Vector Potential for the Magnetic Field

Let me start with two two theorems of Vector Calculus:

Theorem 1: If a vector field has zero curl *everywhere in space,* then that field is a gradient of some scalar field.

Theorem 2: If a vector field has zero divergence *everywhere in space,* then that field is a curl of some other vector field.

The first theorem allows us to introduce the scalar potential for the static electric field,

$$\nabla\times\mathbf{E}(x,y,z)\;=\;0\quad\forall x,y,z\quad\Longrightarrow\quad\mathbf{E}(x,y,z)\;=\;-\nabla V(x,y,z)\text{ for some }V(x,y,z),\tag{1}$$

while the second theorem allows us to introduce the vector potential for the magnetic field,

$$\nabla\cdot\mathbf{B}(x,y,z)\;=\;0\quad\forall x,y,z\quad\Longrightarrow\quad\mathbf{B}(x,y,z)\;=\;\nabla\times\mathbf{A}(x,y,z)\text{ for some }\mathbf{A}(x,y,z).\tag{2}$$

The potentials (1) and (2) have many uses. In particular, they are needed for the Lagrangian or Hamiltonian description of a charged particle's motion in classical mechanics,

$$L(\mathbf{r},\mathbf{v})\;=\;\frac{m}{2}\mathbf{v}^2\;-\;qV(\mathbf{r})\;+\;q\mathbf{v}\cdot\mathbf{A}(\mathbf{r}),\tag{3}$$

$$H(\mathbf{r},\mathbf{p})\;=\;\frac{1}{2m}\big(\mathbf{p}\;-\;q\mathbf{A}(\mathbf{r})\big)^2\;+\;qV(\mathbf{r}),\tag{4}$$

or in quantum mechanics,

$$\widehat{H}\;=\;\frac{1}{2m}\big(\widehat{\mathbf{p}}\;-\;\mathbf{A}(\widehat{\mathbf{r}})\big)\;+\;qV(\widehat{\mathbf{r}}).\tag{5}$$

I shall explain these issues — as well as Aharonov–Bohm effect and Dirac's monopoles — in a couple of exta lectures, one next week, and one after the Thanksgiving break; see also my notes "Dynamics of a Charged Particle and Gauge Transforms" and "Aharonov–Bohm Effect and Dirac Monopoles". But for the current set of notes, I would like to focus on using the vector potential $\mathbf{A}(x,y,z)$ to calculate the magentic field.

Let me start with some general properties of the vector potential. While the electrostatic field $\mathbf{E}(\mathbf{r})$ determines the scalar potential $V(\mathbf{r})$ up to an overall constant term, the magnetic field $\mathbf{B}(\mathbf{r})$ determines the vector potential $\mathbf{A}(\mathbf{r})$ only up to a gradient of an arbitrary scalar field $\Lambda(x, y, z)$. Indeed, the vector potentials $\mathbf{A}(x, y, z)$ and

$$\mathbf{A}'(x, y, z) = \mathbf{A}(x, y, z) + \nabla\Lambda(x, y, z) \tag{6}$$

have the same curl everywhere, so they correspond to the same magnetic field,

$$\mathbf{B}'(x, y, z) = \nabla \times \mathbf{A}'(x, y, z) = \nabla \times \mathbf{A}(x, y, z) + \nabla \times \nabla\Lambda(x, y, z) = \mathbf{B}(x, y, z) + 0. \tag{7}$$

The relations (6) between different vector potentials for the same magnetic field are called the *gauge transforms.*

Despite ambiguity of the vector potential itself, some of its properties are *gauge invariant, i.e.*, the same for all potentials related by gauge transforms. For example, for any closed loop $\mathcal{L}$, the integral

$$\oint_{\mathcal{L}} \mathbf{A} \cdot d\vec{\ell} \tag{8}$$

is gauge invariant; indeed,

$$\oint_{\mathcal{L}} \mathbf{A}'(\mathbf{r}) \cdot d\mathbf{r} - \oint_{\mathcal{L}} \mathbf{A}(\mathbf{r}) \cdot d\mathbf{r} = \oint_{\mathcal{L}} \nabla\Lambda(\mathbf{r}) \cdot d\mathbf{r} = \oint_{\mathcal{L}} d\Lambda(\mathbf{r}) = 0. \tag{9}$$

Physically, the integral (8) in the magnetic flux through the loop $\mathcal{L}$. Indeed, take any surface $\mathcal{S}$ spanning the loop $\mathcal{L}$; by the Stokes' theorem,

$$\Phi_B[\text{through } \mathcal{S}] = \iint_{\mathcal{S}} \mathbf{B} \cdot \mathbf{d^2A} = \iint_{\mathcal{S}} (\nabla \times \mathbf{A}) \cdot \mathbf{d^2A} = \oint_{\mathcal{L}} \mathbf{A} \cdot d\vec{\ell}. \tag{10}$$

We may use eq. (10) to easily find the vector potential for magnetic field which have some symmetries. For example, consider the uniform magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$ inside a long solenoid. By the rotational and translational symmetries of the solenoid, we expect

$$\mathbf{A}(s,\phi,z) \;=\; A(s)\hat{\boldsymbol{\phi}}, \tag{11}$$

while the magnitude $A(s)$ follows from eq. (10): Take a circle of radius $s < R_{\rm solenoid}$, then

$$\oint\limits_{\rm circle} \mathbf{A}\cdot d\vec{\ell} \;=\; A(s)\times 2\pi s, \tag{12}$$

while the magnetic flux through that circle is

$$\Phi_B[\text{circle}] \;=\; B\times\pi s^2, \tag{13}$$

hence

$$A(s) \;=\; \frac{B\times\pi s^2}{2\pi s} \;=\; \tfrac{1}{2}Bs. \tag{14}$$

In Cartesian coordinates, the vector potential becomes

$$\mathbf{A} \;=\; \tfrac{1}{2}Bs\hat{\boldsymbol{\phi}} \;=\; \tfrac{1}{2}B(x\hat{\mathbf{y}} \;-\; y\hat{\mathbf{x}}), \tag{15}$$

which makes it easy to verify

$$\nabla\times\mathbf{A} \;=\; \tfrac{1}{2}B(\hat{\mathbf{x}}\times\hat{\mathbf{y}}) \;-\; \tfrac{1}{2}B(\hat{\mathbf{y}}\times\hat{\mathbf{x}}) \;=\; B\hat{\mathbf{z}} \;=\; \mathbf{B}. \tag{16}$$

