ELECTRIC DIPOLES

In these notes, I write down the electric field of a dipole, and also the net force and the torque on a dipole in the electric field of other charges. For simplicity, I focus on ideal dipoles — also called pure dipoles — where the distance a between the positive and the negative charges is infinitesimal, but the charges are so large that the dipole moment $\mathbf{p}$ is finite.

Electric Field of a Dipole

The potential due to an ideal electric dipole $\mathbf{p}$ is

$$V(\mathbf{r}) \;=\; \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{4\pi\epsilon_0\,r^2}\,, \tag{1}$$

or in terms of spherical coordinates where the North pole ($\theta=0$) points in the direction of the dipole moment $\mathbf{p}$,

$$V(r,\theta) \;=\; \frac{p}{4\pi\epsilon_0}\,\frac{\cos\theta}{r^2}\,. \tag{2}$$

Taking (minus) gradient of this potential, we obtain the dipole's electric field

$$\mathbf{E} \;=\; \frac{p}{4\pi\epsilon_0}\left(\frac{2\cos\theta}{r^3}\,\nabla r\,+\,\frac{\sin\theta}{r^2}\,\nabla\theta\right) \;=\; \frac{p}{4\pi\epsilon_0}\,\frac{1}{r^3}\Big(2\cos\theta\,\hat{\mathbf{r}}\,+\,\sin\theta\,\hat{\boldsymbol{\theta}}\Big). \tag{3}$$

In this formula, the unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ themselves depend on θ and ϕ. Translating them to Cartesian unit vectors, we have

$$\begin{aligned}\hat{\mathbf{r}} \;&=\; \sin\theta\cos\phi\,\hat{\mathbf{x}}\;+\;\sin\theta\sin\phi\,\hat{\mathbf{y}}\;+\;\cos\theta\,\hat{\mathbf{z}},\\ \hat{\boldsymbol{\theta}} \;&=\; \cos\theta\cos\phi\,\hat{\mathbf{x}}\;+\;\cos\theta\sin\phi\,\hat{\mathbf{y}}\;-\;\sin\theta\,\hat{\mathbf{z}},\end{aligned} \tag{4}$$

hence

$$2\cos\theta\,\hat{\mathbf{r}}+\sin\theta\,\hat{\boldsymbol{\theta}} \;=\; 3\sin\theta\cos\theta(\cos\phi\,\hat{\mathbf{x}}+\sin\phi\,\hat{\mathbf{y}})+(2\cos^2\theta-\sin^2\theta\;=\;3\cos^2\theta-1)\,\hat{\mathbf{z}}, \tag{5}$$

and therefore

$$\begin{aligned}
E_x(r,\theta,\phi) &= \frac{p}{4\pi\epsilon_0}\,\frac{3\sin\theta\cos\theta\cos\phi}{r^3}\,, \\
E_y(r,\theta,\phi) &= \frac{p}{4\pi\epsilon_0}\,\frac{3\sin\theta\cos\theta\sin\phi}{r^3}\,, \\
E_z(r,\theta,\phi) &= \frac{p}{4\pi\epsilon_0}\,\frac{3\cos^2\theta-1}{r^3}\,.
\end{aligned}$$

