. \tag{20}$$

To see how this works, note $P_2(x) = \tfrac{3}{2}x^2 - \tfrac{1}{2}$ and hence

$$r^2 \times P_2(\widehat{\mathbf{R}} \cdot \widehat{\mathbf{r}}) \; = \; \tfrac{3}{2}r^2 \times (\widehat{\mathbf{R}} \cdot \widehat{\mathbf{r}})^2 \; - \; \tfrac{1}{2}r^2 \; = \; \tfrac{3}{2}(\widehat{\mathbf{R}} \cdot \mathbf{r})^2 \; - \; \tfrac{1}{2}r^2 \times \widehat{\mathbf{R}}^2 \tag{21}$$

$$\langle\!\langle \text{ note } \widehat{\mathbf{R}} \text{ is a unit vector so } \widehat{\mathbf{R}}^2 = 1 \rangle\!\rangle,$$

$$\text{where } (\widehat{\mathbf{R}} \cdot \mathbf{r})^2 \; = \; \left(\sum_i \widehat{R}_i r_i \right)^2 \; = \; \left(\sum_i \widehat{R}_i r_i \right)\left(\sum_j \widehat{R}_j r_j \right) \; = \; \sum_{i,j} \widehat{R}_i\widehat{R}_j\, r_i r_j \,, \tag{22}$$

$$\widehat{\mathbf{R}}^2 \; = \; \sum_i \widehat{R}_i\widehat{R}_i \; = \; \sum_{i,j} \widehat{R}_i\widehat{R}_j\, \delta_{i,j} \,, \tag{23}$$

thus

$$r^2 \times P_2(\widehat{\mathbf{R}} \cdot \widehat{\mathbf{r}}) = \tfrac{3}{2} \sum_{i,j} \widehat{R}_i \widehat{R}_j \, r_i r_j - \tfrac{1}{2} r^2 \sum_{i,j} \widehat{R}_i \widehat{R}_j \, \delta_{ij}$$
$$= \sum_{i,j} \widehat{R}_i \widehat{R}_j \times \left(\tfrac{3}{2} r_i r_j - \tfrac{1}{2} r^2 \delta_{i,j} \right) \tag{24}$$

and therefore

$$\mathcal{M}_2(\widehat{\mathbf{R}}) = \iiint r^2 \times P_2(\widehat{\mathbf{R}} \cdot \widehat{\mathbf{r}}) \times \rho(\mathbf{r}) \, d^3\mathrm{Vol}$$
$$= \iiint \sum_{i,j} \widehat{R}_i \widehat{R}_j \times \left(\tfrac{3}{2} r_i r_j - \tfrac{1}{2} r^2 \delta_{i,j} \right) \times \rho(\mathbf{r}) \, d^3\mathrm{Vol}$$
$$= \sum_{i,j} \widehat{R}_i \widehat{R}_j \times \iiint \left(\tfrac{3}{2} r_i r_j - \tfrac{1}{2} r^2 \delta_{i,j} \right) \times \rho(\mathbf{r}) \, d^3\mathrm{Vol} \tag{25}$$
$$= \sum_{i,j} \widehat{R}_i \widehat{R}_j \times \mathcal{Q}_{i,j} \,.$$

In terms of the quadrupole moment tensor, the quadrupole potential is

$$V_{\text{quadrupole}}(\mathbf{R}) = \frac{\sum_{i,j} \mathcal{Q}_{i,j} \widehat{R}_i \widehat{R}_j}{4\pi\epsilon_0 \, R^3} \,. \tag{26}$$

3: The octupole moment is a 3-index tensor

$$\mathcal{O}_{i,j,k} = \iiint \left(\tfrac{5}{2} r_i r_j r_k - \tfrac{1}{2} r^2 \left(\delta_{i,j} r_k + \delta_{i,k} r_j + \delta_{j,k} r_i \right) \right) \times \rho(\mathbf{r}) \, d^3\mathrm{Vol} \tag{27}$$

which is totally symmetric WRT any permutations of its induces,

$$\mathcal{O}_{i,j,k} = \mathcal{O}_{j,i,k} = \mathcal{O}_{k,j,i} = \mathcal{Q}_{i,k,j} = \mathcal{O}_{j,k,i} = \mathcal{O}_{k,i,j} \,. \tag{28}$$

