MAGNETIC FIELDS OF STEADY ELECTRIC CURRENTS, BIOT–SAVART–LAPLACE LAW, AND ITS APPLICATIONS

In these notes I explain the magnetic fields of *steady* electric currents. The fields of time-dependent currents are more complicated, and I'll discuss them a few lectures later.

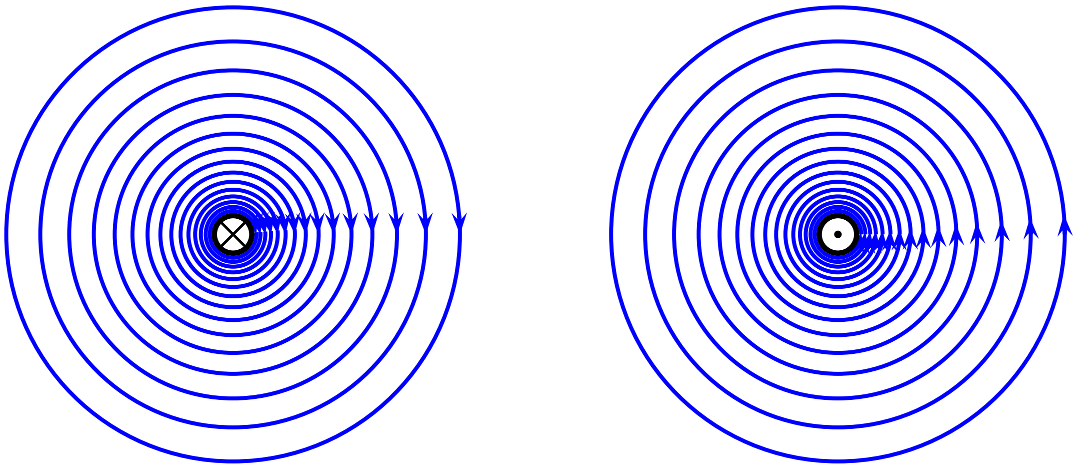
Let's start with the simplest case of an infinite straight wire carrying a current I. The magnetic field of such a wire is

$$\mathbf{B} = \frac{\mu_0 I}{2\pi} \frac{\hat{\phi}}{s} \tag{1}$$

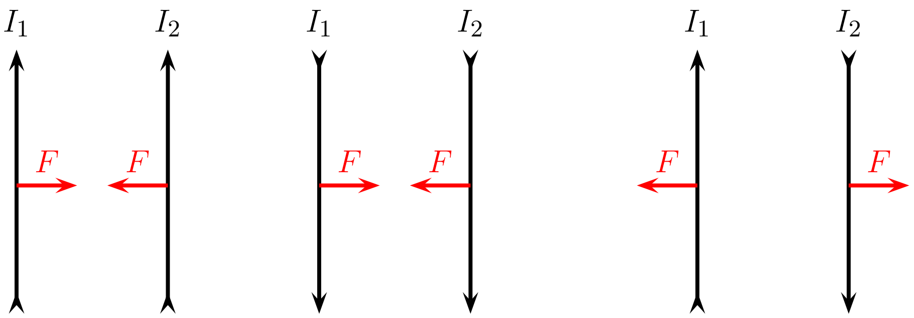
where μ_0 is the fundamental constant of the MKSA system of units called the *vacuum permeability,*

$$\mu_0 = 4\pi \cdot 10^{-7}\ \mathrm{T\,m/A}, \tag{2}$$

s is the distance from the wire, and $\hat{\phi}$ is a unit vector in the circular direction around the wire in the plain $\perp$to the wire. Specifically, if you are looking down the wire and the current flows away from you, then the circular direction $\hat{\phi}$ of the magnetic field is clockwise, while if the current flows toward you, then the magnetic field is counterclockwise. Here are the pictures of the magnetic field lines for the two cases:

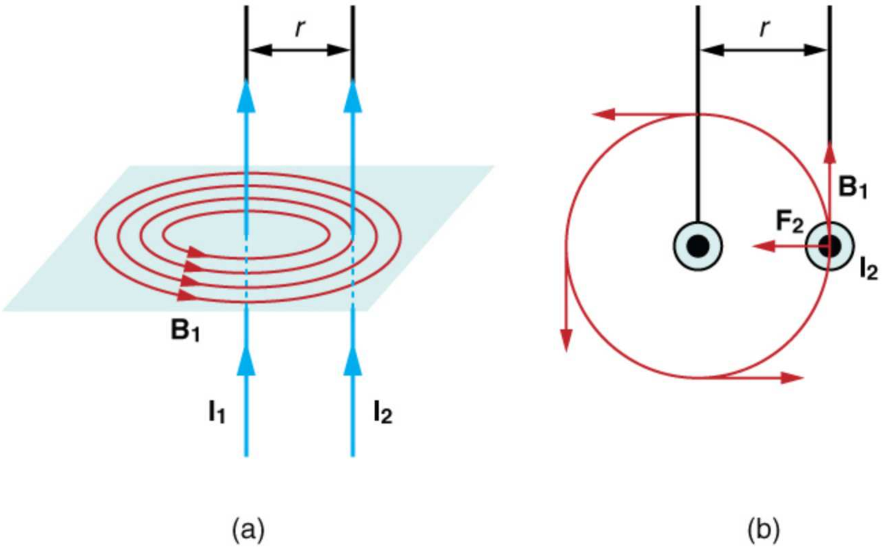


The effect of this magnetic field on another long wire parallel to the first wire is an attractive force if the currents in the two wires flow in the same direction, and a repulsive force if the currents flow in the opposite directions,



(3)

Here is the graphical explanation of the force's direction for the currents in the same direction:



(4)

The magnitude of the force between two wires per unit of wire's length is

$$\frac{F}{L} = \frac{\mu_0}{2\pi} \times \frac{I_1 I_2}{d} \tag{5}$$

where d is the distance between the wires, while the μ_0 constant of the MKSA system of units is *exactly*

$$\mu_0 = 4\pi \cdot 10^{-7}\ \mathrm{T\,m/A} = 4\pi \cdot 10^{-7}\ \mathrm{N/A^2}. \tag{6}$$

In other words, the Ampere — the MKSA unit of electric current — is *defined* such that two long parallel wires separated by 1 m distance and each carrying 1 A current are attracted or repelled with a force of $2 \cdot 10^{-7}$ Newtons per meter of length.