Eq. (15) gives the vector potential inside the long solenoid. Outside the solenoid, the magnetic field is negligible, but the flux through a circle of radius $s > R_{\rm solenoid}$ is non-zero due to the flux inside the solenoid. Thus,

$$\Phi_B[\text{circle}] \;=\; B\times\pi R^2 \tag{17}$$

and hence

$$2\pi s\times A(s) \;=\; \Phi_B \;=\; \pi R^2 B \quad\Longrightarrow\quad A(s) \;=\; \frac{BR^2}{2s}\,. \tag{18}$$

In vector notations,

$$\mathbf{A} \;=\; \frac{BR^2}{2}\,\frac{\hat{\boldsymbol{\phi}}}{s} \;=\; \frac{BR^2}{2}\,\frac{x\hat{\mathbf{y}}\,-\,y\hat{\mathbf{x}}}{x^2+y^2} \;=\; \frac{BR^2}{2}\,\nabla\phi, \tag{19}$$

which agrees with zero magnetic field outside the solenoid,

$$\mathbf{B} \;=\; \nabla\times\mathbf{A} \;=\; \frac{BR^2}{2}\,\nabla\times\nabla\phi \;=\; 0. \tag{20}$$

Equations for the Vector Potential

A static magnetic field of steady currents obeys equations

$$\nabla\cdot\mathbf{B} \;=\; 0, \tag{21}$$
$$\nabla\times\mathbf{B} \;=\; \mu_0\mathbf{J}. \tag{22}$$

In terms of the vector potential $\mathbf{A}(x,y,z)$, the zero-divergence equation (21) is automatic: any $\mathbf{B}=\nabla\times\mathbf{A}$ has zero divergence. On the other hand, the Ampere Law (22) becomes a second-order differential equation

$$\mu_0\mathbf{J} \;=\; \nabla\times(\nabla\times\mathbf{A}) \;=\; \nabla(\nabla\cdot\mathbf{A}) \;-\; \nabla^2\mathbf{A}. \tag{23}$$

Moreover, for any solution $\mathbf{A}(x,y,z)$ of this equation for any given cuurent density $\mathbf{J}(x,y,z)$, there is a whole family of other solutions related to each other by the gauge transforms

$$\mathbf{A}'(x,y,z) \;=\; \mathbf{A}(x,y,z) \;+\; \nabla\Lambda(x,y,z), \quad \text{any } \Lambda(x,y,z). \tag{6}$$

To avoid this redundancy, it is often convenient to impose an extra *gauge-fixing condition* on the vector potential besides $\nabla\times\mathbf{A}=\mathbf{B}$. In magnetostatics, the most commonly used condition is the *transverse gauge* $\nabla \cdot \mathbf{A} = 0$. Note that any vector potential can be gauge-transformed to a potential which obeys the transversality condition. Indeed, suppose $\nabla \cdot \mathbf{A}_0 \neq 0$, then for

$$\Lambda(\mathbf{r}) \;=\; \frac{1}{4\pi} \iiint \frac{(\nabla \cdot \mathbf{A}_0)(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, d^3\mathrm{Vol}' \tag{24}$$

we have

$$\nabla^2 \Lambda(\mathbf{r}) \;=\; -\nabla \cdot \mathbf{A}_0(\mathbf{r}) \tag{25}$$

and therefore $\mathbf{A} = \mathbf{A}_0 + \nabla\Lambda$ — which is gauge-equivalent to the $\mathbf{A}_0$ — has zero divergence,

$$\nabla \cdot \mathbf{A} \;=\; \nabla \cdot \mathbf{A}_0 \;+\; \nabla^2 \Lambda \;=\; 0. \tag{26}$$

In the transverse gauge, $\nabla \times \mathbf{B}$ becomes simply the (minus) Laplacian of the vector potential,

$$\nabla \times \mathbf{B} \;=\; \nabla \times (\nabla \times \mathbf{A}) \;=\; \nabla(\nabla \cdot \mathbf{A}) \;-\; \nabla^2 \mathbf{A} \;\longrightarrow\; -\nabla^2 \mathbf{A}, \tag{27}$$

so the Ampere Law equation (23) becomes the *Poisson equation for the vector potential*,

$$\nabla^2\, \mathbf{A}(x, y, z) \;=\; -\mu_0\, \mathbf{J}(x, y, z). \tag{28}$$

Component by component, it looks exactly like the Poisson equation for the scalar potential of the electrostatics,

$$\nabla^2 V(x, y, z) \;=\; \epsilon_0^{-1}\, \rho(x, y, z), \tag{29}$$

so its solution has a similar Coulomb-like form

$$\mathbf{A}(\mathbf{r}) \;=\; \frac{\mu_0}{4\pi} \iiint \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, dx'\, dy'\, dz'. \tag{30}$$

As written, this formula is for the volume current $\mathbf{J}(x, y, z)$; for a surface current $\mathbf{K}$, it becomes

$$\mathbf{A}(\mathbf{r}) \;=\; \frac{\mu_0}{4\pi} \iint\limits_{\text{surface}} \frac{\mathbf{K}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, d^2A, \tag{31}$$

while for a current in a thin wire we have

$$\mathbf{A}(\mathbf{r}) \;=\; \frac{\mu_0}{4\pi} \int\limits_{\text{wire}} \frac{I\, d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\,. \tag{32}$$

These Coulomb-like equations for the vector potential lead to the appropriate Biot–Savart–Laplace equations for the magnetic field $\mathbf{B}(x,y,z)$ by simply taking the curl of both sides. For example, for the volume current $\mathbf{J}(\mathbf{r}')$,

$$\begin{aligned}\mathbf{B}(\mathbf{r}) \;&=\; \nabla\times\mathbf{A}[\text{from eq. (30)}] \;=\; \nabla\times\left(\frac{\mu_0}{4\pi}\iiint \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, d^3\mathrm{Vol}'\right)\\ &=\; \frac{\mu_0}{4\pi}\iiint \nabla_{\mathbf{r}}\times\left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right) d^3\mathrm{Vol}' \;=\; \frac{\mu_0}{4\pi}\iiint \left(\nabla_{\mathbf{r}}\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\times\mathbf{J}(\mathbf{r}')\, d^3\mathrm{Vol}'\\ &=\; \frac{\mu_0}{4\pi}\iiint \frac{-(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\times\mathbf{J}(\mathbf{r}')\, d^3\mathrm{Vol}' = \frac{\mu_0}{4\pi}\iiint \mathbf{J}(\mathbf{r}')\times\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\, d^3\mathrm{Vol}'.\end{aligned} \tag{33}$$

However, for practical calculations of the magnetic field, it is often easier to first evaluate the Coulomb-like integrals (30)–(32) for the vector potential and then take its curl, instead of directly evaluating the appropriate Biot–Savart–Laplace integral.