Again, its component in the direction $\widehat{\mathbf{R}}$ is simply

$$\mathcal{M}_3(\widehat{\mathbf{R}}) = \sum_{i,j,k} \mathcal{O}_{i,j,k} \widehat{R}_i \widehat{R}_j \widehat{R}_k \,, \tag{29}$$

so the octupole potential is

$$V_{\text{octupole}}(\mathbf{R}) = \frac{\sum_{i,j,k} \mathcal{O}_{i,j,k} \widehat{R}_i \widehat{R}_j \widehat{R}_k}{4\pi\epsilon_0 \, R^4} \,. \tag{30}$$

The proof of eq. (29) and hence (30) follows from the expansion

$$r^3 \times P_3(\widehat{\mathbf{R}} \cdot \widehat{\mathbf{r}}) = \sum_{i,j,k} \widehat{R}_i \widehat{R}_j \widehat{R}_k \times \left(\tfrac{5}{2} r_i r_j r_k - \tfrac{1}{2} r^2 \left(\delta_{i,j} r_k + \delta_{i,k} r_j + \delta_{j,k} r_i \right) \right), \qquad (31)$$

but this time I am skipping the algebra.

$\underline{\ell > 3}$: Similar to the quadrupole and the octupole moments, for larger ℓ the 2^ℓ–pole moment is a ℓ-index tensor totally symmetric in all its indices of general form

$$\mathcal{M}^{(\ell)}_{i,j,\ldots,n} = \iiint \left(\begin{array}{c} \text{homogeneous polynomial} \\ \text{of degree } \ell \text{ in } x, y, z \end{array} \right)_{i,j,\ldots,n} \times \rho(\mathbf{r}) \, d^3\text{Vol} \qquad (32)$$

where the polynomial follows from the expansion of

$$r^\ell \times P_\ell(\widehat{\mathbf{R}} \cdot \widehat{\mathbf{r}}) = \sum_{i,j,\ldots,n} \widehat{R}_i \widehat{R}_j \cdots \widehat{R}_n \times \left(\begin{array}{c} \text{homogeneous polynomial} \\ \text{of degree } \ell \text{ in } x, y, z \end{array} \right)_{i,j,\ldots,n}. \qquad (33)$$

The potential due to the 2^ℓ–pole moment is

$$V_{2^\ell-\text{pole}}(\mathbf{R}) = \frac{\sum_{i,j,\ldots,n} \mathcal{M}^{(\ell)}_{i,j,\ldots,n} \widehat{R}_i \widehat{R}_j \cdots \widehat{R}_n}{4\pi\epsilon_0 \, R^{\ell+1}} \qquad (34)$$

Spherical Harmonic Expansion

In general, the 2^ℓ–pole tensor $\mathcal{M}^{(\ell)}_{i,j,\ldots,n}$ has 3^ℓ components, but many components are related to each other by the total symmetry condition as well as by specific features of the polynomials (33). For example, the quadrupole tensor $\mathcal{Q}_{i,j}$ has 9 components, but they are related by 3 symmetry conditions $\mathcal{Q}_{i,j} = \mathcal{Q}_{j,i}$ and 1 trace condition $\sum_i \mathcal{Q}_{i,i} = 0$, so there are only 5 independent components. In general, for each ℓ the 2^ℓ–pole tensor has only $2\ell+1$ independent components.