In the Gaussian system of units, there is no μ_0. Instead there are factors $1/c$ (where c is the speed of light in vacuum) all over the place, For example, the Lorentz Force on a particle in Gaussian units is

$$\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right), \tag{7}$$

the magnetic force on a current-carrying wire is

$$\mathbf{F} = \frac{1}{c}\int\limits_{\text{wire}} I\,d\vec{\ell}\times\mathbf{B}, \tag{8}$$

the magnetic field of an infinite straight wire is

$$\mathbf{B} = \frac{2I}{c}\frac{\hat{\phi}}{s}, \tag{9}$$

and the force between 2 parallel wires is

$$\frac{F}{L} = \frac{2}{c^2}\times\frac{I_1 I_2}{d}. \tag{10}$$

⋆ ⋆ ⋆

The wires of geometries other than an infinite straight line create magnetic fields much more complicated that (1). For a *steady current* in a wire of most general geometry, there is an integral formula known as the Biot–Savart–Laplace equation or Biot–Savart–Laplace Law:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\limits_{\text{wire}} I\,d\mathbf{r}'\times\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \tag{11}$$

in MKSA units, or

$$\mathbf{B}(\mathbf{r}) = \frac{1}{c}\int\limits_{\text{wire}} I\,d\mathbf{r}'\times\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \tag{12}$$

in Gaussian units. In these formulae, the $\mathbf{r}'$ spans the wire and the $d\mathbf{r}' = d\vec{\ell}$ is the infinitesimal length vector along the wire in the direction of the current. The expression

$$\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = \frac{\text{unit vector from } \mathbf{r}' \text{ to } \mathbf{r}}{\bigl(\text{distance between } \mathbf{r}' \text{ and } \mathbf{r}\bigr)^2} \tag{13}$$

should be familiar to you from the Coulomb Law for the electric field of a charge distribution, for example the electric field of a charged wire is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int\limits_{\text{wire}} \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\,\lambda\,d\ell. \tag{14}$$

But there is a crucial difference between the Biot–Savart–Laplace equation (11) and the Coulomb equation (14) — the cross product of $I d\vec{\ell}$ with the kernel (13) in the BSL equation (11). In these notes, I shall explore the consequences of this vector product for several examples of wire geometries.

Example#1: Infinite Long Wire

For my first example, let me reproduce eq. (1) for the magnetic field of an infinite straight wire from the Biot–Savart–Laplace Law. Let me use the coordinate system where the wire runs along the z axis with the current flowing in the $+\hat{\mathbf{z}}$ direction. Consequently, in the BSL equation

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int\limits_{\text{wire}} I\,d\mathbf{r}' \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \tag{11}$$

we have $\mathbf{r}' = (0,0,z')$ for a variable z' but fixed $x' = y' = 0$, and $I\,d\mathbf{r}' = +I\,dz'\,\hat{\mathbf{z}}$; on the other hand, the coordinates (x,y,z) of the point $\mathbf{r}$ where we measure the magnetic field are completely general. Therefore,

$$\mathbf{r} - \mathbf{r}' = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + (z-z')\hat{\mathbf{z}}, \tag{15}$$

$$I\,dz'\,\hat{\mathbf{z}} \times (\mathbf{r}-\mathbf{r}') = I\,dz'\bigl(x\,\hat{\mathbf{y}} - y\,\hat{\mathbf{x}} + (z-z')\mathbf{0}\bigr), \tag{16}$$

$$|\mathbf{r}-\mathbf{r}'|^2 = x^2 + y^2 + (z-z')^2, \tag{17}$$

$$|\mathbf{r}-\mathbf{r}'|^3 = \bigl(x^2+y^2+(z-z')^2\bigr)^{3/2}, \tag{18}$$

and plugging all these formulae into eq. (11) gives us

$$\mathbf{B}(x,y,z) \;=\; \frac{I\mu_0}{4\pi}\,(x\,\hat{\mathbf{y}}\,-\,y\,\hat{\mathbf{x}})\times\int\limits_{-\infty}^{+\infty}\frac{dz'}{\bigl(x^2+y^2+(z-z')^2\bigr)^{3/2}}\,. \tag{19}$$

To evaluate the integral here, let's change the integration variable from z' to

$$\alpha \;=\; \arctan\frac{s}{z-z'} \quad\Longrightarrow\quad z' \;=\; z\,-\,\frac{s}{\tan\alpha} \;=\; z\,-\,s\times\operatorname{ctan}\alpha \quad\text{where}\;\; s=\sqrt{x^2+y^2}. \tag{20}$$

Consequently,

$$dz' \;=\; +\frac{s\,d\alpha}{\sin^2\alpha}\,, \tag{21}$$

$$x^2+y^2+(z-z')^2 \;=\; s^2\;+\;\frac{s^2}{\tan^2\alpha} \;=\; \frac{s^2}{\sin^2\alpha}\,, \tag{22}$$

$$\begin{aligned}\frac{dz'}{\bigl(x^2+y^2+(z-z')^2\bigr)^{3/2}} \;&=\; \frac{s\,d\alpha}{\sin^2\alpha}\times\frac{\sin^3\alpha}{s^3}\\ &=\; \frac{\sin\alpha\,d\alpha}{s^2} \;=\; \frac{d(-\cos\alpha)}{x^2+y^2}\,,\end{aligned} \tag{23}$$

while the angle α runs from 0 for $z'\to-\infty$ to π for $z'\to+\infty$. Hence,

$$\int\limits_{-\infty}^{+\infty}\frac{dz'}{\bigl(x^2+y^2+(z-z')^2\bigr)^{3/2}} \;=\; \int\limits_0^{\pi}\frac{d(-\cos\alpha)}{x^2+y^2} \;=\; \frac{-\cos(\pi)+\cos(0)}{x^2+y^2} \;=\; \frac{2}{x^2+y^2} \tag{24}$$