Example: Rotating Charged Sphere

Consider a uniformly charged spherical shell of radius R and charge density σ. Let's make this sphere spin around its axis with angular velocity $\boldsymbol{\omega}$. Consequently, a point P on this sphere with radius-vector $\mathbf{r}'$ (counted from the sphere's center) moves with linear velocity $\mathbf{v}=\boldsymbol{\omega}\times\mathbf{r}'$, which makes for the *surface current density*

$$\mathbf{K}(\mathbf{r}') \;=\; \sigma\mathbf{v} \;=\; \sigma\,\boldsymbol{\omega}\times\mathbf{r}'. \tag{34}$$

Let's find the magnetic field of this current, both inside and outside the sphere.

Instead of using the Biot–Savart–Laplace equation, let's start by calculating the vector potential from eq. (31):

$$\mathbf{A}(\mathbf{r}) \;=\; \frac{\mu_0}{4\pi}\iint\limits_{\text{sphere}} \frac{\mathbf{K}(\mathbf{r}')=\sigma\boldsymbol{\omega}\times\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\,d^2A' \;=\; \frac{\mu_0\sigma}{4\pi}\,\boldsymbol{\omega}\times\iint\limits_{\text{sphere}} \frac{\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\,d^2A'. \tag{35}$$

Note: I use vector notations for the angular velocity $\boldsymbol{\omega}$ instead of the spherical coordinates based on the spin axis because evaluating the integral on the RHS of eq. (35) is easier in a different system of spherical coordinates. Indeed, once I pull the $\boldsymbol{\omega}\times$ factor outside the integral, the remaining integral

$$\iint\limits_{\text{sphere}} \frac{\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\,d^2A' \tag{36}$$

depends only on the $\mathbf{r}$ and on the sphere's radius R, hence by spherical symmetry the vector obtaining from the integral (35) must point in the direction of $\mathbf{r}$ — from the center of the sphere towards the point where we evaluate the vector potential. Consequently,

$$\mathbf{A}(\mathbf{r}) \;=\; \frac{\mu_0\sigma}{4\pi}\,(\boldsymbol{\omega}\times\hat{\mathbf{r}})\,\mathcal{I} \tag{37}$$

where $\mathcal{I}$ is the magnitude of the integral in eq. (35), or equivalently its projection on the $\hat{\mathbf{r}}$ axis, thus

$$\mathcal{I} \;=\; \iint\limits_{\text{sphere}} \frac{\mathbf{r}'\cdot\hat{\mathbf{r}}}{|\mathbf{r}-\mathbf{r}'|}\,d^2A'. \tag{38}$$

To take *this* integral, let's use the spherical coordinates where the "north pole" $\theta'=0$ points in the direction of $\mathbf{r}$ so that the θ' coordinate of some point $\mathbf{r}'$ on the sphere is the angle between the vectors $\mathbf{r}'$ and $\mathbf{E}$. Consequently,

$$\mathbf{r}'\cdot\hat{\mathbf{r}} \;=\; R\cos\theta', \quad |\mathbf{r}-\mathbf{r}'|^2 \;=\; \mathbf{r}^2 \;+\; \mathbf{r}'^2 \;-\; 2\mathbf{r}'\cdot\mathbf{r} \;=\; r^2 \;+\; R^2 \;-\; 2Rr\cos\theta', \tag{39}$$

and hence

$$\mathcal{I} \;=\; \iint\limits_{\text{sphere}} \frac{R\cos\theta'}{\sqrt{r^2+R^2-2Rr\cos\theta'}} \times R^2\sin\theta'\,d\theta'\,d\phi'. \tag{40}$$

The integral over $d\phi'$ here is trivial and yields 2π, while in the integral over $d\theta'$ it's convenient to change the integration variable to $c = \cos\theta'$. Thus

$$\mathcal{I} \;=\; 2\pi R^3 \int\limits_0^{\pi} \frac{\cos\theta'\,\sin\theta'\,d\theta'}{\sqrt{r^2 + R^2 - 2Rr\cos\theta'}} \;=\; 2\pi R^3 \int\limits_{-1}^{+1} \frac{c\,dc}{\sqrt{r^2+R^2-2Rrc}}\,. \tag{41}$$

To evaluate the remaining integral, we expand the denominator into Legendre polynomials in c,

$$\frac{1}{\sqrt{r^2+R^2-2Rrc}} \;=\; \begin{cases} \displaystyle\sum_\ell \frac{R^\ell}{r^{\ell+1}} \times P_\ell(c) & \text{for } r > R \text{ (measuring } \mathbf{A} \text{ outside the sphere)},\\[2ex] \displaystyle\sum_\ell \frac{r^\ell}{R^{\ell+1}} \times P_\ell(c) & \text{for } r < R \text{ (measuring } \mathbf{A} \text{ inside the sphere)}, \end{cases} \tag{42}$$

then note that in the numerator $c = P_1(c)$ and therefore

$$\int\limits_{-1}^{+1} P_\ell(c) \times c\,dc \;=\; \frac{2}{2\ell+1} \times \delta_{\ell,1} \;=\; \begin{cases} \frac{2}{3} & \text{for } \ell = 1\\ 0 & \text{for any other } \ell. \end{cases} \tag{43}$$