Likewise, for each ℓ there are $2\ell+1$ independent spherical harmonics $Y_{\ell,m}(\theta, \phi)$, and this is no coincidence. Indeed, instead of describing the angular dependence of the multipoles' components in the direction $\widehat{\mathbf{R}}$ in terms of symmetric multipole tensors, we may expand it in terms of spherical harmonics. The key to this expansion is the following

Lemma: for any integer $\ell = 0, 1, 2, 3, \ldots$ and any two unit vectors $\widehat{\mathbf{a}}$ and $\widehat{\mathbf{b}}$, the Legendre polynomial of their dot product (*i.e.*, of the cosine of the angle between these vectors) expands into products of spherical harmonics according to

$$P_\ell(\widehat{\mathbf{a}} \cdot \widehat{\mathbf{b}}) \;=\; \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{+\ell} Y_{\ell,m}(\widehat{\mathbf{a}}) Y_{\ell,m}^*(\widehat{\mathbf{b}}). \tag{35}$$

Proving this lemma is best done in the quantum-mechanical language of Dirac brackets and projection operators. Since some students may be unfamiliar with this language, the proof is postponed to the end of these notes as *optional reading*.

Meanwhile, let's apply the Lemma to the vectors $\widehat{\mathbf{R}}$ and $\widehat{\mathbf{r}}$ in the context of eq. (12):

$$
\begin{aligned}
\mathcal{M}_\ell(\widehat{\mathbf{R}}) &= \iiint r^\ell \times P_\ell(\widehat{\mathbf{R}} \cdot \widehat{\mathbf{r}}) \times \rho(\mathbf{r}) \, d^3\text{Vol} \\[2mm]
&= \iiint r^\ell \times \frac{4\pi}{2\ell+1} \left(\sum_{m=-\ell}^{+\ell} Y_{\ell,m}(\widehat{\mathbf{R}}) Y_{\ell,m}^*(\widehat{\mathbf{r}}) \right) \times \rho(\mathbf{r}) \, d^3\text{Vol} \\[2mm]
&= \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{+\ell} Y_{\ell,m}(\widehat{\mathbf{R}}) \times \iiint r^\ell \times Y_{\ell,m}^*(\widehat{\mathbf{r}}) \times \rho(\mathbf{r}) \, d^3\text{Vol}.
\end{aligned}
\tag{36}
$$

Hence, let's define the *spherical harmonics of multipoles* according to

$$\mathcal{M}_{\ell,m} \;\overset{\text{def}}{=}\; \frac{4\pi}{2\ell+1} \iiint r^\ell \times Y_{\ell,m}^*(\theta, \phi) \times \rho(r, \theta, \phi) \, d^3\text{Vol}. \tag{37}$$

Then, the 2^ℓ–pole potential has form

$$V_{2^\ell\text{--pole}}(R, \Theta, \Phi) \;=\; \frac{1}{4\pi\epsilon_0} \sum_{m=-\ell}^{+\ell} \mathcal{M}_{\ell,m} \times \frac{Y_{\ell,m}(\Theta, \Phi)}{R^{\ell+1}} \tag{38}$$

and the entire potential expands into

$$V_{\text{net}}(R, \Theta, \Phi) \;=\; \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} \mathcal{M}_{\ell,m} \times \frac{Y_{\ell,m}(\Theta, \Phi)}{R^{\ell+1}} \,. \tag{39}$$

Axial Symmetry

For the axially symmetric charge distributions $\rho(r, \theta, \phi) = \rho(r, \theta$ only), expanding the electric multipoles into spherical harmonics becomes particularly simple: for each ℓ, only the $m = 0$ harmonic may have a non-zero coefficient $\mathcal{M}_{\ell,0} \neq 0$; all the other $\mathcal{M}_{\ell,m}$ with $m \neq 0$ must vanish. Indeed, for the axially symmetric charges, the integral (37) becomes

$$\mathcal{M}_{\ell,m} = \frac{4\pi}{2\ell+1} \int\limits_0^\infty dr\, r^2 \times \int\limits_0^\pi d\theta\, \sin\theta \times r^\ell \rho(r,\theta) \times \int\limits_0^{2\pi} d\phi\, Y^*_{\ell,m}(\theta,\phi), \tag{40}$$

and since $Y_{\ell,m}(\theta, \phi) = e^{im\phi} \times$ a function of θ, the ϕ integral vanishes for $m \neq 0$,

$$\int\limits_0^{2\pi} d\phi\, Y^*_{\ell,m}(\theta,\phi) = 0 \quad \text{for } m \neq 0. \tag{41}$$