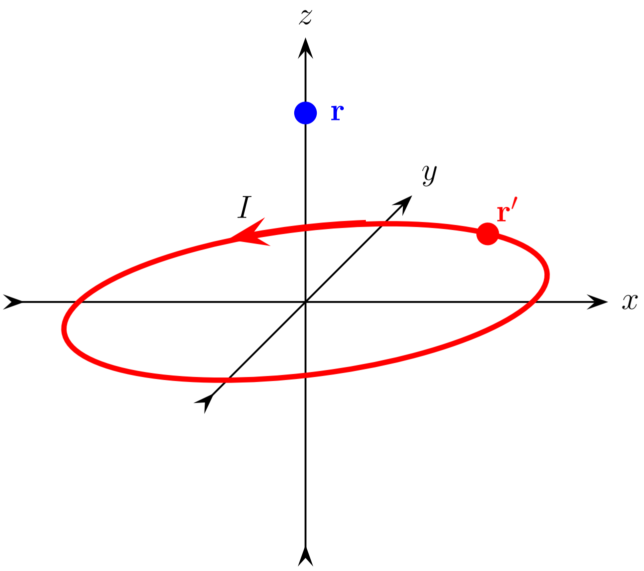
and therefore

$$\begin{aligned}\mathbf{B}(x,y,z) \;&=\; \frac{\mu_0 I}{2\pi}\,\frac{x\,\hat{\mathbf{y}}\,-\,y\,\hat{\mathbf{x}}}{x^2+y^2} \quad\langle\!\langle\,\text{in Cartesian coordinates}\,\rangle\!\rangle\\ &=\; \frac{\mu_0 I}{2\pi}\,\frac{\hat{\phi}}{s} \quad\langle\!\langle\,\text{in cylindrical coordinates}\,\rangle\!\rangle.\end{aligned} \tag{25}$$

Thus, the basic formula (1) for the infinite straight wire indeed follows from the general Biot–Savart–Laplace equation..

EXAMPLE#2: CIRCULAR RING

For the next example, consider a wire shaped into a circular ring of radius R. For simplicity, let me limit the calculation of the magnetic field to the axis of the ring, otherwise we would have to deal with elliptic integrals. Let's use the coordinate system where the ring lies in the xy plane while its symmetry axis is the z axis, thus



(26)

Along the circular wire,

$$\mathbf{r}' = R\cos\phi\,\hat{\mathbf{x}} + R\sin\phi\,\hat{\mathbf{y}}, \tag{27}$$

$$d\mathbf{r}' = R(-\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}})\,d\phi, \tag{28}$$

while the points $\mathbf{r}$ where we measure the magnetic field are restricted to $\mathbf{r} = z\,\hat{\mathbf{z}}$, hence

$$\mathbf{r} - \mathbf{r}' = -R\cos\phi\,\hat{\mathbf{x}} - R\sin\phi\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}, \tag{29}$$

$$\begin{aligned}(-\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}})\times(-\cos\phi\,\hat{\mathbf{x}} - \sin\phi\,\hat{\mathbf{y}}) &= \\ &= \sin\phi\cos\phi\big(\hat{\mathbf{x}}\times\hat{\mathbf{x}} - \hat{\mathbf{y}}\times\hat{\mathbf{y}} = \mathbf{0}\big) \\ &\quad + \sin^2\phi\big(\hat{\mathbf{x}}\times\hat{\mathbf{y}} = +\hat{\mathbf{z}}\big) - \cos^2\phi\big(\hat{\mathbf{y}}\times\hat{\mathbf{x}} = -\hat{\mathbf{z}}\big) \\ &= (\sin^2\phi + \cos^2\phi)\hat{\mathbf{z}} = \hat{\mathbf{z}},\end{aligned} \tag{30}$$

$$
\begin{aligned}
(-\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}})\times\hat{\mathbf{z}} &= -\sin\phi\big(\hat{\mathbf{x}}\times\hat{\mathbf{z}} = -\hat{\mathbf{y}}\big) + \cos\phi\big(\hat{\mathbf{y}}\times\hat{\mathbf{z}} = +\hat{\mathbf{x}}\big)\\
&= \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}},
\end{aligned} \tag{31}
$$

$$
I\,d\mathbf{r}'\times(\mathbf{r}-\mathbf{r}') = IR\Big(z\cos\phi\,\hat{\mathbf{x}} + z\sin\phi\,\hat{\mathbf{y}} + R\,\hat{\mathbf{z}}\Big)\,d\phi, \tag{32}
$$

$$
|\mathbf{r}-\mathbf{r}'|^2 = R^2 + z^2, \tag{33}
$$

$$
|\mathbf{r}-\mathbf{r}'|^3 = \big(R^2+z^2\big)^{3/2}. \tag{34}
$$

Plugging all these formulae into the Biot–Savart–Laplace equation, we obtain

$$
\begin{aligned}
\mathbf{B}(0,0,z) &= \frac{\mu_0}{4\pi}\int\limits_{\text{wire}} I\,d\mathbf{r}'\times\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\\
&= \frac{\mu_0}{4\pi}\,\frac{IR}{\big(R^2+z^2\big)^{3/2}}\int\limits_0^{2\pi} d\phi\Big(z\cos\phi\,\hat{\mathbf{x}} + z\sin\phi\,\hat{\mathbf{y}} + R\,\hat{\mathbf{z}}\Big),
\end{aligned} \tag{35}
$$

where the integral evaluates to

$$
\begin{aligned}
\int\limits_0^{2\pi} d\phi\Big(z\cos\phi\,\hat{\mathbf{x}} + z\sin\phi\,\hat{\mathbf{y}} + R\hat{\mathbf{z}}\Big) &= z\,\hat{\mathbf{x}}\times\int\limits_0^{2\pi} d\phi\,\cos\phi + z\,\hat{\mathbf{y}}\times\int\limits_0^{2\pi} d\phi\,\sin\phi + R\,\hat{\mathbf{z}}\times\int\limits_0^{2\pi} d\phi\\
&= z\,\hat{\mathbf{x}}\times 0 + z\,\hat{\mathbf{y}}\times 0 + R\,\hat{\mathbf{z}}\times 2\pi\\
&= 2\pi R\,\hat{\mathbf{z}}.
\end{aligned} \tag{36}
$$

