BOUNDARY PROBLEMS FOR DIELECTRICS

In the bulk of a uniform dielectric, the bound charges shadow the free charges,

$$\rho_b \;=\; -\frac{\epsilon-1}{\epsilon}\times\rho_f \quad\Longrightarrow\quad \rho_{\text{net}} \;=\; +\frac{1}{\epsilon}\times\rho_f. \tag{1}$$

Alas, the bound surface charges on the outer boundary of a dielectric are more complicated. Therefore, dielectrics and their boundaries distort the electric fields of free charges even when they are placed completely outside the dielectric. In these notes, I give two examples of such dielectric boundary problems.

But first, a few general rules:

1. In vacuum or inside any uniform dielectric, the electric potential $V(x,y,z)$ obeys the Poisson equation

$$\nabla^2 V(x,y,z) \;=\; -\frac{\rho_{\text{net}}}{\epsilon_0} \;=\; -\frac{\rho_f(x,y,z)}{\epsilon(x,y,z)\epsilon_0} \tag{2}$$

   where $\epsilon(x,y,z)$ depends only on whether (x,y,z) is inside or outside the dielectric.

2. The potential $V(x,y,z)$ is continuous across the dielectric boundaries. This assures continuity of the tangential components $\mathbf{E}^{\|}$ of the electric tension field.

3. The normal component of the tension field is not continuous, but $D^{\perp}$ — and hence $\epsilon\times E^{\perp} = D^{\perp}/\epsilon_0$ — should be continuous.

- Thus, as one approaches the dielectric's surface from the inside or from the outside the dielectric.

$$\lim V^{\text{outside}} \;=\; \lim V^{\text{inside}} \quad\text{but}\quad \lim E_{\perp}^{\text{outside}} \;=\; \epsilon\times\lim E_{\perp}^{\text{inside}} \tag{3}$$

  Or for a boundary between two kinds of dielectric with different dielectric constants,

$$\lim V^{(1)} \;=\; \lim V^{(2)} \quad\text{but}\quad \epsilon_1\times\lim E_{\perp}^{(1)} \;=\; \epsilon_2\times\lim E_{\perp}^{(2)}\,. \tag{4}$$

Example#1: A Dielectric Ball in External Electric Field

Consider a solid ball made of a uniform dielectric in an external electric field $\mathbf{E}_0$. That is, far away from the ball

$$\text{for } r \to \infty, \quad V \;\approx\; -E_0 \times z \;=\; -E_0 \times r\,\cos\theta, \tag{5}$$

but closer to the ball the field is distorted by the bound charges on the ball's surface.

To solve this problem, I'll start by writing down the general formulae for the potential $V(r,\theta,\phi)$ inside and outside the ball using the separation of coordinates method, and then I shall apply the boundary conditions (3) at the surface of the ball. Since there are no free charges anywhere inside or outside the ball, the potential obeys the Laplace equation $\nabla^2 V(r,\theta,\phi) = 0$. Also, by the axial symmetry of the problem, we may take V to be ϕ–independent. Hence separating the dependence of V on the remaining r and θ coordinates, we have

$$\begin{aligned}
\text{for } r < R, \quad V^{\text{inside}}(r,\theta) \;&=\; \sum_{\ell=0}^{\infty} \left(A_\ell \times r^\ell \,+\, \frac{B_\ell}{r^{\ell+1}} \right) \times P_\ell(\cos\theta), \\
\text{for } r > R, \quad V^{\text{outside}}(r,\theta) \;&=\; \sum_{\ell=0}^{\infty} \left(C_\ell \times r^\ell \,+\, \frac{D_\ell}{r^{\ell+1}} \right) \times P_\ell(\cos\theta),
\end{aligned} \tag{6}$$

for some constant coefficients $A_\ell, B_\ell, C_\ell, D_\ell$. In particular, the B_ℓ coefficients govern the asymptotic behavior of the potential at $r \to 0$ — the center of the dielectric ball. In the example at hand, there are no charges or multipoles at the center, so the potential should be completely regular for $r \to 0$, which means

$$\text{all } B_\ell \;=\; 0. \tag{7}$$

Likewise, the C_ℓ coefficients govern the asymptotic behavior of the potential far away from the ball:

$$\text{for } r \to \infty, \quad V^{\text{outside}}(r,\theta) \approx \sum_{\ell=}^{\infty} C_\ell \times r^\ell \times P_\ell(\cos\theta); \tag{8}$$

