The non-textbook problem:

(a) The electric current density $\mathbf{J}$ and the electric charge density ρ are related by the *continuity equation*

$$\nabla\cdot\mathbf{J}(x,y,z,t)\;+\;\frac{\partial\rho(x,y,z,t)}{\partial t}\;=\;0. \tag{1}$$

A *steady* current is time-independent, which requires time-independent $\partial\rho/\partial t$, hence

$$\rho(x,y,z,t)\;=\;F(x,y,z)\;+\;t\times G(x,y,z) \tag{2}$$

for some time-independent functions $F(x,y,z)$ and $G(x,y,z)$. But since the electric charge cannot keep accumulating all the time, we must have $G=0$ and thus time-independent $\rho(x,y,z)$. Consequently, *a steady current density must have zero divergence,*

$$\nabla\cdot\mathbf{J}(x,y,z)\;=\;0. \qquad (3).$$

Let's check this condition for the current in question:

$$\mathbf{J}(x,y,z)\;=\;k\bigl(x\,\hat{\mathbf{x}}\,+\,y\,\hat{\mathbf{y}}\,-\,2z\,\hat{\mathbf{z}}\bigr), \tag{4}$$

$$\begin{aligned}\nabla\cdot\mathbf{J}\;&=\;\frac{\partial J_x}{\partial x}\;+\;\frac{\partial J_y}{\partial y}\;+\;\frac{\partial J_z}{\partial z}\\ &=\;k\;+\;k\;-\;2k\;=\;0.\end{aligned} \tag{5}$$

We see that the current (4) is divergence-less, so *it may be steady.*

Note: we cannot prove that the current in question is indeed steady — we do not have any information about its time dependence or independence, — but at least it may be steady without causing a charge buildup.

(b) By Gauss theorem, the net flux of a divergence-less current through a complete surface of some finite volume — like the cylinder in question — must be zero. So let's check the inflow/outflow of the current (4) through each side of the cylinder, namely its top disk, the bottom disk, and the cylindrical wall.

The net current through the top end of the cylinder at $z = H$ is

$$\begin{aligned} I_{\text{top}} &= \iint\limits_{\text{top}} \mathbf{J}\cdot\mathbf{d^2A} = \iint\limits_{\text{top}} J_z(x,y,z=H)\,dx\,dy \\ &= \iint\limits_{\text{top}} (-2kH)\,dx\,dy = -2kH\times\text{Area(top)} = -2KH\times\pi R^2 \\ &= -2\pi kHR^2. \end{aligned} \tag{6}$$

The overall $-$ sign here indicates that the current flows <u>into</u> the cylinder.

On the other hand, the net current through the bottom end of the cylinder is zero because at $z = 0$, $J_z = 0$. Indeed,

$$I_{\text{bot}} = \iint\limits_{\text{bot}} \mathbf{J}\cdot\mathbf{d^2A} = -\iint\limits_{\text{bot}} J_z(x,y,z=0)\,dx\,dy = 0. \tag{7}$$

Thus, we see that the current flows in from the top of the cylinder but it does not flow out through the bottom end. Instead, it flows out through the outer cylindrical wall.

Indeed, in the cylindrical coordinates (s,ϕ,z),

$$\mathbf{J} = ks\,\hat{\mathbf{s}} - 2kz\,\hat{\mathbf{z}}, \tag{8}$$

hence its component $\perp$ to the cylindrical wall at $s = R$ is

$$\mathbf{J}_\perp = +kR. \tag{9}$$

Consequently, the net current through the wall is

$$\begin{aligned} I_{\text{wall}} &= \iint\limits_{\text{wall}} \mathbf{J}\cdot\mathbf{d^2A} = \iint\limits_{\text{wall}} \mathbf{J}_\perp\,d^2A \\ &= +kR\times\text{Area(wall)} = +kR\times 2\pi RH \\ &= +2\pi kHR^2. \end{aligned}$$

Altogether, the net inflow/outflow of the current into/out from the cylinder is zero, as it should be for a divergence-less current. But there is a current

$$I = 2\pi k H R^2 \tag{10}$$

flowing *through* the cylinder: It flows in through the top disk, then flows out through the outer wall.