Altogether, the magnetic filed along the ring's axis is

$$
\mathbf{B}(0,0,z) = \frac{\mu_0 I}{2}\,\frac{R^2}{(R^2+z^2)^{3/2}}\,\hat{\mathbf{z}}\,. \tag{37}
$$

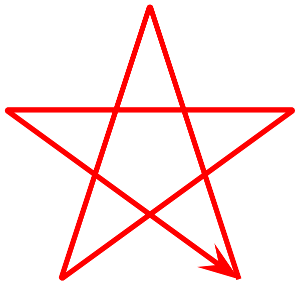
In particular, at the center of the ring, the magnetic field is

$$
\mathbf{B}(\text{center}) = \frac{\mu_0 I}{2R}\,\hat{\mathbf{z}}. \tag{38}
$$

Note: on the diagram (26), the current in the wire flows counterclockwise; consequently, the magnetic field (37) points up, in the $+\hat{\mathbf{z}}$ direction. For a clockwise current, we would have an opposite sign of $I\,d\mathbf{r}'$ and hence opposite direction $-\hat{\mathbf{z}}$ of the magnetic field — down. This is an example of the *right screw rule* for the current loops: turn a right screw (almost all the screws are right) in the direction of the current in the loop, and the screw will move in the direction of the **B** field. Equivalently, you may use the *right hand rule:* curl the fingers of your right hand around the loop in the direction of the current, and your thumb will point the direction of the **B** field.

SEGMENTS:

In many cases, a wire is made of several segments. Each segment has a simple geometric shape — a piece of a straight line, or a circular arc — but the overall geometry can be quite elaborate. For example, consider a star made of 5 straight-line segments,



(39)

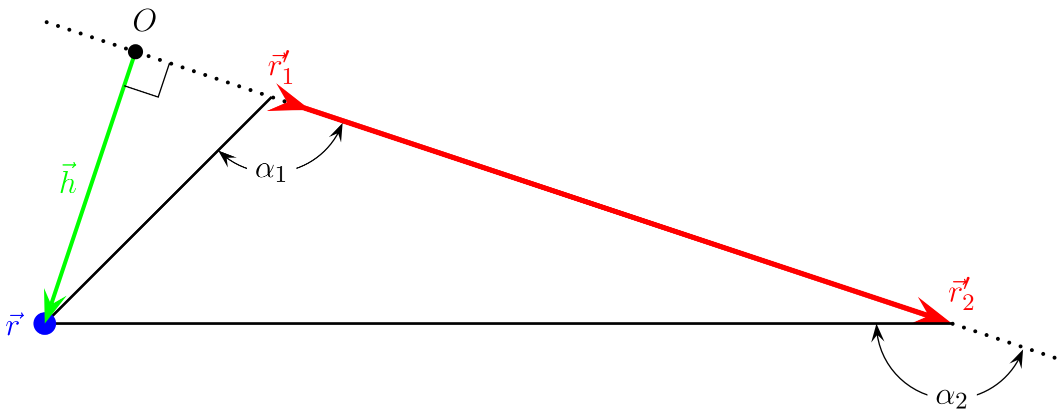
For a wire like this, the Biot–Savart–Laplace integral over the whole wire becomes a sum of integrals over the individual segments,

$$\mathbf{B}(\mathbf{r}) \;=\; \frac{\mu_0 \times I}{4\pi} \times \sum_i^{\text{segments}} \int\limits_{\text{segment}\#i} \frac{d\mathbf{r}' \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\,. \tag{40}$$

Let's work out the integrals here for the straight-line and the circular-arc segments, and than we shall see a few interesting combinations.

Example#3: Straight-Line Segment:

Consider a wire segment which follows a straight line from point $\mathbf{r}'_1$ to point $\mathbf{r}'_1$. Let's picture the triangle made by the two ends of this segment and by the point $\mathbf{r}$ where we measure the magnetic field:



(41)

Since the wire segment is straight, the infinitesimal vector $d\vec{\ell} = d\mathbf{r}'$ along the segment has a fixed direction, same as $\mathbf{r}'_2 - \mathbf{r}'_1$. Consequently, the vector product in the numerator of the BSL integral remains constant along the whole segment. Indeed,

$$\begin{aligned} d\mathbf{r}' \times (\mathbf{r} - \mathbf{r}') &= d\mathbf{r}' \times (\mathbf{r} - \mathbf{R}_O) \;-\; d\mathbf{r}' \times (\mathbf{r}' - \mathbf{R}_O) \\ &\qquad \langle\!\langle \text{ where the second term vanishes since } d\mathbf{r}' \parallel (\mathbf{r}' - \mathbf{R}_O) \parallel (\mathbf{r}'_2 - \mathbf{r}'_1) \,\rangle\!\rangle \\ &\equiv d\mathbf{r}' \times (\mathbf{r} - \mathbf{R}_O) \\ &= d\vec{\ell} \times \vec{h}, \end{aligned} \tag{42}$$

where $d\vec{\ell} = d\mathbf{r}'$ is the infinitesimal length element along the straight segment, and $\vec{h} = \mathbf{r} - \mathbf{R}_O$ is the *height* of the triangle (41). In other words, $\vec{h}$ is the line to the point $\mathbf{r}$ where we measure the magnetic field from the wire segment — or from the extrapolated straight line of the wire segment — in the direction $\perp$ to the segment.