Consequently,

$$\mathcal{I} \;=\; 2\pi R^3 \times \frac{2}{3} \times \begin{cases} \dfrac{R}{r^2} & \text{outside the sphere,}\\[2ex] \dfrac{r}{R^2} & \text{inside the sphere,} \end{cases} \tag{44}$$

and plugging this result into eq. (37), we finally arrive at the vector potential:

$$\text{Outside the sphere,}\quad \mathbf{A} \;=\; \frac{\mu_0 \sigma R^4}{3}\,\frac{\boldsymbol{\omega} \times \hat{\mathbf{r}}}{r^2}\,, \tag{45}$$

$$\text{Inside the sphere,}\quad \mathbf{A} \;=\; \frac{\mu_0 \sigma R}{3}\, r(\boldsymbol{\omega} \times \hat{\mathbf{r}}). \tag{46}$$

Now that we finally got the vector potential, the magnetic field obtains by taking its curl. By the double vector product formula,

$$\nabla\times(r\boldsymbol{\omega}\times\hat{\mathbf{r}}) \;=\; \nabla\times(\boldsymbol{\omega}\times\mathbf{r}) \;=\; \boldsymbol{\omega}(\nabla\cdot\mathbf{r}) \;-\; (\boldsymbol{\omega}\cdot\nabla)\mathbf{r} \;=\; \boldsymbol{\omega}(3) \;-\; \boldsymbol{\omega} \;=\; 2\boldsymbol{\omega}\,, \tag{47}$$

hence

$$\text{inside the sphere,}\quad \mathbf{B} \;=\; \nabla\times\left(\mathbf{A} \;=\; \frac{\mu_0\sigma R}{3}\,r(\boldsymbol{\omega}\times\hat{\mathbf{r}})\right) \;=\; \frac{2\mu_0\sigma R}{3}\,\boldsymbol{\omega}. \tag{48}$$

Note uniformilty of this field inside the sphere!

As the outside the sphere, by the Leibniz rule

$$\begin{aligned}\nabla\times\left(\frac{\boldsymbol{\omega}\times\hat{\mathbf{r}}}{r^2}\right) \;&=\; \nabla\times\left(\frac{\boldsymbol{\omega}\times\mathbf{r}}{r^3}\right) \;=\; \left(\nabla\frac{1}{r^3}\right)\times(\boldsymbol{\omega}\times\mathbf{r}) \;+\; \frac{1}{r^3}\nabla\times(\boldsymbol{\omega}\times\mathbf{r})\\ &=\; \frac{-3\hat{\mathbf{r}}}{r^4}\times(\boldsymbol{\omega}\times\mathbf{r}) \;+\; \frac{1}{r^3}\,(2\boldsymbol{\omega}) \;=\; \frac{1}{r^3}\Big(2\boldsymbol{\omega} \;-\; 3\hat{\mathbf{r}}\times(\boldsymbol{\omega}\times\hat{\mathbf{r}})\Big)\end{aligned} \tag{49}$$

where

$$\hat{\mathbf{r}}\times(\boldsymbol{\omega}\times\hat{\mathbf{r}}) \;=\; \boldsymbol{\omega}(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}) \;-\; \hat{\mathbf{r}}(\boldsymbol{\omega}\cdot\hat{\mathbf{r}}) \;=\; \boldsymbol{\omega} \;-\; \hat{\mathbf{r}}(\boldsymbol{\omega}\cdot\hat{\mathbf{r}}), \tag{50}$$

$$2\boldsymbol{\omega} \;-\; 3\hat{\mathbf{r}}\times(\boldsymbol{\omega}\times\hat{\mathbf{r}}) \;=\; 3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\boldsymbol{\omega}) \;-\; \boldsymbol{\omega}. \tag{51}$$

Altogether,

$$\nabla\times\left(\frac{\boldsymbol{\omega}\times\hat{\mathbf{r}}}{r^2}\right) \;=\; \frac{1}{r^3}\big(3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\boldsymbol{\omega}) \;-\; \boldsymbol{\omega}\big) \tag{52}$$

and therefore

$$\text{outside the sphere,}\quad \mathbf{B} \;=\; \nabla\times\left(\mathbf{A} \;=\; \frac{\mu_0\sigma R}{3}\left(\frac{\boldsymbol{\omega}\times\hat{\mathbf{r}}}{r^2}\right)\right) \;=\; \frac{\mu_0\sigma R^4}{3r^3}\,\big(3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\boldsymbol{\omega}) - \boldsymbol{\omega}\big). \tag{53}$$

Curiously, this looks like the field of a pure magnetic dipole with dipole moment

$$\mathbf{m} \;=\; \frac{4\pi}{3}\,R^4\sigma\boldsymbol{\omega} \;=\; \frac{Q_{\text{net}}R^2}{2}\,\boldsymbol{\omega}. \tag{54}$$

I shall explain the magnetic dipoles, quadrupoles, *etc.*, later in these notes.

Example: Flat Current Sheet

For our next example, consider a flat current sheet in the xy plane with uniform current density $\mathbf{K}$ in the $\hat{\mathbf{y}}$ direction. In terms of the 3D current density,

$$\mathbf{J}(x,y,z) \;=\; K\delta(z)\,\hat{\mathbf{y}}. \tag{55}$$

Consequently, the Poisson equation for the vector potential of the current sheet is

$$\nabla^2\mathbf{A} \;=\; -\mu_0 K\delta(z)\hat{\mathbf{y}}. \tag{56}$$

Thanks to the symmetries of this equation, we may look for a solution of the form

$$\mathbf{A}(x,y,z) \;=\; A(z\text{ only})\,\hat{\mathbf{y}} \tag{57}$$

where $A(z)$ obey the 1D Poisson equation

$$\frac{d^2A}{dz^2} \;=\; -\mu_0 K\delta(z). \tag{58}$$

Despite the delta function on the RHS, the solution of this differential equation is continuous at $z=0$, namely

$$A(z) \;=\; -\tfrac{1}{2}\mu_0 K\times|z|\,, \tag{59*}$$

although its derivative has a discontinuity,

$$\text{disc}\left(\frac{dA}{dz}\right) \;=\; -\mu_0 K. \tag{60}$$

$\star$ A general solution of eq. (58) is $A(z)=-\frac{1}{2}\mu_0K\times|z|+\alpha z+\beta$ for arbitrary constants α and β, but the upside-down symmetry $z\to-z$ of the current sheet requires $\alpha=0$, while β is physically irrelevant.