Comparing this asymptotics to eq. (5), we immediately see that in our example

$$C_1 \;=\; -E_0, \quad \text{all other } C_\ell \;=\; 0. \tag{9}$$

The remaining coefficients A_ℓ and D_ℓ follow from the boundary conditions (3) at the surface of the dielectric ball. In particular

$$V^{\text{inside}}(R,\theta) \;=\; V^{\text{outside}}(R,\theta) \quad \text{for all } \theta \tag{10}$$

calls for

$$A_\ell \times R^\ell \;+\; \frac{B_\ell}{R^{\ell+1}} \;=\; C_\ell \times R^\ell \;+\; \frac{D_\ell}{R^{\ell+1}}\,. \tag{11}$$

Also, the electric field $\perp$ to the surface is the radial component

$$E_r \;=\; -\frac{\partial V}{\partial r} = \sum_{\ell=0}^{\infty} P_\ell(\cos\theta) \times \begin{cases} \left(-\ell A_\ell \times r^{\ell-1} \;+\; \dfrac{(\ell+1)B_\ell}{r^{\ell+2}}\right) & \text{inside the ball,} \\ \left(-\ell C_\ell \times r^{\ell-1} \;+\; \dfrac{(\ell+1)D_\ell}{r^{\ell+2}}\right) & \text{outside the ball.} \end{cases} \tag{12}$$

Hence, requiring that

$$E_r^{\text{outside}}(R,\theta) \;=\; \epsilon \times E_r^{\text{inside}}(R,\theta) \quad \text{for all } \theta \tag{13}$$

calls for

$$-\epsilon\ell A_\ell \times R^{\ell-1} \;+\; \frac{\epsilon(\ell+1)B_\ell}{R^{\ell+2}} \;=\; -\ell C_\ell \times R^{\ell-1} \;+\; \frac{(\ell+1)D_\ell}{R^{\ell+2}}\,. \tag{14}$$

It remains to solve the equations (11) and (14) for the A_ℓ and D_ℓ coefficients on terms of the B_ℓ and C_ℓ. For all the $\ell \neq 1$ modes, the solution is trivial:

$$B_\ell \;=\; C_\ell \;=\; 0 \quad\Longrightarrow\quad A_\ell \;=\; D_\ell \;=\; 0. \tag{15}$$

For the remaining $\ell = 1$ mode, we have $B_1 = 0$, $C_1 = -E_0$, hence

$$A_1 \times R \;=\; -E_0 \times R \;+\; \frac{D_1}{R^2} \quad \text{and} \quad -\epsilon A_1 \;=\; +E_0 \;+\; \frac{2D_1}{R^3}\,, \tag{16}$$

and therefore

$$\begin{aligned}
\epsilon A_1 + 2A_1 &= \left(-E_0 - \frac{2D_1}{R_3}\right) + 2\left(-E_0 + \frac{D_1}{R_3}\right) = -3E_0 \\
\Longrightarrow \quad A_1 &= -\frac{3}{\epsilon+2} \times E_0, \\
\epsilon D_1 + 2D_1 &= \epsilon\Big(A_1 \times R^3 + E_0 \times R^3\Big) + \Big(-\epsilon A_1 \times R^3 - E_0 \times R^3\Big) \\
&= (\epsilon - 1) \times E_0 \times R^3 \\
\Longrightarrow \quad D_1 &= \frac{\epsilon - 1}{\epsilon + 2} \times E_0 R^3.
\end{aligned} \tag{17}$$

Altogether, outside the ball

$$V(r,\theta) = E_0 \left(-r + \frac{\epsilon - 1}{\epsilon + 2} \times \frac{R^3}{r^2}\right) \times \cos\theta, \tag{18}$$

or in terms of the electric field

$$\mathbf{E}^{\text{outside}} = \mathbf{E}_0 + \mathbf{E}[\text{pure dipole } \mathbf{p}] \tag{19}$$

where $\mathbf{p}$ — the net dipole moment induced in the dielectric ball — is

$$\mathbf{p} = 4\pi\epsilon_0 \frac{\epsilon - 1}{\epsilon + 2} R^3 \mathbf{E}_0 \,. \tag{20}$$