Note: if we measure the magnetic field at a point $\mathbf{r}$ which happens to lie right on the extrapolated straight line of the wire segment, then $\vec{h} = 0$ and hence $d\vec{\ell} \times \vec{h} \equiv 0$. Consequently, the whole BSL integral vanishes regardless of the denominator's details, and the magnetic field of the segment is zero. Thus, *straight segments 'pointing' directly towards or directly away from* $\mathbf{r}$ *do not contribute to the magnetic field at* $\mathbf{r}$.

For $\vec{h} \neq 0$, the direction of the magnetic field is the direction of the vector product $d\vec{\ell} \times \vec{h}$ in the numerator of the BSL integral. This direction is $\perp$ to the wire and to the $\vec{h}$; in other words, the direction of $\mathbf{B}(\mathbf{r})$ is $\perp$ to the whole triangle (41). The specific perpendicular obtains from the right screw rule: If from your point of view, the current flows in the clockwise direction around $\mathbf{r}$ — as it does on figure (41)— then take the perpendicular which points away from you. OOH, if you see the current flows counterclockwise around $\mathbf{r}$, then take the perpendicular which points towards you.

Now that we know the direction of the magnetic field, let's find its magnitude

$$B = \frac{\mu_0 I}{4\pi} \times \int\limits_{\ell_1}^{\ell_2} \frac{d\ell \times h}{|\mathbf{r}' - \mathbf{r}|^3} \tag{43}$$

In this formula, ℓ is the coordinate along the wire; let's I take it's origin $\ell = 0$ to be the point O where the height $\vec{h}$ of the triangle touches the wire or the extrapolated line of the wire. In terms of this ℓ,

$$|\mathbf{r}' - \mathbf{r}|^2 = \ell^2 + h^2 \implies |\mathbf{r}' - \mathbf{r}|^3 = \left(\ell^2 + h^2\right)^{3/2}, \tag{44}$$

so the BSL integral (43) becomes

$$B = \frac{\mu_0 I}{4\pi} \times \int\limits_{\ell_1}^{\ell_2} \frac{h \times d\ell}{\left(\ell^2 + h^2\right)^{3/2}}. \tag{45}$$

To evaluate *this* integral, we proceed similarly to eqs. (20) through (24): we change the integration variable ℓ to the angle

$$\alpha = \operatorname{arc\,ctan} \frac{-\ell}{h} \implies \ell = -h \times \operatorname{ctan} \alpha, \tag{46}$$

hence

$$\frac{d\ell}{\left(\ell^2 + h^2\right)^{3/2}} = \frac{d(-\cos\alpha)}{h^2} \tag{47}$$

and therefore

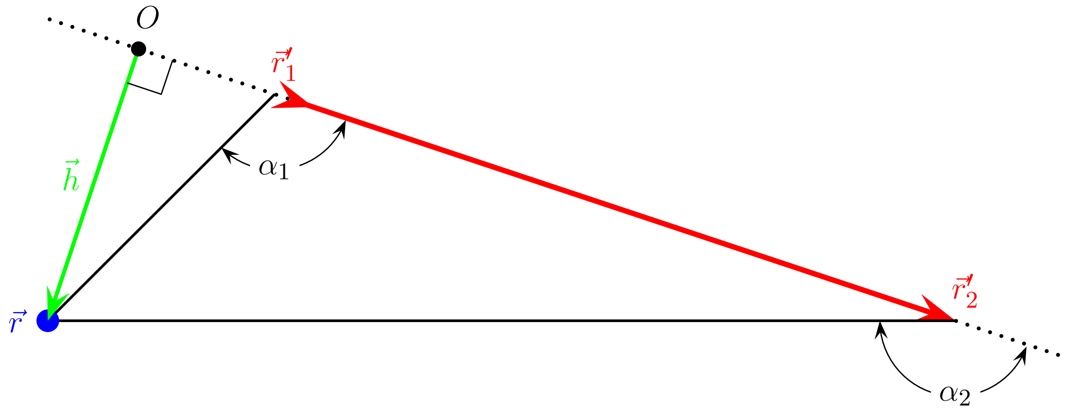
$$\int_{\ell_1}^{\ell_2} \frac{h \times d\ell}{\left(\ell^2 + h^2\right)^{3/2}} = \int_{\alpha_1}^{\alpha_2} \frac{d(-\cos\alpha)}{h} = \frac{\cos\alpha_1 - \cos\alpha_2}{h} \tag{48}$$

where the angles α_1 and α_2 are exactly as shown on the diagram (41).

Altogether, the magnetic field of a straight wire segment is

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 \times I}{4\pi h} \times (\cos\alpha_1 - \cos\alpha_2) \times \mathbf{n} \tag{49}$$

where the height h and the angles α_1 and α_2 are as shown on figure below



(41)

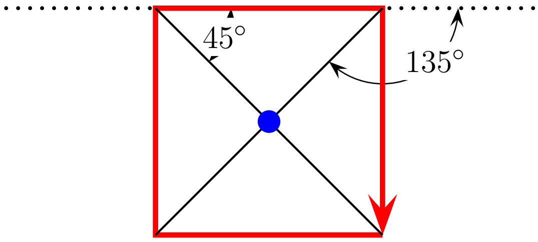
and $\mathbf{n}$ is the unit vector $\perp$ to the whole triangle in the direction given by the right-screw rule.