For the remaining $m = 0$ components, the spherical harmonics $Y_{\ell,0}(\theta, \phi)$ are proportional to the Legendre polynomials,

$$Y_{\ell,0}(\theta, \phi) = \sqrt{\frac{2\ell+1}{4\pi}} \times P_\ell(\cos\theta), \tag{42}$$

so the multipole expansion (39) becomes

$$\begin{aligned}
V_{\text{net}}(R, \Theta, \Phi) &= \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^\infty \sqrt{\frac{2\ell+1}{4\pi}} \mathcal{M}_{\ell,0} \times \frac{P_\ell(\cos\Theta)}{R^{\ell+1}} \\
&= \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^\infty \mathcal{M}_{\text{axial}}^{(\ell)} \times \frac{P_\ell(\cos\Theta)}{R^{\ell+1}}
\end{aligned} \tag{43}$$

where

$$\begin{aligned}
\mathcal{M}_{\text{axial}}^{(\ell)} &= \sqrt{\frac{2\ell+1}{4\pi}} \times \mathcal{M}_{\ell,0} = \sqrt{\frac{4\pi}{2\ell+1}} \iiint Y_{\ell,0}(\theta)\rho(r,\theta)\, d^3\text{Vol} \\
&= \iiint r^\ell P_\ell(\cos\theta) \times \rho(r,\theta)\, d^3\text{Vol}.
\end{aligned} \tag{44}$$

In terms of the multipole tensors $\mathcal{M}^{(\ell)}_{i,j,\ldots,n}$,

$$\mathcal{M}_{\text{axial}}^{(\ell)} = \mathcal{M}^{(\ell)}_{z,z,\ldots,z}. \tag{45}$$

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Appendix: Proving the Multipole Expansion Theorem

In this Appendix I prove the theorem (2). That is, I show that the series on the LHS of

$$\sum_{\ell=0}^{\infty} \frac{r^{\ell}}{R^{\ell+1}} \times P_{\ell}(\cos\theta) \;=\; \frac{1}{\sqrt{R^2 - 2Rr\cos\theta + r^2}} \;=\; \frac{1}{|\mathbf{R} - \mathbf{r}|}. \tag{46}$$

converges for any $r < R$ and that the sum is precisely the expression on the RHS.

This proof is optional reading for the students in my ElectroDynamics class, as it involves complex analysis techniques some students might have not learned yet. (But if you have not, please take a complex analysis class before you graduate, all physicists should be familiar with it.) Specifically, I use the contour integrals in the complex plane and the residue method for taking such integrals:

$$\oint_{\Gamma} \frac{dz}{2\pi i} \frac{f(z)}{(z-x)^{n+1}} \;=\; \text{Residue}\left[\frac{f(z)}{(z-x)^{n+1}}\right]_{@z=x} \;=\; \frac{1}{n!} \left.\frac{d^n f(z)}{dz^n}\right|_{@z=x} \tag{47}$$

provided the contour Γ circles x and that the function $f(x)$ is analytic and has no singularities inside the contour Γ.

My starting point is the Rodriguez formula for the Legendre polynomials,

$$P_{\ell}(x) \;=\; \frac{1}{2^{\ell}\,\ell!} \frac{d^{\ell}}{dx^{\ell}} (x^2 - 1)^{\ell}. \tag{48}$$