As to the inside of the ball,

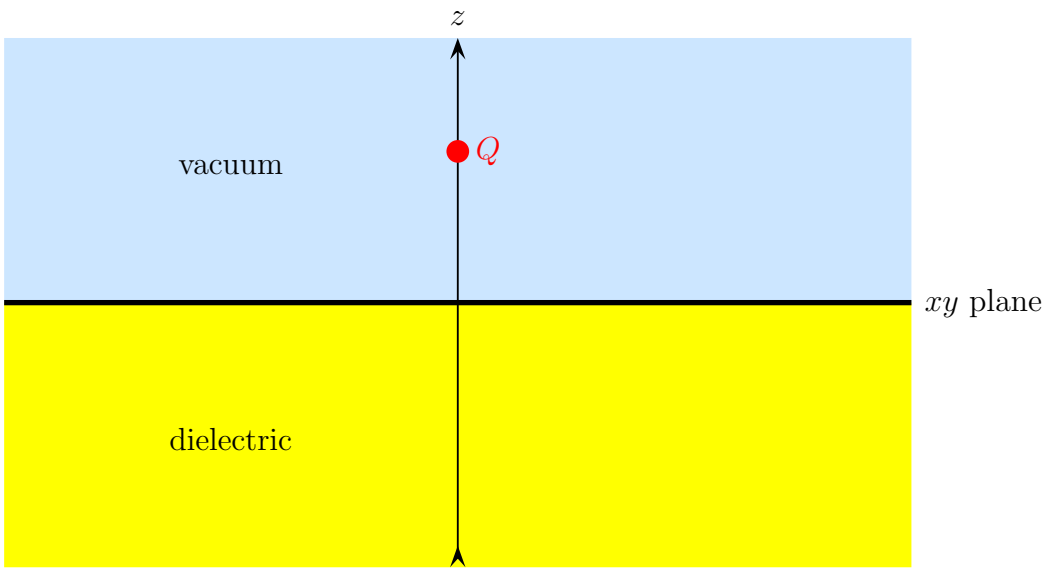
$$V^{\text{inside}} = -\frac{3}{\epsilon+2} \times E_0 \times r \cos\theta, \tag{21}$$

which means *uniform* electric field

$$\mathbf{E}^{\text{inside}} = +\frac{3}{\epsilon+2} \mathbf{E}_0 \,. \tag{22}$$

Example#2: A Point Charge Outside a Dielectric Half–Space

Consider a large and thick slab of a uniform dielectric. Let's put a point charge Q above the upper surface of the slab, but so close to it that the bottom surface of the slab — as well as the left, right, forward, and backward surfaces — are comparatively so much further away from the charge that we may approximate them as being infinitely far away. In this approximation, the dielectric fill up the whole bottom half ($z < 0$) of the 3D space, while upper half-space remains empty except for the point charge Q at $x = y = 0$, $z = +a$. Pictorially,



(23)

Our task is to find the potential and hence the electric field both inside and outside the dielectric.

Before we address this problem, let's for the moment replace the dielectric at $z < 0$ with a conductor. For that situation, the simpest solution is in terms of the mirror image charge $Q_I = -Q$ at $x = y = 0$, $z = -a$. Specifically,

$$\begin{aligned} \text{for } z > 0, \quad \mathbf{E}(x,y,z) &= \mathbf{E}[Q](x,y,z) + \mathbf{E}[Q_I](x,y,z) \\ \text{but for } z < 0, \quad \mathbf{E}(x,y,z) &= 0. \end{aligned}$$

Of course, in reality there is no image charge inside a conductor. Instead, there is a surface

charge at $z = 0$ — the top of the conductor —

$$\sigma_c(x,y) \;=\; -\frac{Q}{2\pi} \times \frac{a}{\left(x^2+y^2+a^2\right)^{3/2}} \tag{24}$$

whose electric field looks like the field of the image charge above the surface while below the surface it cancels the field of Q,

$$\mathbf{E}[\sigma_c](x,y,z) \;=\; \begin{cases} \mathbf{E}[Q_I](x,y,x) & \text{for } z>0, \\ -\mathbf{E}[Q](x,y,z) & \text{for } z<0. \end{cases} \tag{25}$$

Now let's go back to the dielectric case. Similar to the conductor, there would not be any charges inside the dielectric but only on the surface. Indeed, in a uniform dielectric the bound volume charges shadow the free charges, $\rho_b = -\frac{\epsilon-1}{\epsilon}\rho_f$, and since in our case there are no free charges inside the dielectric, there also would not be any bound charges below the surface. However, on the surface we would generally have some bound charges, hence

$$\text{at } z=0, \quad \sigma(x,y) \;=\; \sigma_b(x,y) \;=\; P_z(x,y,0). \tag{26}$$

Altogether, we have these charges on the $z = 0$ plane, and the point charge Q above the plane, and this is it — there are no other charges anywhere else.