Note: in the limit of infinitely long segment in both directions, $\alpha_1 \to 0$, $\alpha_2 \to \pi$, hence $\cos\alpha_1 - \cos\alpha_2 \to 2$, and the magnetic field of the segment agrees with the formula for an infinite wire,

$$B_\infty = \frac{\mu_0 \times I}{2\pi h}. \tag{50}$$

EXAMPLE#4: A SQUARE LOOP

Consider a closed loop of wire in the shape of an $a \times a$ square:



(51)

Let's calculate the magnetic field at the center of the square (shown in blue).

The square wire consists of 4 similar straight-line segments, so all we need is to evaluate eq. (49) for the magnetic field due to each segment, and then total up the 4 segments' contributions. For each segment, $h = \frac{1}{2}a$, $\alpha_1 = 45°$, $\alpha_2 = 135°$, hence

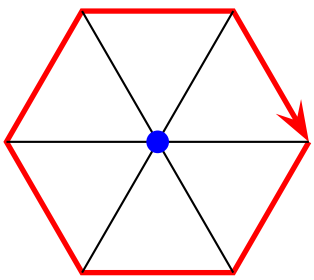
$$B_{1\,\text{segment}} = \frac{\mu_0 \times I}{4\pi(a/2)} \times \left(\cos 45° - \cos 135° = \sqrt{2}\right) = \frac{\sqrt{2}}{2\pi} \times \frac{\mu_0 I}{a}. \tag{52}$$

Also, for each segment the triangle spanning the wire and the center of the square where we measure **B** lies in the plane of the square, so the direction of the magnetic field due to each segment is $\perp$ to the whole square. Specifically, the magnetic field points into the page since in each segment the current flows clockwise around the center. Thus, altogether, *the magnetic field points into the screen and its magnitude is*

$$B^{\text{whole}}_{\text{square}} = 4 \times B_{1\,\text{segment}} = \frac{2\sqrt{2}}{\pi} \times \frac{\mu_0 I}{a}. \tag{53}$$

EXAMPLE#5: SYMMETRIC N-SIDED POLYGON

In this example the wire also makes a complete loop, this time in the shape of symmetric N–sided polygon with side a, for example

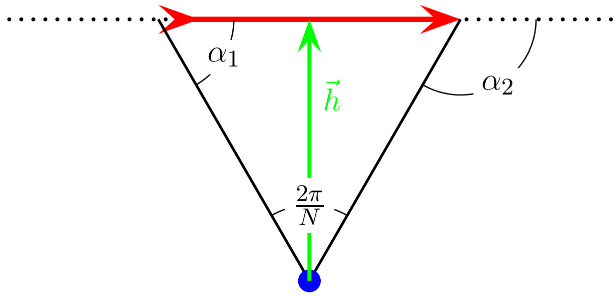


(54)

Again, we focus on the magnetic field at the center of the polygon, so by symmetry each segment of the wire contributes a similar $B_{1\,\text{segment}}$. All these contributions are directed $\perp$ to the polygon, specifically into the screen, hence

$$\mathbf{B}_{\text{polygon}} \;=\; N \times B_{1\,\text{segment}} \times \mathbf{n} \tag{55}$$

where $\mathbf{n}$ is the unit vector pointing into the page. Now let's draw a single segment of the wire and the triangle connecting it to the center point where the magnetic field is measured:



Simple geometry+trigonometry for this triangle gives us

$$\alpha_{1,2} \;=\; \frac{\pi}{2} \mp \frac{\pi}{N}\,, \tag{56}$$

$$\cos\alpha_1 - \cos\alpha_2 \;=\; 2\sin\frac{\pi}{N}\,, \tag{57}$$

$$h \;=\; \frac{a}{2} \times \operatorname{ctan}\frac{\pi}{N}\,, \tag{58}$$

and therefore

$$B_{1\,\text{segment}} \;=\; \frac{\mu_0 I}{4\pi} \times \frac{\cos\alpha_1 - \cos\alpha_2}{h} \;=\; \frac{\mu_0 I}{4\pi} \times \frac{2\tan\frac{\pi}{N}}{a} \times 2\sin\frac{\pi}{N}\,. \tag{59}$$

Finally, combining all N segments, we find *the magnetic field at the center of the polygon is*

$$\begin{aligned} B \;=\; N \times B_{1\,\text{segment}} \;&=\; N \times \frac{\mu_0 \times I}{4\pi a} \times 2\sin\frac{\pi}{N} \times 2\tan\frac{\pi}{N} \\ &=\; \frac{\mu_0 \times I}{\text{perimeter} = Na} \times \frac{N^2}{\pi} \times \sin\frac{\pi}{N} \times \tan\frac{\pi}{N}\,. \end{aligned} \tag{60}$$

To check this formula, we first plug in $N = 4$ for the square and compare with eq. (53)

from the previous example:

$$\text{for } N=4,\quad \frac{N^2}{\pi}\times\sin\frac{\pi}{N}\times\tan\frac{\pi}{N}\;=\;\frac{16}{\pi}\times\frac{\sqrt{2}}{2}\times 1\;=\;\frac{8\sqrt{2}}{\pi}\,, \tag{61}$$

hence

$$B[\text{from eq. (60)}]\;=\;\frac{\mu_0 I}{4a}\times\frac{8\sqrt{2}}{\pi}\;=\;\frac{2\sqrt{2}}{\pi}\times\frac{\mu_0 I}{a}\,, \tag{62}$$

in complete agreement with eq. (53) for the square.