In light of the residue-method formula (47), we may turn the ℓth derivative in this formula into a complex contour integral

$$P_{\ell}(x) \;=\; \frac{1}{2^{\ell}} \oint_{\Gamma} \frac{dz}{2\pi i} \frac{(z^2 - 1)^{\ell}}{(z-x)^{\ell+1}} \tag{49}$$

where Γ is some closed contour which circles x. Now let's plug this formula into the series

on the LHS of eq. (46):

$$
\begin{aligned}
\text{series} \ &= \ \sum_{\ell=0}^{\infty} \frac{r^{\ell}}{R^{\ell+1}} \times P_{\ell}(x) \\
&= \ \sum_{\ell=0}^{\infty} \frac{r^{\ell}}{R^{\ell+1}} \times \frac{1}{2^{\ell}} \oint_{\Gamma} \frac{dz}{2\pi i} \frac{(z^2-1)^{\ell}}{(z-x)^{\ell+1}} \\
&\quad \langle\!\langle\, \text{putting the sum inside the integral} \,\rangle\!\rangle \\
&= \ \oint_{\Gamma} \frac{dz}{2\pi i} \sum_{\ell=0}^{\infty} \frac{r^{\ell}}{R^{\ell+1}} \times \frac{(z^2-1)^{\ell}}{2^{\ell}(z-x)^{\ell+1}} \\
&= \ \oint_{\Gamma} \frac{dz}{2\pi i} \frac{1}{R(z-x)} \times \sum_{\ell=0}^{\infty} \left(\frac{r(z^2-1)}{2R(z-t)} \right)^{\ell} \\
&= \ \oint_{\Gamma} \frac{dz}{2\pi i} \frac{1}{R(z-x)} \times \frac{1}{1 - \frac{r(z^2-1)}{2R(z-x)}} \\
&= \ \oint_{\Gamma} \frac{dz}{2\pi i} \frac{-2}{rz^2 - 2Rz + 2Rx - r} \ .
\end{aligned}
\tag{50}
$$

Note: before the summation, each term on the third line has poles at $z = x$ and at $z = \infty$, but after the summation, both poles have moved to the roots of the quadratic equation

$$
rz^2 - 2Rz + 2Rx - r \ = \ 0,
\tag{51}
$$

thus

$$
z_{1,2} \ = \ \frac{R \pm \sqrt{R^2 - 2rRx + r^2}}{r} \ ; \qquad \text{for } r \ll R, \quad z_1 \approx \frac{2R}{r} \to \infty, \quad \text{while} \quad z_2 \approx x. \tag{52}
$$

This tells us how to choose the integration contour Γ: It should circle around x and have enough room to accommodate the shifting of the pole from x to z_2, but it should not include the other pole at z_1 which have moved in from the infinity. Consequently, evaluating the integral on the bottom line of eq. (50) by the residue method, we have

$$
\oint_{\Gamma} \frac{dz}{2\pi i} \frac{-2}{rz^2 - 2Rz + 2Rx - r} \ = \ \text{Residue}\left[\frac{-2}{rz^2 - 2Rz + 2Rx - r} \right]_{@z=z_2} \ . \tag{53}
$$

Specifically,

$$\frac{-2}{rz^2 - 2Rz + 2Rx - r} = \frac{-2}{r} \times \frac{1}{(z - z_1)(z - z_2)},$$ (54)

so the residue of this function at $z = z_2$ is simply

$$\text{Residue} = \frac{-2}{r} \times \frac{1}{z_2 - z_1} = \frac{-2}{r} \times \frac{r}{-2\sqrt{R^2 - 2rRx + r^2}} = +\frac{1}{\sqrt{R^2 - 2rRx + r^2}}.$$ (55)

Thus,

$$\text{the series} = \sum_{\ell=0}^{\infty} \frac{r^\ell}{R^{\ell+1}} \times P_\ell(x) = \frac{1}{\sqrt{R^2 - 2rRx + r^2}},$$ (56)

exactly as on the RHS of eq. (46).

To complete the proof, consider the convergence of the multipole expansion (46). For any physical angle θ ranging between 0 and π, the $x = \cos\theta$ ranges between $+1$ and -1, and for all such x, all the Legendre polynomials $P_\ell(x)$ take values between -1 and $+1$. Consequently,

$$\left| \begin{matrix} \ell^{\text{th}} \text{ term in the} \\ \text{multipole expansion} \end{matrix} \right| < \frac{r^\ell}{R^{\ell+1}} = \frac{(r/R)^\ell}{R},$$ (57)

so the series on the LHS of eq. (46) converges for any $r < R$.