The hard question is to find the surface charge density $\sigma_b(x,y)$, and I claim that the solution is similar to the charges (24) on a conducting plane but with a smaller coefficient $\nu < 1$,

$$\sigma_b(x,y) \;=\; \nu \times \sigma_c(x,y) \;=\; -\nu \times \frac{Q}{2\pi} \times \frac{a}{\left(x^2+y^2+a^2\right)^{3/2}}\,. \tag{27}$$

In a couple of pages we shall see that

$$\nu \;=\; \frac{\epsilon-1}{\epsilon+1}\,, \tag{28}$$

but for the moment let's allow a general $0 < \nu < 1$ and calculate the net electric field due to the bound charges (27) as well as the point charge Q. In light of eq. (25), the field generated

by just the bound charges (27) is

$$\mathbf{E}[\sigma_b](x,y,z) \;=\; \begin{cases} +\nu\mathbf{E}[Q_I](x,y,x) & \text{for } z>0, \\ -\nu\mathbf{E}[Q](x,y,z) & \text{for } z<0. \end{cases} \tag{29}$$

hence the net electric field in the system is

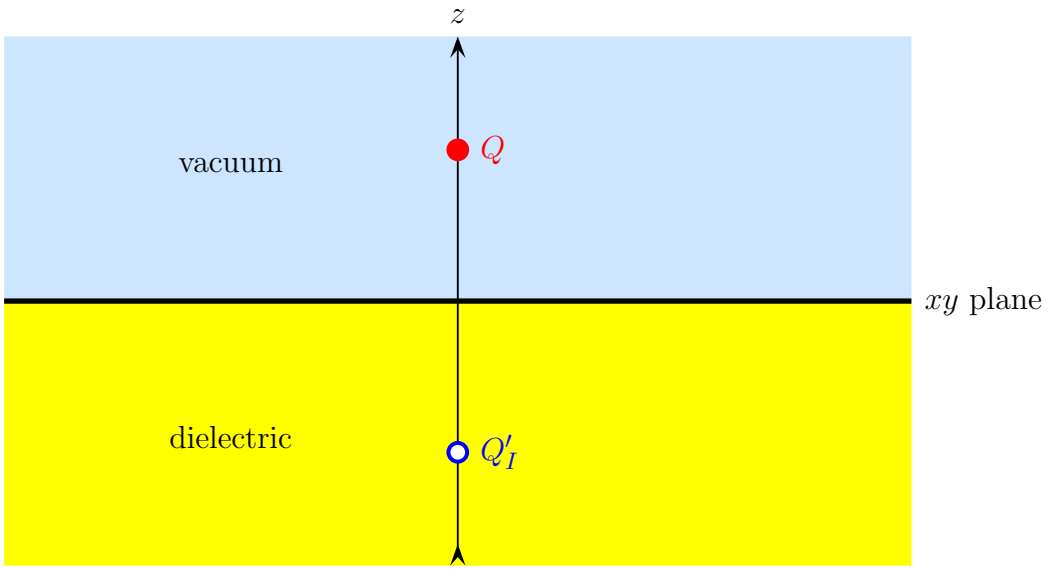
$$\begin{aligned} &\text{For } z<0 \text{ (inside the dielectric)}, \\ \mathbf{E}^{\rm net}(x,y,z) \;&=\; \mathbf{E}[Q](x,y,z) \;-\; \nu\mathbf{E}[Q](x,y,z) \\ &=\; (1-\nu)\times\mathbf{E}[Q](x,y,z), \end{aligned} \tag{30}$$

$$\begin{aligned} &\text{For } z>0 \text{ (above the dielectric)}, \\ \mathbf{E}^{\rm net}(x,y,z) \;&=\; \mathbf{E}[Q](x,y,z) \;+\; \nu\mathbf{E}[Q_I](x,y,z) \\ &=\; \mathbf{E}[Q](x,y,z) \;+\; \mathbf{E}[Q'_I](x,y,z) \end{aligned} \tag{31}$$

where Q'_I is the *reduced* image charge

$$Q'_I \;=\; \nu\times Q_I \;=\; -\nu\times Q \quad \text{located at } x=y=0,\; z=-a. \tag{32}$$