Second, let's take a large N limit in which the polygon becomes a circular ring of perimeter $Na=2\pi R$. In this limit,

$$\lim_{N\to\infty}\left(\frac{N^2}{\pi}\times\sin\frac{\pi}{N}\times\tan\frac{\pi}{N}\right)\;=\;\pi, \tag{63}$$

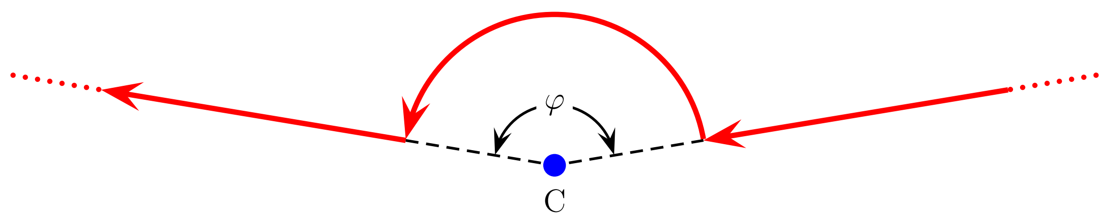
hence the magnetic field at the center of the polygon becomes

$$B[\text{from eq. (60)}]\;\to\;\frac{\mu_0\times I}{2\pi R}\times\pi\;=\;\frac{\mu_0\times I}{2R}\,, \tag{64}$$

which indeed agrees with the magnetic field (38) at the center of a circular ring.

Example#6: A Circular Arc

As our final example, let's calculate the magnetic field at the center of a circular arc. More generally, consider a wire comprised of a semicircle and two straight segments



(65)

and calculate the magnetic field at point C at the center of the circular arc.

Note: besides being at the center of the arc, the point C happens to lie on the straight-line extrapolations of the straight segments of the wire. Consequently, the two straight segments do not contribute to the magnetic field $\mathbf{B}(C)$ at that point; instead, the entire field at point C comes from the circular arc segment only.

Let's parametrize the arc segment by the angle ϕ from the point C; ϕ ranges from ϕ_0 to $\phi_0 + \varphi$. In terms of ϕ,

$$\begin{aligned} \mathbf{r}' &= R\cos\phi\,\hat{\mathbf{x}} \;+\; R\sin\phi\,\hat{\mathbf{y}}, &(66)\\ \mathbf{r} - \mathbf{r}' &= -R\cos\phi\,\hat{\mathbf{x}} \;-\; R\sin\phi\,\hat{\mathbf{y}}, &(67)\\ d\mathbf{r}' &= R(-\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}})\,d\phi, &(68) \end{aligned}$$

hence in the numerator of the Biot–Savart–Laplace integral

$$\begin{aligned} d\mathbf{r}' \times (\mathbf{r} - \mathbf{r}') &= R^2\,d\phi \begin{pmatrix} \sin\phi\cos\phi(\hat{\mathbf{x}}\times\hat{\mathbf{x}} + \hat{\mathbf{y}}\times\hat{\mathbf{y}} = \mathbf{0}) + \sin^2\phi(\hat{\mathbf{x}}\times\hat{\mathbf{y}} = +\hat{\mathbf{z}}) \\ - \cos^2\phi(\hat{\mathbf{y}}\times\hat{\mathbf{x}} = -\hat{\mathbf{z}}) \end{pmatrix} &(69)\\ &= R^2\,d\phi\big(\sin^2\phi + \cos^2\phi)\hat{\mathbf{z}} = R^2\,d\phi\,\hat{\mathbf{z}}. \end{aligned}$$

Note: the direction of this vector product is always vertically Up, $\perp$ to the plane of the ring, so the magnetic field's direction is going to be vertically Up.

As to the denominator of the BSL formula, the whole circular arc is at constant distance $|\mathbf{r} - \mathbf{r}'| \equiv R$ from the ring's center, so the denominator is a constant R^3. Altogether,

$$\int\limits_{\text{arc}} \frac{d\mathbf{r}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} = \int\limits_{\phi_0}^{\phi_0+\varphi} \frac{R^2\,\hat{\mathbf{z}}\,d\phi}{R^3} = \frac{\varphi}{R}\,\hat{\mathbf{z}}\,, \qquad (70)$$

so the magnetic field at point C is

$$\mathbf{B}(\mathrm{C}) = \frac{\mu_0 I}{4\pi R} \times \varphi \times \hat{\mathbf{z}}\,. \qquad (71)$$

Note: the φ angle in this formula should be taken in radians.

Magnetic Fields of Thick Conductors

The Biot–Savart–Laplace formula

$$\mathbf{B}(\mathbf{r}) \;=\; \frac{\mu_0}{4\pi}\int\limits_{\text{wire}} I\,d\mathbf{r}'\times\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \tag{11}$$

gives the magnetic field of a steady current flowing through a thin wire which may be approximated by an infinitely thin line, straight or curved. For such a wire, it does not matter how the current is distributed across the wire's cross-section, only the net current I enter the formula.

But sometimes we have currents flowing through the volume of a conductor which is too thick to be approximated as a line. For such conductors,

$$\int\limits_{\text{wire}} I\,d\mathbf{r}' \;\xrightarrow[\text{becomes}]{}\; \iiint\limits_{\substack{\text{conductor's}\\ \text{volume}}} d^3\mathrm{Vol}'\,\mathbf{J}(\mathbf{r}') \tag{72}$$

where $\mathbf{J}(\mathbf{r}')$ is the *current density* at the point $\mathbf{r}'$. Consequently, the Biot–Savart–Laplace equation for such current densities generalizes to

$$\mathbf{B}(\mathbf{r}) \;=\; \frac{\mu_0}{4\pi}\iiint d^3\mathrm{Vol}\;\mathbf{J}(\mathbf{r}')\times\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\,. \tag{73}$$

Likewise, for a steady current flowing along a conducting surface with density $\mathbf{K}(\mathbf{r}')$, the BSL equation becomes

$$\mathbf{B}(\mathbf{r}) \;=\; \frac{\mu_0}{4\pi}\iint d^2A\;\mathbf{K}(\mathbf{r}')\times\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\,. \tag{74}$$

Note: the volume current density J is defined as current per unit of area $\perp$to the current, while the surface current density K is defined as current per unit of *length* $\perp$to the current. As a vector, $\mathbf{K}$ must be tangent to the surface; for example, for the surface spanning the (x, y) plane, we may have any K_x and K_y components, but the K_z component must vanish.