Problem **5.9**:

(a) The current loop on figure 5.23(a) comprises two straight segments and two circular arcs centered on the point P. The magnetic fields due to such wire segments are discussed in detail in my notes on Biot–Savart–Laplace Law and its applications. In particular, for the straight segments along lines crossing the point P — as both straight segments of figure 5.23(a) do — the magnetic field vanishes at point P. Consequently,

$$\mathbf{B}(P) = \mathbf{B}[\text{inner arc}](P) + \mathbf{B}[\text{outer arc}](P) + 0 + 0. \tag{11}$$

As to the circular arcs,

$$\mathbf{B}[\text{inner arc}](P) = \frac{\mu_0 I}{4\pi}\frac{\phi}{a}(+\hat{\mathbf{z}}) \tag{12}$$

where ϕ is the angle spanned by the arc — which seems to be close to $\pi/2$, although I am not sure if it meant to be exactly $\pi/2$ or not — while $\hat{\mathbf{z}}$ is the unit vector $\perp$ to the page, specifically out from the page towards your eyes.

Similarly, for the outer arc,

$$\mathbf{B}[\text{inner arc}](P) = \frac{\mu_0 I}{4\pi}\frac{\phi}{b}(-\hat{\mathbf{z}}) \tag{13}$$

where the $-\hat{\mathbf{z}}$ direction (into the page) stems from the clockwise direction of the current in the outer arc. (Unlike the counterclockwise direction in the inner arc.)

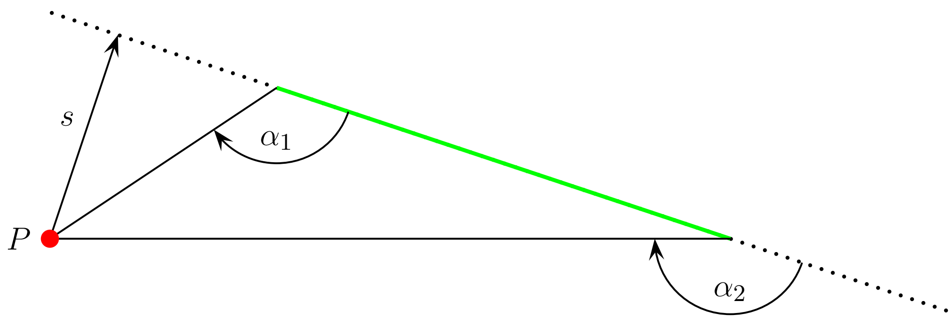
Altogether, adding the fields of both arcs (as well as straight segments, which do not contribute), we get the net magnetic field

$$\mathbf{B}(P) = \frac{\mu_0 I}{4\pi}\left(\frac{\phi}{a} - \frac{\phi}{b}\right)\hat{\mathbf{z}}. \tag{14}$$

(b) The current loop on figure 5.23(b) comprises 2 straight segments, one circular arc centered at P, and some closing segment at infinity (not shown) whose magnetic field may be neglected since it's so far away. Thus,

$$\mathbf{B}(P) \;=\; \mathbf{B}[\text{top straight wire}](P) + \mathbf{B}[\text{bottom straight wire}](P) + \mathbf{B}[\text{circular arc}](P). \quad (15)$$

For each straight-line wire segment with geometry like this



(16)

the magnetic field at point P is

$$\mathbf{B}[\text{straight segment}](P) \;=\; \frac{\mu_0 I}{4\pi s} \times \left(\cos\alpha_1 \;-\; \cos\alpha_2\right)\hat{\mathbf{n}} \quad (17)$$

where s is the distance from P to the straight line along the wire segment, and $\hat{\mathbf{n}}$ is a unit vector $\perp$ to the whole plane including the wire and the point P. For each straight wire on figure 5.23(b), $\hat{\mathbf{n}}$ is $\perp$ to the page, and since the currents in both segments run clockwise around P, $\hat{\mathbf{n}} = -\hat{\mathbf{z}}$ points into the page. Also, both segments have $s = R$, while

For the top straight wire,

$$\alpha_1 \;=\; 90^\circ, \quad \alpha_2 \;=\; 180^\circ \quad\Longrightarrow\quad \cos\alpha_1 \;-\; \cos\alpha_2 \;=\; 1, \quad (18)$$

For the bottom straight wire,

$$\alpha_1 \;=\; 0^\circ, \quad \alpha_2 \;=\; 90^\circ \quad\Longrightarrow\quad \cos\alpha_1 \;-\; \cos\alpha_2 \;=\; 1. \quad (19)$$

Consequently,

$$\mathbf{B}[\text{top}](P) \;=\; \mathbf{B}[\text{bottom}](P) \;=\; \frac{\mu_0 I}{4\pi R}\,(-\hat{\mathbf{z}}). \tag{20}$$