Here is the picture showing both the original charge Q (solid red circle) and the reduced image charge Q'_I (open blue circle):



(33)

Now let's prove that eqs. (30) and (31) indeed solve the problem. First, both above and

below the $z = 0$ surface the net electric field obeys the electrostatic equations $\nabla \times \mathbf{E} = 0$ (this should be obvious) and $\epsilon_0 \nabla \cdot \mathbf{E} = \rho$. Indeed,

$$\begin{aligned} \text{for } z > 0, \quad \nabla \cdot \mathbf{E}^{\text{net}}(x,y,z) &= Q \times \delta(x)\delta(y)\delta(z-a) \;+\; Q'_I \times \delta(x)\delta(y)\delta(z+a) \\ &= Q \times \delta(x)\delta(y)\delta(z-a) \;+\; 0, \\ \text{for } z < 0, \quad \nabla \cdot \mathbf{E}^{\text{net}}(x,y,z) &= (1-\nu) \times Q \times \delta(x)\delta(y)\delta(z-a) \;=\; 0. \end{aligned} \tag{34}$$

Second, we must check the boundary conditions (3) for $z = 0$. Here we may save a lot of calculations by using the reflection symmetry $z \to -z$, hence for the original mirror charge $Q_I = -Q$,

$$\text{at } z = 0, \quad \mathbf{E}_{x,y}[Q_I] \;=\; -\mathbf{E}_{x,y}[Q] \quad \text{while} \quad \mathbf{E}_z[Q_I] \;=\; +\mathbf{E}[Q]. \tag{35}$$

For the reduced mirror charge Q'_I, these relations become

$$\begin{aligned} \mathbf{E}_{x,y}[Q'_I](x,y,0) &= -\nu \times \mathbf{E}_{x,y}[Q](x,y,0) \\ \text{while} \quad \mathbf{E}_z[Q'_I](x,y,0) &= +\nu \times \mathbf{E}_z[Q](x,y,0), \end{aligned} \tag{36}$$

hence the net electric field just above the boundary is

$$\begin{aligned} {}^{\text{net}}\mathbf{E}^{\text{above}}_{x,y}(x,y,0) &= (1-\nu) \times \mathbf{E}_{x,y}[Q](x,y,0), \\ {}^{\text{net}}\mathbf{E}^{\text{above}}_{z}(x,y,0) &= (1+\nu) \times \mathbf{E}_{z}[Q](x,y,0). \end{aligned} \tag{37}$$

At the same time, the net electric field just below the boundary is simply

$${}^{\text{net}}\mathbf{E}^{\text{below}}_{x,y,z}(x,y,0) \;=\; (1-\nu) \times \mathbf{E}_{x,y,z}[Q](x,y,0) \tag{38}$$

for all 3 components. Comparing these fields to each other, we see that

$${}^{\text{net}}\mathbf{E}^{\text{above}}_{x,y}(x,y,0) \;=\; {}^{\text{net}}\mathbf{E}^{\text{below}}_{x,y}(x,y,0), \tag{39}$$

$$\text{but} \quad {}^{\text{net}}\mathbf{E}^{\text{above}}_{z}(x,y,0) \;=\; \frac{1+\nu}{1-\nu} \times {}^{\text{net}}\mathbf{E}^{\text{below}}_{z}(x,y,0). \tag{40}$$

By inspection, eq. (39) is precisely the boundary condition for the tangential electric field, while eq. (40) looks like the boundary condition for the normal electric fields (3), provided

$$\frac{1+\nu}{1-\nu} \;=\; \epsilon. \tag{41}$$

The last condition sets the value of the ν coefficient we should use for our solution, namely

$$\nu \;=\; \frac{\epsilon-1}{\epsilon+1}\,. \tag{42}$$

Quod erat demonstrandum.