This is general behavior of the vector potential for all kinds of 2D current sheets, flat or curved, with uniform or non-uniform 2D currents: *The vector potential is continuous across the current sheet, but its normal derivative has a discontinuity,*

$$\operatorname{disc}\left(\frac{\partial\mathbf{A}}{\partial x_{\text{normal}}}\right) \;=\; -\mu_0\mathbf{K}. \tag{61}$$

Consequently, the magnetic field has a discontinuity

$$\operatorname{disc}(\mathbf{B}) \;=\; \mu_0\mathbf{K}\times\mathbf{n} \tag{62}$$

where $\mathbf{n}$ is the unit vector $\perp$ to the current sheet.

Multipole Expansion for the Vector Potential

Suppose electric current I flows through a closed wire loop of some complicated shape, and we want to find its magnetic field far away from the wire. Let's work through the vector potential according to the Coulomb-like formula

$$\mathbf{A}(\mathbf{r}) \;=\; \frac{\mu_0}{4\pi}\oint\limits_{\text{wire}}\frac{I\,d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\,. \tag{63}$$

Far away from the wire, we may expand the denominator here into a power series in (r'/r), thus

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} \;=\; \sum_{\ell=0}^{\infty}\frac{r'^{\ell}}{r^{\ell+1}}\times P_\ell(\cos\alpha) \tag{64}$$

where α is the angle between the vectors $\mathbf{r}$ and $\mathbf{r}'$,

$$\cos\alpha \;=\; \hat{\mathbf{r}}\cdot\hat{\mathbf{r}}'. \tag{65}$$

Plugging the expansion (64) into eq. (63) for the vector potential, we obtain

$$\mathbf{A}(\mathbf{r}) \;=\; \frac{\mu_0 I}{4\pi}\sum_{\ell=0}^{\infty}\frac{1}{r^{\ell+1}}\oint\limits_{\text{wire}} r'^{\ell}P_\ell(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')\,d\mathbf{r}' \tag{66}$$

— the *expansion of the vector potential into magnetic multipole terms.* Let me write down more explicit formulae for the three leading terms,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\left[\begin{array}{ll} \frac{1}{r}\oint d\mathbf{r}' & \langle\!\langle\,\text{monopole}\,\rangle\!\rangle \\ +\ \frac{1}{r^2}\oint(\hat{\mathbf{r}}\cdot\mathbf{r}')\,d\mathbf{r}' & \langle\!\langle\,\text{dipole}\,\rangle\!\rangle \\ +\ \frac{1}{r^3}\oint\left(\frac{3}{2}(\hat{\mathbf{r}}\cdot\mathbf{r}')^2-\frac{1}{2}r'^2\right)d\mathbf{r}' & \langle\!\langle\,\text{quadrupole}\,\rangle\!\rangle \\ +\ \cdots & \langle\!\langle\,\text{higher multipoles}\,\rangle\!\rangle \end{array}\right]. \tag{67}$$

Naively, the leading term in this expansion is the monopole term for $\ell = 0$ (the top line in eq. (67)), but it vanishes for any closed current loop,

$$\oint d\mathbf{r}' = 0 \tag{68}$$

Thus, *the magnetic multipole expansion starts with the dipole term* — which dominates the magnetic field at large distances from the wire loop. (Except when the dipole moment happens to vanish.)

Let's simplify the dipole term in (67) using a bit of vector calculus, Let $\mathbf{c}$ be some constant vector. Then

$$\begin{aligned} \mathbf{c}\cdot\oint(\hat{\mathbf{r}}\cdot\mathbf{r}')\,d\mathbf{r}' &= \oint(\hat{\mathbf{r}}\cdot\mathbf{r}')\,\mathbf{c}\cdot d\mathbf{r}' \\ &\quad\langle\!\langle\,\text{by Stokes' theorem}\,\rangle\!\rangle \\ &= \iint\Big(\nabla_{\mathbf{r}'}\times\big((\hat{\mathbf{r}}\cdot\mathbf{r}')\,\mathbf{c}\big)\Big)\cdot\mathbf{d^2A} \end{aligned} \tag{69}$$

where

$$\nabla_{\mathbf{r}'}\times\big((\hat{\mathbf{r}}\cdot\mathbf{r}')\,\mathbf{c}\big) = \big(\nabla_{\mathbf{r}'}(\hat{\mathbf{r}}\cdot\mathbf{r}')\big)\times\mathbf{c} = \hat{\mathbf{r}}\times\mathbf{c}, \tag{70}$$

and hence

$$\begin{aligned} \mathbf{c}\cdot\oint(\hat{\mathbf{r}}\cdot\mathbf{r}')\,d\mathbf{r}' &= \iint(\hat{\mathbf{r}}\times\mathbf{c})\cdot\mathbf{d^2A} = (\hat{\mathbf{r}}\times\mathbf{c})\cdot\oint\mathbf{d^2A} \\ &= (\hat{\mathbf{r}}\times\mathbf{c})\cdot\mathbf{a} \quad \langle\!\langle\,\text{where }\mathbf{a}\text{ is the vector area of the loop}\,\rangle\!\rangle \\ &= (\mathbf{a}\times\hat{\mathbf{r}})\cdot\mathbf{c}. \end{aligned} \tag{71}$$

Since $\mathbf{c}$ here can be *any* constant vector, it follows that

$$\oint (\hat{\mathbf{r}} \cdot \mathbf{r}')\, d\mathbf{r}' \;=\; \mathbf{a} \times \hat{\mathbf{r}}. \tag{72}$$

Finally, plugging this integral into the dipole term in the expansion (67), we arrive at

$$\mathbf{A}_{\text{dipole}}(\mathbf{r}) \;=\; \frac{\mu_0}{4\pi}\,\frac{I\mathbf{a} \times \hat{\mathbf{r}}}{r^2} \;=\; \frac{\mu_0}{4\pi}\,\frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2} \tag{73}$$

where $\mathbf{m} = I\mathbf{a}$ is the *magnetic dipole moment* of the current loop.