Moreover, if we analytically continue the series to complex r, it would converge for all $|r| < R$; in other words, it has *radius of convergence* $= R$. Indeed, as a function of complex r, the $1/\sqrt{\cdots}$ on the RHS of (46) has singularities at

$$r_{1,2} = R\cos\theta \pm iR\sin\theta, \qquad |r_{1,2}| = R,$$

and that's what sets the radius of convergence to $|r| < R$.

For $r > R$ we may no longer expand the inverse distance into powers of r/R. Instead, we may expand it into powers of the inverse ratio R/r:

$$\text{For } r > R, \quad \frac{1}{\sqrt{R^2 + r^2 - 2Rr\cos\theta}} = \sum_{\ell=0}^{\infty} \frac{R^\ell}{r^{\ell+1}} \times P_\ell(\cos\theta),$$ (58)

which works exactly like eq. (2) once we exchange $r \leftrightarrow R$.

Physically, the expansion (2) is useful for potentials far outside complicated charged bodies, while the inverse expansion (58) is useful for potentials deep inside a cavity.

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Appendix: Proving the Lemma about Spherical Harmonics

This Appendix is *optional reading* for the students who want to know how to prove the lemma (35). My proof has 2 steps: First, I shall prove that the sum

$$\sum_{m=-\ell}^{+\ell} Y_{\ell,m}(\theta_a, \phi_a) Y_{\ell,m}^*(\theta_b, \phi_b) \tag{59}$$

depends only on the angle Θ_{ab} *between* the directions (θ_a, ϕ_a) and (θ_b, ϕ_b) but remains invariant under simultaneous rotations of both directions. Second, I'll use this invariance to evaluate the sum and to show that it agrees with eq. (35).

The first step is best described in the quantum mechanical language of Dirac brackets and operators. Specifically, consider a quantum particle living in 2 curved dimensions, specifically on a sphere of some fixed radius $r = $ const. The position a of such a particle can be described by two spherical angles (θ_a, ϕ_a), or equivalently by a unit vector $\mathbf{a}$ pointing towards the particle from the sphere's center. Consequently, the quantum states of such a particle are described by wave-functions $\psi(\mathbf{a}) = \psi(\theta_a, \phi_a)$.

Now consider the Hilbert space of such wave-functions. The spherical harmonics provide a complete orthonormal basis for this Hilbert space:

$$\begin{aligned}
\text{any } \psi(\theta, \phi) &= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} \langle \ell, m | \psi \rangle \times Y_{\ell,m}(\theta, \phi) \\
\text{for } \langle \ell, m | \psi \rangle &= \iint Y_{\ell,m}^*(\theta, \phi) \psi(\theta, \phi) \, d^2\Omega(\theta, \phi).
\end{aligned} \tag{60}$$

Physically, the states $|\ell, m\rangle$ are eigenstates of the angular momentum operators $\hat{\mathbf{L}}^2$ and $\hat{L}_z$,

$$\hat{L}_z |\ell, m\rangle = \hbar m |\ell, m\rangle, \quad \hat{\mathbf{L}}^2 |\ell, m\rangle = \hbar^2 \ell(\ell+1) |\ell, m\rangle. \tag{61}$$

Consequently, the operator

$$\hat{\Pi}_\ell = \sum_{m=-\ell}^{+\ell} |\ell, m\rangle \langle \ell, m| \tag{62}$$

is the projector operator on all states with definite $\mathbf{L}^2$, namely $\hbar^2 \ell(\ell+1)$.