As to the circular arc segment, its magnetic field is

$$\mathbf{B}[\text{arc}] \;=\; \frac{\mu_0 I}{4\pi}\,\frac{\phi=\pi}{R}\,(-\hat{\mathbf{z}}), \tag{21}$$

which also points into the page. Altogether, the net field at point P is

$$\mathbf{B}(P) \;=\; \frac{\mu_0 I}{4\pi}\times\frac{2+\pi}{R}\times(-\hat{\mathbf{z}}). \tag{22}$$

<u>Problem **5.11**</u>:

A densely wound solenoid can be approximated by nearly-continuous sequence of circular rings, each carrying the same current I, so the net magnetic field at point P on the axis of the solenoid is

$$\mathbf{B}^{\text{net}}(P) \;=\; \sum_{\text{rings}} \mathbf{B}[\text{ring}](P) \;\approx\; \int\limits_{x_1}^{x_2} \mathbf{B}[\text{ring}\,@\,x](P)\times\frac{N}{L}\,dx \tag{23}$$

where x is the coordinate along the solenoid's axis and N/L is the density of rings, *i.e.*, the density of the solenoid's winding.

For simplicity, let's count x left from the point P, so that the ring $@\,x$ has its center at distance x from P. The magnetic field of this ring at P is calculated in my notes on Biot–Savart–Laplace Law and its applications as

$$B[\text{ring}\,@\,x](P) \;=\; \frac{I\mu_0}{2}\times\frac{a^2}{(a^2+x^2)^{3/2}}. \tag{24}$$

The direction of this field is $\perp$ to the ring, *i.e.*, along the solenoid's axis, $\mathbf{B}=\pm B\hat{\mathbf{x}}$, where the $\pm$ sign depends on the direction of the current in the ring. The problem does not specify this direction, but it must be the same direction for all the rings in the solenoid.

Plugging the single ring's field (22) into eq. (23) for the solenoid, we find

$$\mathbf{B}^{\text{solenoid}} = \frac{I\mu_0}{2}\times(\pm\hat{\mathbf{x}})\times\frac{N}{L}\int\limits_{x_1}^{x_2}\frac{a^2\,dx}{(a^2+x^2)^{3/2}}\,. \tag{25}$$

It remains to evaluate the integral in this formula. Following the textbook suggestion on figure 5.25, let's change the integration variable from x to

$$\theta = \arctan\frac{a}{x} \quad\Longrightarrow\quad x = \frac{a}{\tan\theta} = a\times\operatorname{ctan}\theta\,. \tag{26}$$

Consequently,

$$dx = -\frac{a\,d\theta}{\sin^2\theta}\,, \tag{27}$$

$$a^2 + x^2 = a^2\times\left(1+\operatorname{ctan}^2\theta\right) = \frac{a^2}{\sin^2\theta}\,, \tag{28}$$

$$\frac{1}{(a^2+x^2)^{3/2}} = \frac{\sin^3\theta}{a^3}\,, \tag{29}$$

$$\frac{a^2\,dx}{(a^2+x^2)^{3/2}} = -\sin\theta\,d\theta = +d(\cos\theta), \tag{30}$$

and therefore

$$\int\limits_{x_1}^{x_2}\frac{a^2\,dx}{(a^2+x^2)^{3/2}} = \int\limits_{\theta_1}^{\theta_2}\frac{d(\cos\theta)}{a} = \cos\theta_2 - \cos\theta_1\,. \tag{31}$$

Finally, plugging this result into eq. (25), we arrive at

$$\mathbf{B}^{\text{solenoid}} = \frac{NI\mu_0}{2L}\big(\cos\theta_2-\cos\theta_1\big)\,(\pm\hat{\mathbf{x}}) = \frac{NI\mu_0}{2L}\big(\cos\theta_1-\cos\theta_2\big)\,(\mp\hat{\mathbf{x}}). \tag{32}$$

In particularly, inside an infinite solenoid,

$$\theta_1\approx 0,\quad \theta_2\approx 180^\circ \quad\Longrightarrow\quad \cos\theta_1 - \cos\theta_2 \approx 2, \tag{33}$$

and therefore

$$\mathbf{B}^{\text{solenoid}} = \frac{NI\mu_0}{L}\,(\mp\hat{\mathbf{x}}). \tag{34}$$

PS: The direction of the magnetic field in eqs. (32) and (34) — $+\hat{\mathbf{x}}$ or $-\hat{\mathbf{x}}$ — follows from the right hand rule: Wrap the finders of your right hand around the solenoid so that they point in the direction of the current, then your thumb points in the direction of the magnetic field. For example, if in the solenoid on figure 5.25 the current flows up in the front side of the solenoid (the side facing you) and flows down on the back side, then the magnetic field points to the left of the page.

