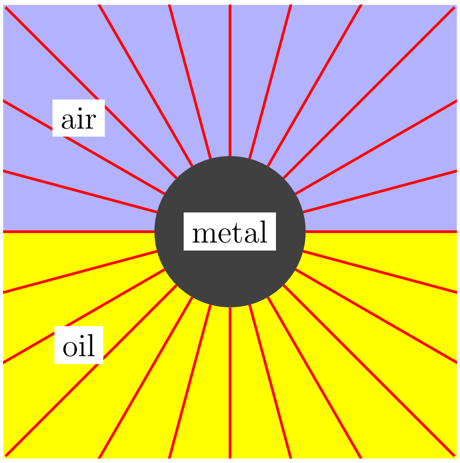
PHY–352 K. Mid-term Exam #2. November 20, 2018.

1. A charged metal sphere or radius R floats equator-deep in a pool of oil, which is a linear dielectric with susceptibility χ. The upper half of the sphere sticks out in the air (which you may approximate as vacuum). Despite the asymmetry between the oil and the air, the electric field outside the metal is spherically symmetric,



$$\mathbf{E} = \frac{k}{r^2}\,\hat{\mathbf{r}} \tag{1}$$

for some constant k.

   (a) Verify that this electric field obeys the boundary conditions at the oil-air interface.
   (b) Find all the free charge densities and the bound charge densities in the system. (Both surface and volume densities, if any.)
   (c) If the metal sphere is used as a capacitor plate (with the other "plate" at the infinity), what is the capacitance of this capacitor?
   (d) Find the net electric energy of the system.

2. Calculate the net magnetic force between two infinitely long wires that are **not** parallel to each other. For example, one wire runs along the z axis while the other wire spans

$$\mathbf{r} = a\,\hat{\mathbf{x}} + \ell\sin\theta\,\hat{\mathbf{y}} + \ell\cos\theta\,\hat{\mathbf{z}}, \quad \text{fixed } a,\theta, \quad \text{variable } \ell \text{ from } -\infty \text{ to } +\infty. \tag{2}$$

Note: since the wires are not parallel, calculate the net force rather than force per unit of length. To save your time, here are some useful formulae:

$$\frac{\hat{\phi}}{s} = \frac{x\,\hat{\mathbf{y}} - y\,\hat{\mathbf{x}}}{x^2+y^2}, \qquad \int_{-\infty}^{+\infty} \frac{d\ell}{a^2 + c^2\ell^2} = \frac{\pi}{ac}, \qquad \int_{-\infty}^{+\infty} \frac{\ell\,d\ell}{a^2 + c^2\ell^2} = 0. \tag{3}$$

3. Find the magnetic field of a helical current density

$$\mathbf{J}(x,y,z) \;=\; J_0\Big(\cos(kz)\,\hat{\mathbf{x}} + \sin(kz)\,\hat{\mathbf{y}}\Big) \qquad \langle\!\langle \text{constant } J_0,\ \text{constant } k \rangle\!\rangle \tag{4}$$

flowing through the whole space. Note several symmetries of this current, including invariance under simultaneous translations in z directions and rotations around the z axis.

(a) Can this current be steady?

(b) Without taking any integrals or solving any differential equations, use the symmetries of the current density (4) to argue that the vector potential should have form

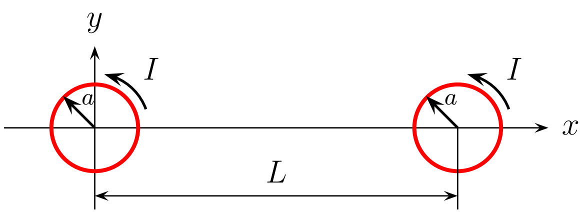
$$\mathbf{A}(x,y,z) \;=\; A_0\Big(\cos(kz)\,\hat{\mathbf{x}} + \sin(kz)\,\hat{\mathbf{y}}\Big) \tag{5}$$

for some constant A_0.

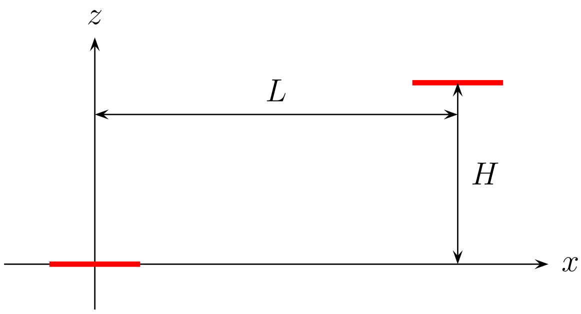
(c) Check that the vector potential (5) obeys the Poisson equation $\nabla^2\mathbf{A} = -\mu_0\mathbf{J}$ for the appropriate value of A_0 and find that value.

(d) Finally, write down the magnetic field $\mathbf{B}$ for the vector potential (5).

4. Consider two similar circular wires. Both circles are horizontal, have same small radius a, and both wires carry the same current I, The horizontal distance between the two circles' centers is L while the vertical distance is H:



top view



side view

Assume $a \ll H, L$ and approximate both current loops as magnetic dipoles.

Find the torque on the each wire loop relative to its own center.