EXAMPLE#7: INFINITE FLAT CURRENT SHEET

As our final example, consider infinite flat current sheet — *i.e.*, a 2D conducting surface — carrying a uniform surface current $\mathbf{K}$. Let's choose our coordinates such that the current flows in $+\hat{\mathbf{y}}$ direction along the surface spanning the (x,y) plane. Then the magnetic field at some generic point $\mathbf{r}=(x,y,z)$ is given by eq. (74), specifically,

$$\mathbf{B}(x,y,z) = \frac{\mu_0 K}{4\pi}\iint\limits_{\text{plane}} dx'\,dy'\,\frac{\hat{\mathbf{y}}\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\,. \tag{75}$$

In the numerator inside the integral here,

$$\mathbf{r} - \mathbf{r}' = (x-x')\hat{\mathbf{x}} + (y-y')\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}, \tag{76}$$

$$\hat{\mathbf{y}}\times(\mathbf{r}-\mathbf{r}') = (x-x')(-\hat{\mathbf{z}}) + (y-y')\mathbf{0} + z(+\hat{\mathbf{x}}) = -(x-x')\,\hat{\mathbf{z}} + z\,\hat{\mathbf{x}}, \tag{77}$$

while in the denominator

$$|\mathbf{r}-\mathbf{r}'|^3 = \big((x-x')^2+(y-y')^2+z^2\big)^{3/2}. \tag{78}$$

To simplify these expressions, let's change the integration variables x' and y' to the polar coordinates (s,ϕ) centered at (x,y), thus

$$x' = x + s\times\cos\phi, \tag{79}$$

$$y' = y + s\times\sin\phi, \tag{80}$$

$$dx'\,dy' = s\,ds\,d\phi, \tag{81}$$

$$\hat{\mathbf{y}}\times(\mathbf{r}-\mathbf{r}') = -s\cos\phi\,\hat{\mathbf{z}} + z\,\hat{\mathbf{x}}, \tag{82}$$

$$|\mathbf{r}-\mathbf{r}'|^3 = \big(s^2+z^2\big)^{3/2}. \tag{83}$$

Plugging all these formulae into eq. (75), we arrive at

$$\mathbf{B}(s,y,z) = \frac{\mu_0 K}{4\pi}\int\limits_0^\infty ds\,s\int\limits_0^{2\pi} d\phi\,\frac{-s\,\cos\phi\,\hat{\mathbf{z}} + z\,\hat{\mathbf{x}}}{(s^2+z^2)^{3/2}}\,. \tag{84}$$

Integrating over the polar angle ϕ, we immediately obtain

$$\int_0^{2\pi} d\phi\,\bigl(-s\,\cos\phi\,\hat{\mathbf{z}} + z\,\hat{\mathbf{x}}\bigr) \;=\; -s\,\hat{\mathbf{z}}\int_0^{2\pi}\cos\phi\,d\phi + z\,\hat{\mathbf{x}}\int_0^{2\pi}d\phi \;=\; -s\hat{\mathbf{z}}\times 0 + z\hat{\mathbf{x}}\times 2\pi \;=\; 2\pi z\,\hat{\mathbf{x}}, \tag{85}$$

hence

$$\mathbf{B}(x,y,z) \;=\; \frac{\mu_0 K}{2}\times\hat{\mathbf{x}}\times\int_0^\infty \frac{zs\,ds}{(s^2+z^2)^{3/2}}\,. \tag{86}$$

Note: the magnetic field everywhere points in either $+\hat{\mathbf{x}}$ or $-\hat{\mathbf{x}}$ direction, depending on the sign of the remaining integral in this formula.

To evaluate this integral, we change variables from s to $t = s^2 + z^2$, thus

$$zs\,ds \;=\; \frac{z}{2}\,dt, \quad (z^2+s^2)^{3/2} \;=\; t^{3/2}, \tag{87}$$

hence

$$\int_0^\infty \frac{zs\,ds}{(s^2+z^2)^{3/2}} \;=\; \frac{z}{2}\int_{z^2}^\infty \frac{dt}{t^{3/2}} \;=\; \frac{z}{2}\times\left[\frac{-2}{\sqrt{t}}\right]_{z^2}^{+\infty} \;=\; \frac{z}{2}\times\left[0 - \frac{-2}{\sqrt{z^2}}\right] \;=\; \frac{z}{\sqrt{z^2}} \;=\; \mathrm{sign}(z), \tag{88}$$

and therefore

$$\mathbf{B}(x,y,z) \;=\; \frac{\mu_0 K}{2}\times\mathrm{sign}(z)\,\hat{\mathbf{x}} \tag{89}$$

Note: the magnetic field of the infinite current sheet is completely uniform above the sheet ($z > 0$) and likewise completely uniform below the sheet ($z < 0$), but at the current sheet itself ($z = 0$) there is a discintinuous jump. Its magnitude B of the field is the same $\mu_0 K/2$ above and below the sheet, but the directions are opposite: above the sheet, the magnetic field points in the $+\hat{\mathbf{x}}$ direction while above the sheet it points in the $-\hat{\mathbf{x}}$ direction. Relative to the currents direction $+\hat{\mathbf{y}}$, the magnetic field above the sheet points 90° to the right of the current, while below the sheet it points 90° to the left of the current.

In a later lecture we shall learn that the discontinuity of the magnetic field across the current sheet

$$\mathrm{disc}(\mathbf{B}) \;=\; \mu_0 \mathbf{K} \times \mathbf{n} \tag{90}$$

follows from the Ampere Law, just like the discontinuity of the electric field across a charged surface

$$\mathrm{disc}(\mathbf{E}) \;=\; \frac{\sigma}{\epsilon_0}\,\mathbf{n} \tag{91}$$

follows from the Gauss Law. However, the discontinuities of the electric and the magnetic fields have different directions: while the electric discontinuity disc($\mathbf{E}$) is $\perp$to the whole charged surface, the magnetic discontinuity disc($\mathbf{B}$) is tangent to the current sheet but $\perp$to the current's direction.