Problem **5.6**:

When a volume charge with density ρ moves with velocity $\mathbf{v}$, it creates a current density

$$\mathbf{J} = \rho\mathbf{v}. \tag{35}$$

Likewise, when a surface charge with density σ moves parallel to the surface, it creates a surface current density

$$\mathbf{K} = \sigma\mathbf{v}. \tag{36}$$

Now let's apply these formulae to the rotating charges at hand.

(a) When a disk such as phonograph record or CD rotates with angular velocity ω around its center, a point at distance r from the center moves in a circle with linear speed $v = \omega r$. In vector notations,

$$\mathbf{v} = \vec{\omega} \times \mathbf{r} = \omega r\,\hat{\phi}. \tag{37}$$

Consequently, a uniformly charged disk with surface charge density σ gives rise to the surface current density

$$\mathbf{K} = \sigma\omega r\,\hat{\phi}. \tag{38}$$

(b) For a rigid sphere rotating around its axis, a point with spherical coordinates (r, θ, ϕ) moves along the latitude circle of radius $r\sin\theta$, so its velocity is

$$\mathbf{v} = \omega \times r\sin\theta \times \hat{\phi}. \tag{39}$$

Consequently, when the sphere has a uniform charge density ρ, its rotation gives rise to current density

$$\mathbf{J} = \rho\omega\, r\sin\theta\,\hat{\phi}. \tag{40}$$

Problem **5.12**:

When a spherical shell of radius R rotates at angular velocity ω, a point with spherical coordinates (θ, ϕ) moves in the latitude circle with speed

$$v = \frac{\text{circle length}}{\text{rotation period}} = \frac{2\pi R \sin\theta}{2\pi/\omega} = \omega R \sin\theta, \tag{41}$$

or in terms of the velocity vector

$$\mathbf{v} = \omega R \sin\theta\, \hat{\phi}. \tag{42}$$

For the uniformly charged sphere with surface charge density

$$\sigma = \frac{Q}{4\pi R^2} \tag{43}$$

the rotation gives rise to the surface current density

$$\mathbf{K} = \sigma \mathbf{v} = \frac{Q\omega \sin\theta}{4\pi R}\, \hat{\phi}, \tag{44}$$

Now consider the magnetic field of this density. By the Biot–Savart–Laplace Law,

$$\mathbf{B}(P) = \frac{\mu_0}{4\pi} \iint \mathbf{K} \times \frac{\mathbf{r}_P - \mathbf{r}(\theta,\phi)}{|\mathbf{r}_P - \mathbf{r}(\theta,\phi)|^3}\, d^2A(\theta,\phi). \tag{45}$$

For the point P at the center of the sphere,

$$\frac{\mathbf{r}_P - \mathbf{r}(\theta,\phi)}{|\mathbf{r}_P - \mathbf{r}(\theta,\phi)|^3} = -\frac{\hat{\mathbf{r}}(\theta,\phi)}{R^2} \tag{46}$$

where $\hat{\mathbf{r}}(\theta,\phi)$ is simply the unit vector in the direction (θ,ϕ). Consequently,

$$\mathbf{K} \times \frac{\mathbf{r}_P - \mathbf{r}(\theta,\phi)}{|\mathbf{r}_P - \mathbf{r}(\theta,\phi)|^3} = \frac{Q\omega \sin\theta}{4\pi R} \frac{1}{R^2} \left(-\hat{\phi} \times \hat{\mathbf{r}}\right) = \frac{Q\omega \sin\theta}{4\pi R^3} (-\hat{\theta}). \tag{47}$$

Integrating this expression over the area of the sphere, we have

$$d^2A(\theta,\phi) = R^2 \sin\theta\, d\theta\, d\phi, \tag{48}$$

hence

$$\mathbf{B}(\text{center}) \;=\; \frac{\mu_0}{4\pi}\iint \frac{Q\omega\,\sin\theta}{4\pi R^3}\,(-\hat{\theta})\,R^2\,\sin\theta\,d\theta\,d\phi \;=\; \frac{Q\omega\mu_0}{16\pi^2 R}\int\limits_0^{\pi}\sin^2\theta\int\limits_0^{2\pi}d\phi(-\hat{\theta}). \qquad (49)$$