I am going to skip over the higher multipoles in these notes. Instead, let me consider replacing a single wire loop with a circuit of several connected wires. In this case, we may use the Kirchhoff Law to express the whole circuit as several overlapping loops with independent currents; if a wire belongs to several loops, the current in that wire is the algebraic sum of the appropriate loop currents. By the superposition principle, the vector potential of the whole circuit is the sum of vector potentials of the individual loops, and as long as the whole circuit occupies small volume of size $\ll r$, we may expand each loop's $\mathbf{A}$ into multipoles, exactly as we did it for a single loop. In general, the leading contribution is the *net dipole term*,

$$\mathbf{A}_{\text{dipole}}(\mathbf{r}) \;=\; \sum_i^{\text{loops}} \frac{\mu_0}{4\pi}\,\frac{\mathbf{m}_i \times \hat{\mathbf{r}}}{r^2} \;=\; \frac{\mu_0}{4\pi}\,\frac{\mathbf{m}_{\text{net}} \times \hat{\mathbf{r}}}{r^2} \tag{74}$$

where

$$\mathbf{m}_{\text{net}} \;=\; \sum_i^{\text{loops}} \mathbf{m}_i \;=\; \sum_i^{\text{loops}} I_i\, \mathbf{a}_i \tag{75}$$

is the net dipole moment of the whole circuit.

Now suppose instead of a circuit of thin wires we have some current density $\mathbf{J}(\mathbf{r}')$ flowing through the volume of some thick conductor. However, the conductor's size is much smaller than the distance r to where we want to calculate the vector potential and the magnetic field. In this case, we may use the multipole expansion, but the algebra is a bit different from what we had for a thin wire:

$$\mathbf{A}(\mathbf{r}) \;=\; \frac{\mu_0}{4\pi}\iiint\frac{\mathbf{J}(\mathbf{r})}{|\mathbf{r}-\mathbf{r}'|}\,d^3\mathrm{Vol}' \;=\; \frac{\mu_0}{4\pi}\sum_{\ell=0}^{\infty}\frac{1}{r^{\ell+1}}\iiint r'^{\ell}P_\ell(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')\,\mathbf{J}(\mathbf{r}')\,d^3\mathrm{Vol}', \tag{76}$$

or in a more explicit form

$$\mathbf{A}(\mathbf{r}) \;=\; \frac{\mu_0 I}{4\pi}\left[\begin{array}{ll} \displaystyle\frac{1}{r}\iiint\mathbf{J}(\mathbf{r}')\,d^3\mathrm{Vol}' & \langle\!\langle\,\text{monopole}\,\rangle\!\rangle\\ \displaystyle +\;\frac{1}{r^2}\iiint(\hat{\mathbf{r}}\cdot\mathbf{r}')\,\mathbf{J}(\mathbf{r}')\,d^3\mathrm{Vol}' & \langle\!\langle\,\text{dipole}\,\rangle\!\rangle\\ \displaystyle +\;\frac{1}{r^3}\iiint\left(\tfrac{3}{2}(\hat{\mathbf{r}}\cdot\mathbf{r}')^2-\tfrac{1}{2}r'^2\right)\mathbf{J}(\mathbf{r}')\,d^3\mathrm{Vol}' & \langle\!\langle\,\text{quadrupole}\,\rangle\!\rangle\\ +\;\cdots & \langle\!\langle\,\text{higher multipoles}\,\rangle\!\rangle \end{array}\right]. \tag{77}$$

The monopole term here vanishes just as it did for the wire loop, albeit in a less obvious way. To see how this works, pick a constant vector $\mathbf{c}$ and take the divergence

$$\nabla_{\mathbf{r}'}\cdot\big((\mathbf{c}\cdot\mathbf{r}')\mathbf{J}(\mathbf{r}')\big) \;=\; \mathbf{c}\cdot\mathbf{J} \;+\; (\mathbf{c}\cdot\mathbf{r})\,(\nabla\cdot\mathbf{J}), \tag{78}$$

where the second term on the RHS vanishes for a steady — and hence divergence-less — current. Consequently,

$$\begin{aligned} \mathbf{c}\cdot\iiint\limits_{\mathcal{V}}\mathbf{J}(\mathbf{r}')d^3\mathrm{Vol}' &= \iiint\limits_{\mathcal{V}}(\mathbf{c}\cdot\mathbf{J}(\mathbf{r}'))\,d^3\mathrm{Vol}'\\ &\quad\langle\!\langle\,\text{by eq. (78)}\,\rangle\!\rangle\\ &= \iiint\limits_{\mathcal{V}}\nabla_{\mathbf{r}'}\cdot\big((\mathbf{c}\cdot\mathbf{r}')\mathbf{J}(\mathbf{r}')\big)\,d^3\mathrm{Vol}'\\ &\quad\langle\!\langle\,\text{by Gauss theorem}\,\rangle\!\rangle\\ &= \iint\limits_{\mathcal{S}}\big((\mathbf{c}\cdot\mathbf{r}')\mathbf{J}(\mathbf{r}')\big)\cdot\mathbf{d^2A}' \end{aligned} \tag{79}$$

where $\mathcal{S}$ is the surface of the volume $\mathcal{V}$. That volume must include the whole conductor, but we may also make it a bit bigger, which would put the surface $\mathcal{S}$ outside the conductor.

But then there would be no current along or across $\mathcal{S}$, so the integral on the bottom line of (79) must vanish. Consequently, the top line of eq. (79) must vanish too, and since $\mathbf{c}$ is an arbitrary constant vector, this means zero monopole moment,

$$\iiint\limits_{\mathcal{V}} \mathbf{J}(\mathbf{r}')d^3\mathrm{Vol}' \;=\; 0. \tag{80}$$