Since the $\hat{\mathbf{L}}^2$ operator is invariant under all 3D rotations of the sphere about its center, the projection operators $\hat{\Pi}_\ell$ must also be invariant under rotations. Consequently, for any two definite-positions states $|\mathbf{a}\rangle = |\theta_a, \phi_a\rangle$ and $|\mathbf{b}\rangle = |\theta_b, \phi_b\rangle$, the Dirac sandwich

$$\langle \mathbf{a}| \hat{\Pi}_\ell |\mathbf{b}\rangle \tag{63}$$

must be invariant under *similtaneous* rotations of the unit vectors $\mathbf{a}$ and $\mathbf{b}$. Therefore, this Dirac sandwich may depend only on the *relative* angle Θ_{ab} between the unit vectors $\mathbf{a}$ and $\mathbf{b}$, but it cannot depend on where the vectors $\mathbf{a}$ and $\mathbf{b}$ point in absolute terms, thus

$$\langle \mathbf{a}| \hat{\Pi}_\ell |\mathbf{b}\rangle = F_\ell(\Theta_{ab} \text{ only}), \tag{64}$$

On the other hand, by construction of the operator (62),

$$\langle \mathbf{a}| \hat{\Pi}_\ell |\mathbf{b}\rangle = \sum_{m=-\ell}^{+\ell} \langle \mathbf{a}|\ell, m\rangle \langle \ell, m|\mathbf{b}\rangle = \sum_{m=-\ell}^{+\ell} Y_{\ell,m}(\theta_a, \phi_a) \times Y_{\ell,m}^*(\theta_b, \phi_b), \tag{65}$$

hence

$$\sum_{m=-\ell}^{+\ell} Y_{\ell,m}(\theta_a, \phi_a) \times Y_{\ell,m}^*(\theta_b, \phi_b) = F_\ell(\Theta_{ab} \text{ only}). \tag{66}$$

This completes the first step of the proof.

The second step is to find the specific form of the functions $F_\ell(\Theta_{ab})$. To do that, let's evaluate the sums (66) for a particularly simple choice of point $\mathbf{b}$, namely be the North pole of the sphere, $\theta_b = 0$, ϕ_b undefined. Meanwhile, the point $\mathbf{a}$ can be anywhere on the sphere. For our choice of point $\mathbf{b}$, the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$ is simply the latitude of $\mathbf{a}$, $\Theta_{ab} = \theta_a$, so we may evaluate the $F_\ell(\Theta_{ab})$ as

$$F_\ell(\Theta_{ab} = \theta_a) = \sum_{m=-\ell}^{+\ell} Y_{\ell,m}(\theta_a, \phi_a) Y_{\ell,m}^*(\theta_b = 0). \tag{67}$$

The spherical harmonics $Y_{\ell,m}(\theta, \phi)$ have general form

$$Y_{\ell,m}(\theta, \phi) = e^{im\phi} \times (\sin\theta)^{|m|} \times \text{Polynomial}(\cos\theta), \tag{68}$$

so thanks to the $(\sin\theta)^{|m|}$ factor, all the harmonics with $m \neq 0$ vanish at the poles. Conse-

quently, only the $\ell = 0$ term contributes to the sum (67), thus

$$F(\Theta_{ab} = \theta_a) \;=\; Y_{\ell,0}(\theta_a) \times Y_{\ell,0}^{*}(\theta_b = 0) \;=\; \frac{2\ell+1}{4\pi} \times P_\ell(\cos\theta_a) \times P_\ell(1) \qquad (69)$$

where the second equality follows from the relation (42) of the $Y_{\ell,0}$ harmonics to the Legendre polynomials. Moreover, the Legendre polynomials are normalized so that $P_\ell(1) = 1$ for all ℓ, hence

$$F(\Theta_{ab} = \theta_a) \;=\; \frac{2\ell+1}{4\pi} \times P_\ell(\cos\theta_a). \qquad (70)$$

This completes the second step of the proof.

Altogether, we have

$$\sum_{m=-\ell}^{+\ell} Y_{\ell,m}(\theta_a, \phi_a) \times Y_{\ell,m}^{*}(\theta_b, \phi_b) \;=\; \frac{2\ell+1}{4\pi} \times P_\ell(\cos\Theta_{ab}), \qquad (71)$$

quod erat demonstrandum.