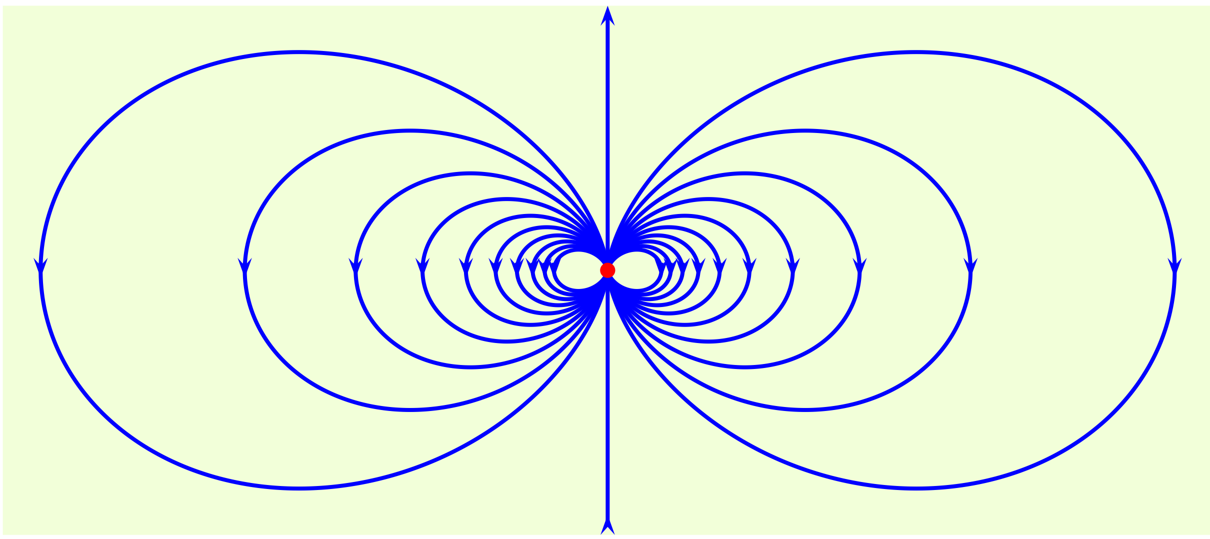
In terms of the (x,y,z) coordinates

$$\begin{aligned}
E_x(x,y,z) &= \frac{p}{4\pi\epsilon_0}\,\frac{3xz}{(x^2+y^2+z^2)^{5/2}}\,, \\
E_y(x,y,z) &= \frac{p}{4\pi\epsilon_0}\,\frac{3yz}{(x^2+y^2+z^2)^{5/2}}\,, \\
E_z(x,y,z) &= \frac{p}{4\pi\epsilon_0}\,\frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}\,,
\end{aligned} \tag{6}$$

or in vector notations,

$$\mathbf{E}(\mathbf{r}) = \frac{3(\mathbf{p}\cdot\widehat{\mathbf{r}})\widehat{\mathbf{r}}-\mathbf{p}}{4\pi\epsilon_0\,r^3}\,. \tag{7}$$

Note that along the dipole axis the electric field points in the direction of the dipole moment $\mathbf{p}$, while in the plane $\perp$ to the dipole axis the field points in the opposite direction from the dipole moment. To get a more general pocture of the dipole's electric field, here is the diagram of the electric field lines in the xz plane:



Force and Torque on a Dipole

Now consider an ideal dipole $\mathbf{p}$ placed in an electric field $\mathbf{E}(x,y,z)$ due to some other sources. If this electric field is uniform, there is no net force on the dipole but there is a net torque. Indeed, the force $\mathbf{F}_+ = +q\mathbf{E}$ acting on the positive charge cancels the opposite force $\mathbf{F}_- = -q\mathbf{E} = -\mathbf{F}_+$ acting on the negative charge — so the net force is zero — but the two forces are acting at different points, which causes a torque. Specifically, the net torque of the two forces is

$$\vec{\tau} \;=\; \mathbf{r}_+ \times \mathbf{F}_+ \;+\; \mathbf{r}_- \times \mathbf{F}_- \;=\; (\mathbf{r}_+ - \mathbf{r}_-) \times q\mathbf{E} \;=\; q(\mathbf{r}_+ - \mathbf{r}_-) \times \mathbf{E}, \tag{8}$$

or in terms of the dipole moment $\mathbf{p} = q(\mathbf{r}_+ - \mathbf{r}_-)$,

$$\vec{\tau} \;=\; \mathbf{p} \times \mathbf{E}. \tag{9}$$

This torque vanishes when the dipole moment $\mathbf{p}$ is parallel to the electric field $\mathbf{E}$. Otherwise, the torque twists the dipole trying to make it align with the field, $\mathbf{p} \to \mathbf{p}' \upuparrows \mathbf{E}$.