Next, consider the dipole term in (77) and try to rewrite it in the form (73) for some dipole moment vector $\mathbf{m}$. This time, the algebra is a bit more complicated. For an arbitrary but constant vector $\mathbf{c}$, we have

$$\mathbf{c}\cdot\big(\hat{\mathbf{r}}\times(\mathbf{J}\times\mathbf{r}')\big) \;=\; (\mathbf{c}\cdot\mathbf{J})\,(\hat{\mathbf{r}}\cdot\mathbf{r}') \;-\; (\mathbf{c}\cdot\mathbf{r}')\,(\hat{\mathbf{r}}\cdot\mathbf{J}), \tag{81}$$

$$\nabla_{\mathbf{r}'}\big((\mathbf{c}\cdot\mathbf{r}')\,(\hat{\mathbf{r}}\cdot\mathbf{r}')\,\mathbf{J}(\mathbf{r}')\big) \;=\; (\mathbf{c}\cdot\mathbf{J})\,(\hat{\mathbf{r}}\cdot\mathbf{r}') \;+\; (\mathbf{c}\cdot\mathbf{r}')\,(\hat{\mathbf{r}}\cdot\mathbf{J}) \;+\; (\mathbf{c}\cdot\mathbf{r}')\,(\hat{\mathbf{r}}\cdot\mathbf{r}')\,(\nabla\cdot\mathbf{J}), \tag{82}$$

⟪ where the last term vanishes for a steady current. ⟫

⟪ which has $\nabla\cdot\mathbf{J}=0$ ⟫

and hence

$$(\mathbf{c}\cdot\mathbf{J})\,(\hat{\mathbf{r}}\cdot\mathbf{r}') \;=\; \tfrac{1}{2}\,\mathbf{c}\cdot\big(\hat{\mathbf{r}}\times(\mathbf{J}\times\mathbf{r}')\big) \;+\; \tfrac{1}{2}\,\nabla_{\mathbf{r}'}\big((\mathbf{c}\cdot\mathbf{r}')\,(\hat{\mathbf{r}}\cdot\mathbf{r}')\,\mathbf{J}(\mathbf{r}')\big). \tag{83}$$

Consequently, dotting $\mathbf{c}$ with the dipole integral, we obtain

$$\begin{aligned}
\mathbf{c}\cdot\iiint\limits_{\mathcal{V}}(\hat{\mathbf{r}}\cdot\mathbf{r}')\,\mathbf{J}(\mathbf{r}')\,d^3\mathrm{Vol}' \;&=\; \iiint\limits_{\mathcal{V}}(\mathbf{c}\cdot\mathbf{J})\,(\hat{\mathbf{r}}\cdot\mathbf{r}')\,d^3\mathrm{Vol}'\\
&=\; \frac{1}{2}\iiint\limits_{\mathcal{V}}\Big(\mathbf{c}\cdot\big(\hat{\mathbf{r}}\times(\mathbf{J}\times\mathbf{r}')\big)\Big)d^3\mathrm{Vol}'\\
&\qquad+\;\frac{1}{2}\iiint\limits_{\mathcal{V}}\Big(\nabla_{\mathbf{r}'}\big((\mathbf{c}\cdot\mathbf{r}')\,(\hat{\mathbf{r}}\cdot\mathbf{r}')\,\mathbf{J}(\mathbf{r}')\big)\Big)d^3\mathrm{Vol}'\\
&=\; \frac{1}{2}\,\mathbf{c}\cdot\left(\hat{\mathbf{r}}\times\iiint\limits_{\mathcal{V}}(\mathbf{J}(\mathbf{r}')\times\mathbf{r}')\,d^3\mathrm{Vol}'\right)\\
&\qquad+\;\frac{1}{2}\iint\limits_{\mathcal{S}}(\mathbf{c}\cdot\mathbf{r}')\,(\hat{\mathbf{r}}\cdot\mathbf{r}')\,\mathbf{J}(\mathbf{r}')\cdot\mathbf{d^2A}'
\end{aligned} \tag{84}$$

Similar to what we did for the monopole term, let's take the integration volume $\mathcal{V}$ a bit larger that the whole conductor, so its surface $\mathcal{S}$ is completely outside the conductor. Then on the last line of eq. (84) the current $\mathbf{J}$ vanishes everywhere on the surface, which kills the surface integral. This veaves us with

$$\mathbf{c}\cdot\iiint\limits_{\text{conductor}+}(\hat{\mathbf{r}}\cdot\mathbf{r}')\,\mathbf{J}(\mathbf{r}')\,d^3\mathrm{Vol}' \;=\; \frac{1}{2}\,\mathbf{c}\cdot\left(\hat{\mathbf{r}}\times\iiint\limits_{\text{conductor}+}(\mathbf{J}(\mathbf{r}')\times\mathbf{r}')\,d^3\mathrm{Vol}'\right), \tag{85}$$

and since $\mathbf{c}$ is an arbitrary constant vector,

$$\begin{aligned}\iiint\limits_{\text{conductor}+}(\hat{\mathbf{r}}\cdot\mathbf{r}')\,\mathbf{J}(\mathbf{r}')\,d^3\mathrm{Vol}' \;&=\; \frac{\hat{\mathbf{r}}}{2}\times\iiint\limits_{\text{conductor}+}\mathbf{J}(\mathbf{r}')\times\mathbf{r}'\,d^3\mathrm{Vol}'\\ &=\;\left(\frac{1}{2}\iiint\limits_{\text{conductor}+}\mathbf{r}'\times\mathbf{J}(\mathbf{r}')\,d^3\mathrm{Vol}'\right)\times\hat{\mathbf{r}}.\end{aligned} \tag{86}$$

Plugging this formula into the dipole term in the vector potential (77), we arrive at

$$\mathbf{A}_{\text{dipole}}(\mathbf{r}) \;=\; \frac{\mu_0}{4\pi r^2}\iiint\limits_{\text{conductor}+}(\hat{\mathbf{r}}\cdot\mathbf{r}')\,\mathbf{J}(\mathbf{r}')\,d^3\mathrm{Vol}' \;=\; \frac{\mu_0}{4\pi}\,\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2} \tag{87}$$

— exactly as in eq. (73) for the current loop — for the *magnetic dipole moment*

$$\mathbf{m} \;=\; \frac{1}{2}\iiint\limits_{\text{conductor}+}\mathbf{r}'\times\mathbf{J}(\mathbf{r}')\,d^3\mathrm{Vol}'. \tag{88}$$