It remains to perform the angular integrals in this formula. Please note that the unit vector $\hat{\theta}$ depends on θ and ϕ, so we should re-express it in terms of the Cartesian components $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$:

$$\hat{\theta} \;=\; -\sin\theta\,\hat{\mathbf{z}} \;+\; \cos\theta\big(\cos\phi\,\hat{\mathbf{x}} \;+\; \sin\phi\,\hat{\mathbf{y}}\big). \qquad (50)$$

Integrating the components of this vector over ϕ, we get

$$\int\limits_0^{2\pi}\sin\theta\,d\phi \;=\; 2\pi\sin\theta, \quad \int\limits_0^{2\pi}\cos\theta\,\cos\phi\,d\phi \;=\; 0, \quad \int\limits_0^{2\pi}\cos\theta\,\sin\phi\,d\phi \;=\; 0, \qquad (51)$$

and therefore

$$\int\limits_0^{2\pi}(-\hat{\theta})\,d\phi \;=\; +2\pi\sin\theta\,\hat{\mathbf{z}} \;+\; 0\hat{\mathbf{x}} \;+\; 0\hat{\mathbf{y}}. \qquad (52)$$

Consequently,

$$\int\limits_0^{\pi}\sin^2\theta\int\limits_0^{2\pi}d\phi(-\hat{\theta}) \;=\; 2\pi\hat{\mathbf{z}}\int\limits_0^{\pi}\sin^3\theta\,d\theta \qquad (53)$$

where in the remaining integral

$$\sin^3\theta\,d\theta \;=\; \sin^2\theta\times d(-\cos\theta) \;=(1-\cos^2\theta)\,d(-\cos\theta) \;=\; d\big(-\cos\theta \;+\; \tfrac{1}{3}\cos^3\theta\big), \qquad (54)$$

hence

$$\int\limits_0^{\pi}\sin^3\theta\,d\theta \;=\; \big(-\cos\theta \;+\; \tfrac{1}{3}\cos^3\theta\big)\Big|_{\theta=0}^{\theta=\pi} \;=\; \big(+1-\tfrac{1}{3}\big) \;-\; \big(-1+\tfrac{1}{3}\big) \;=\; \tfrac{4}{3}. \qquad (55)$$

Altogether,

$$\mathbf{B}(\text{center}) \;=\; \frac{Q\omega\mu_0}{16\pi^2 R}\times(2\pi\hat{\mathbf{z}})\times\frac{4}{3} \;=\; \frac{Q\omega\mu_0}{6\pi R}\,\hat{\mathbf{z}}. \qquad (56)$$

ALTERNATIVE SOLUTION:

The rotating charged sphere acts as a continuous sequence of current-carrying circular loops. A loop at latitude θ has radius $r = R\sin\theta$ and carries current

$$dI \;=\; K \times R\,d\theta \;=\; \frac{Q\omega\sin\theta}{4\pi}\,d\theta \tag{57}$$

The magnetic field of such a ring at the center of the sphere is

$$dB \;=\; \frac{\mu_0\,dI}{2} \times \frac{r^2}{(r^2+z^2)^{3/2}} \;=\; \frac{\mu_0 Q\omega\sin\theta\,d\theta}{8\pi} \times \frac{R^2\sin^2\theta}{R^3} \;=\; \frac{\mu_0 Q\omega}{8\pi R}\,\sin^3\theta\,d\theta. \tag{58}$$

Assuming the sphere is positively charged and spins counterclockwise (when viewed from above the North pole $\theta = 0$), the current (57) is counterclockwise in the xy plane, and the magnetic field (58) points due North, in the $+\hat{\mathbf{z}}$ direction, Thus

$$d\mathbf{B}[\text{single ring}] \;=\; \frac{\mu_0 Q\omega}{8\pi R}\,\hat{\mathbf{z}}\,\sin^3\theta\,d\theta, \tag{59}$$