When the electric field $\mathbf{E}(x,y,z)$ is not uniform, the two charges of the dipole feel slightly different electric fields, so the net force on the dipole does not quite vanish:

$$\mathbf{F}^{\text{net}} \;=\; q\big(\mathbf{E}(\mathbf{r}_+) \;-\; \mathbf{E}(\mathbf{r}_-)\big) \;\neq\; 0. \tag{10}$$

but for small displacements $\mathbf{a} = \mathbf{r}_+ - \mathbf{r}_-$ between the charges, we may expand the difference between the electric fields acting on them into a power series in $\mathbf{a}$. Let $\mathbf{r}_\pm = \mathbf{r} \pm \frac{1}{2}\mathbf{a}$ where $\mathbf{r}$ is the center of the dipole; then

$$\mathbf{E}(\mathbf{r}_\pm) \;=\; \mathbf{E}(\mathbf{r}) \;\pm\; (\tfrac{1}{2}\mathbf{a}\cdot\nabla)\mathbf{E}\Big|_{@\mathbf{r}} \;+\; \tfrac{1}{2}(\tfrac{1}{2}\mathbf{a}\cdot\nabla)^2\mathbf{E}\Big|_{@\mathbf{r}} \;\pm\; \tfrac{1}{6}(\tfrac{1}{2}\mathbf{a}\cdot\nabla)^3\mathbf{E}\Big|_{@\mathbf{r}} \;+\; \cdots, \tag{11}$$

hence the difference

$$\mathbf{E}(\mathbf{r}_+) \;-\; \mathbf{E}(\mathbf{r}_-) \;=\; (\mathbf{a}\cdot\nabla)\mathbf{E}\Big|_{@\mathbf{r}} \;+\; \tfrac{1}{24}(\mathbf{a}\cdot\nabla)^3\mathbf{E}\Big|_{@\mathbf{r}} \;+\; \cdots, \tag{12}$$

so the net force on the dipole is

$$\mathbf{F}^{\text{net}} \;=\; q(\mathbf{a}\cdot\nabla)\mathbf{E}\Big|_{@\mathbf{r}} \;+\; \frac{q}{24}(\mathbf{a}\cdot\nabla)^3\mathbf{E}\Big|_{@\mathbf{r}} \;+\; \cdots. \tag{13}$$

For a physical dipole with a finite distance a between the two charges, we must generally take into account all the subleading terms in this expansion. But for an ideal dipole we take the limit $a \to 0$ while $p = q \times a$ stays finite, so in this limit $q \times a^n \to 0$ for any $n > 1$. Consequently, the leading term $q(\mathbf{a} \cdot \nabla)\mathbf{E}$ in eq. (13) stays finite, but all the subleading terms $q(\mathbf{a} \cdot \nabla)^n \mathbf{E}$ vanish in the pure dipole limit. Thus, the net force on an ideal dipole is simply

$$\mathbf{F}^{\text{net}} = (\mathbf{p} \cdot \nabla)\mathbf{E}(\mathbf{r}). \tag{14}$$

Similar to the net force, the net potential energy of a dipole obtains as

$$U^{\text{net}} = qV(\mathbf{r}_+) - qV(\mathbf{r}_-) \tag{15}$$

where

$$V(\mathbf{r}_\pm) = V(\mathbf{r}) \pm (\tfrac{1}{2}\mathbf{a} \cdot \nabla)V(\mathbf{r}) + \tfrac{1}{2}(\tfrac{1}{2}\mathbf{a} \cdot \nabla)^2 V(\mathbf{r}) \pm \tfrac{1}{6}(\tfrac{1}{2}\mathbf{a} \cdot \nabla)^3 V(\mathbf{r}) + \cdots \tag{16}$$

and therefore

$$U^{\text{net}} = q(\mathbf{a} \cdot \nabla)V(\mathbf{r}) + \tfrac{1}{24}q(\mathbf{a} \cdot \nabla)^3 V(\mathbf{r}) + \cdots \tag{17}$$

Again, for a real dipole with a finite distance a between the two charges we should generally take into account all terms in this series, but for a pure dipole with $a \to 0$ (but finite $p = qa$) the subleading terms become negligible compared to the leading term

$$q\mathbf{a} \cdot \nabla V(\mathbf{r}) = -\mathbf{p} \cdot \mathbf{E}(\mathbf{r}). \tag{18}$$

Thus, an ideal dipole with moment $\mathbf{p}$ located at point $\mathbf{r}$ has net potential energy

$$U(\mathbf{r}, \mathbf{p}) = -\mathbf{p} \cdot \mathbf{E}(\mathbf{r}). \tag{19}$$