Let me conclude this section with the dipole term in the magnetic field,

$$\mathbf{B}_{\text{dipole}}(\mathbf{r}) \;=\; \nabla\times\mathbf{A}_{\text{dipole}}(\mathbf{r}) \;=\; \frac{\mu_0}{4\pi}\,\frac{3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}\,-\,\mathbf{m}}{r^3}\,, \tag{89}$$

where the algebra of taking the curl is exactly as in eq. (52) earlier in these notes. In spherical coordinates centered at the dipole and aligned with the dipole moment,

$$\mathbf{A} \;=\; \frac{\mu_0 m}{4\pi}\,\frac{\sin\theta}{r^2}\,\hat{\boldsymbol{\phi}}, \tag{90}$$

$$\mathbf{B} \;=\; \frac{\mu_0 m}{4\pi}\,\frac{2\cos\theta\,\hat{\mathbf{r}}\,+\,\sin\theta\hat{\boldsymbol{\theta}}}{r^3}\,. \tag{91}$$

Force and Torque on a Magnetic Dipole

The **B** field (89) of a magnetic dipole looks exactly like the **E** field of an electric dipole:

$$\mathbf{B}_{\text{dipole}} = \frac{\mu_0}{4\pi}\frac{3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}}{r^3}, \quad \mathbf{E}_{\text{dipole}} = \frac{1}{4\pi\epsilon_0}\frac{3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}}{r^3}. \tag{92}$$

Likewise, the force and the torque on a pure magnetic dipole in an external magnetic field have exactly similar form to the force and the torque on an electric dipole in an external electric field (see my notes on electric dipoles for details). In particular, *in a uniform external* **B** *field, there is no net force on a dipole but there is a net torque,*

$$\mathbf{F}_{\text{net}} = 0, \quad \boldsymbol{\tau}_{\text{net}} = \mathbf{m}\times\mathbf{B}. \tag{93}$$

In fact, in a uniform **B** field these formulae are exact for any closed current loop rather than just a pure dipole. For the net force, this is trivial,

$$\mathbf{F}_{\text{net}} = \oint I d\vec{\ell}\times\mathbf{B} = I\left(\oint d\vec{\ell} = 0\right)\times\mathbf{B} = 0. \tag{94}$$

For the net torque, we need some algebra first:

$$d\big(\mathbf{r}\times(\mathbf{r}\times\mathbf{B})\big) = d\mathbf{r}\times(\mathbf{r}\times\mathbf{B}) + \mathbf{r}\times(d\mathbf{r}\times\mathbf{B}), \tag{95}$$

$$\begin{aligned} \mathbf{B}\times(d\mathbf{r}\times\mathbf{r}) &= -d\mathbf{r}\times(\mathbf{r}\times\mathbf{B}) - \mathbf{r}\times(\mathbf{B}\times d\mathbf{r}) \quad \langle\!\langle \text{ by the Jacobi identity } \rangle\!\rangle \\ &= -d\mathbf{r}\times(\mathbf{r}\times\mathbf{B}) + \mathbf{r}\times(d\mathbf{r}\times\mathbf{B}), \end{aligned} \tag{96}$$

$$\text{hence } \mathbf{r}\times(d\mathbf{r}\times\mathbf{B}) = \tfrac{1}{2}d\big(\mathbf{r}\times(\mathbf{r}\times\mathbf{B})\big) + \tfrac{1}{2}\mathbf{B}\times(d\mathbf{r}\times\mathbf{r}). \tag{97}$$

Consequently,

$$\begin{aligned} \boldsymbol{\tau}_{\text{net}} &= \int \mathbf{r}\times d\mathbf{F} = \oint \mathbf{r}\times(I\,d\mathbf{r}\times\mathbf{B}) \\ &\quad \langle\!\langle \text{ in light of eq. (97) } \rangle\!\rangle \\ &= \frac{I}{2}\oint d\big(\mathbf{r}\times(\mathbf{r}\times\mathbf{B})\big) + \frac{I}{2}\oint \mathbf{B}\times(d\mathbf{r}\times\mathbf{r}) \\ &\quad \langle\!\langle \text{ where the first } \oint \text{ of a total differential is zero } \rangle\!\rangle \\ &= 0 + \mathbf{B}\times\frac{I}{2}\oint d\mathbf{r}\times\mathbf{r} \\ &= \left(\frac{I}{2}\oint \mathbf{r}\times d\mathbf{r}\right)\times\mathbf{B}. \end{aligned} \tag{98}$$

But for any closed loop $\frac{1}{2}\oint \mathbf{r} \times d\mathbf{r}$ is the vector area $\mathbf{a}$ of the surface spanning that loop, hence

$$\frac{I}{2}\oint \mathbf{r} \times d\mathbf{r} \;=\; I\mathbf{a} \;=\; \mathbf{m} \tag{99}$$

— the magnetic moment of the lopp, — so the net torque is indeed

$$\vec{\tau}_{\rm net} \;=\; \mathbf{m} \times \mathbf{B}. \tag{100}$$

When the external magnetic field is non-uniform the net force on a current loop does not vanish. For a small loop, the net force is related to the magnetic moment as

$$\mathbf{F}_{\rm net} \;=\; \nabla(\mathbf{m} \cdot \mathbf{B}) \tag{101}$$

where the gradient acts only on the components of $\mathbf{B}$ and not on the $\mathbf{m}$. Let me skip the proof of this formula and simply say that it is completely similar to the force on an electric dipole in a non-uniform electric field,

$$\mathbf{F}_{\rm net} \;=\; (\mathbf{p} \cdot \nabla)\mathbf{E} \;=\; \nabla(\mathbf{p} \cdot \mathbf{E}). \tag{102}$$

(Note that $\nabla(\mathbf{p} \cdot \mathbf{E}) - (\mathbf{p} \cdot \nabla)\mathbf{E} \;=\; \mathbf{p} \times (\nabla \times \mathbf{E}) = 0$ since $\nabla \times \mathbf{E} = 0$.)

For atoms and molecules, the magnetic dipole moment is fixed by the quantum effects. Consequently, the magnetic force (101) on an atom or a molecules acts as a potential force, with a potential energy

$$U(x, y, z) \;=\; -\mathbf{m} \cdot \mathbf{B}(x, y, z). \tag{103}$$

The same potential energy — or rather its variation when the magnetic moment $\mathbf{m}$ changes its direction — is also responsible for magnetic the torque $\boldsymbol{\tau} = \mathbf{m} \times \mathbf{B}$.