The potential energy (19) accounts for the mechanical work of the force (14) when the dipole is moved around and also for the work of the torque (9) when the dipole is rotated; thus, *both the force* (14) *and the torque* (9) *are conservative.* To see how this works, consider infinitesimal dosplacements and rotations of the dipole,

$$\mathbf{r} \to \mathbf{r} + \vec{\alpha}, \quad \mathbf{p} \to \mathbf{p} + \vec{\varphi} \times \mathbf{p} \tag{20}$$

for some infinitesimal vectors $\vec{\alpha}$ and $\vec{\varphi}$. The work of the force (14) and the torque (9) due to such combined displacement and rotation is

$$\delta W = \vec{\alpha} \cdot \mathbf{F} + \vec{\varphi} \cdot \vec{\tau} = \vec{\alpha} \cdot \big[(\mathbf{p} \cdot \nabla)\mathbf{E}(\mathbf{r})\big] + \vec{\varphi} \cdot \big[\mathbf{p} \times \mathbf{E}(\mathbf{r})\big], \tag{21}$$

so let's check that the infinitesimal variation of the energy (19) agrees with

$$\delta W = -\delta U(\mathbf{r}, \mathbf{p}). \tag{22}$$

Indeed,

$$\begin{aligned} -\delta U &= +\delta\mathbf{p} \cdot \mathbf{E}(\mathbf{r}) + \mathbf{p} \cdot \big[\delta\mathbf{E}(\mathbf{r}) = (\delta\mathbf{r} \cdot \nabla)\mathbf{E}(\mathbf{r})\big] \\ &= (\vec{\varphi} \times \mathbf{p}) \cdot \mathbf{E}(\mathbf{r}) + \mathbf{p} \cdot \big[(\vec{\alpha} \cdot \nabla)\mathbf{E}(\mathbf{r})\big], \end{aligned} \tag{23}$$

where the first term has form $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$, thus

$$1^{\text{st}}\,\text{term} = (\vec{\varphi} \times \mathbf{p}) \cdot \mathbf{E}(\mathbf{r}) = \vec{\varphi} \cdot (\mathbf{p} \times \mathbf{E}(\mathbf{r})) \equiv \vec{\varphi} \times \tau, \tag{24}$$

which is precisely the torque term in the work (21). As to the second term in eq. (23),

$$\begin{aligned} 2^{\text{nd}}\,\text{term} &= \mathbf{p} \cdot \big[(\vec{\alpha} \cdot \nabla)\mathbf{E}(\mathbf{r})\big] = -\mathbf{p} \cdot \big[(\vec{\alpha} \cdot \nabla)\nabla V(\mathbf{r})\big] \\ &= -(\vec{\alpha} \cdot \nabla)(\mathbf{p} \cdot \nabla)V(\mathbf{r}) = -\vec{\alpha} \cdot \big[(\mathbf{p} \cdot \nabla)\nabla V(\mathbf{r})\big] \\ &= +\vec{\alpha} \cdot \big[(\mathbf{p} \cdot \nabla)\mathbf{E}(\mathbf{r})\big] \equiv \vec{\alpha} \cdot \mathbf{F}, \end{aligned} \tag{25}$$

which is precisely the force term in the work (21). And this proves that the force (14) and the torque (9) on the dipole are indeed conservative and their work is accounted by the potential energy (19).

To be precise, the torque (9) is *the torque relative to the dipole center* $\mathbf{r}$. In a non-uniform electric field, the torque relative to some other point $\mathbf{r}_0$ has an extra term due to the net force (14) on the dipole, thus

$$\vec{\tau}^{\,\mathrm{net}} \;=\; (\mathbf{r}-\mathbf{r}_0)\times\mathbf{F}^{\mathrm{net}} \;+\; \vec{\tau}^{\,\mathrm{relative\,to\,}\mathbf{r}} \;=\; (\mathbf{r}-\mathbf{r}_0)\times(\mathbf{p}\cdot\nabla)\mathbf{E}(\mathbf{r}) \;+\; \mathbf{p}\times\mathbf{E}(\mathbf{r}). \tag{26}$$

This net torque may also be obtained from the potential energy U — or rather its infinitesimal variation under simultaneous rotations of the dipole moment vector $\mathbf{p}$ and of radius vector $\mathbf{r}-\mathbf{r}_0$ of the dipole from the reference point $\mathbf{r}_0$ — but I am not going to work it out in these notes.