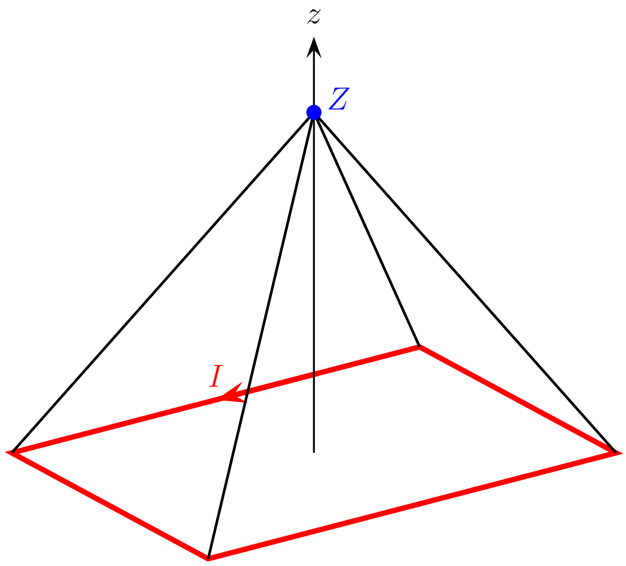
and therefore the net magnetic field at the center is the sum — or rather the integral — of the fields due to all such rings, thus

$$\mathbf{B}(\text{center}) \;=\; \frac{\mu_0 Q\omega}{8\pi R}\,\hat{\mathbf{z}}\int_0^\pi \sin^3\theta\,d\theta. \tag{60}$$

Finally, performing the integral here exactly as in eqs. (54) and (55), we arrive at

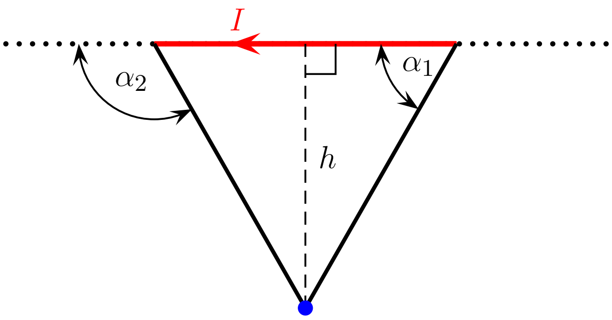
$$\mathbf{B}(\text{center}) \;=\; \frac{\mu_0 Q\omega}{6\pi R}\,\hat{\mathbf{z}}. \tag{61}$$

Problem **5.36**:



(62)

The net magnetic field at point Z is the sum of magnetic fields due to each side of the square current loop. As explained in my notes Biot–Savart–Laplace Law (example#3), the magnetic field of a straight wire segment has a simple form in terms of the triangle formed by the two ends of the segment and the point where we measure the field,



$$\mathbf{B}(\bullet) = \frac{\mu_0 I}{4\pi}\,\frac{\cos\alpha_1 - \cos\alpha_2}{h}\,\mathbf{n} \tag{63}$$

where $\mathbf{n}$ is the unit vector $\perp$ to the plane of the triangle.

For the problem at hand, the triangles form sides of the pyramid (62) whose base is the square current loop. Each triangle is isosceles with base w and height

$$h = \sqrt{z^2 + (w/2)^2}\,, \tag{64}$$

hence

$$\alpha_1 = \arctan\frac{h}{(w/2)}\,, \qquad \alpha_2 = \pi - \alpha_1, \tag{65}$$

and therefore

$$\cos\alpha_1 \;-\; \cos\alpha_2 \;=\; 2\cos\alpha_1 \;=\; \frac{2}{\sqrt{1+\tan^2\alpha_1}} \;=\; \frac{w}{\sqrt{h^2+(w/2)^2}} \tag{66}$$

Altogether,

$$\mathbf{B}[1\text{ side}] \;=\; \frac{\mu_0 I}{4\pi}\,\frac{w}{h\sqrt{h^2+(w/2)^2}}\,\mathbf{n}\,. \tag{67}$$

Now consider the directions of the magnetic field from each side of the square. Each side of the pyramid (62) makes angle

$$\beta \;=\; \arctan\frac{z}{(w/2)} \;=\; \arccos\frac{(w/2)}{h} \;=\; \arcsin\frac{z}{h} \tag{68}$$

with the horizontal, so the unit vector $\mathbf{n}$ makes angle β with the vertical. By the right hand rule, $\mathbf{n}$ sticks out and up from the pyramid side, hence

$$n_z \;=\; +\cos\beta \;=\; +\frac{(w/2)}{h} \tag{69}$$

while

$$|\mathbf{n}_{\text{horizontal}}| \;=\; \sin\beta \;=\; \frac{z}{h}\,. \tag{70}$$

Moreover, for each side of the pyramid, the horizontal component of the $\mathbf{n}$ vector points in a different direction, East, North, West, and South, or in terms of xy components,

$$\mathbf{n}_1 \;=\; +\sin\beta\,\hat{\mathbf{x}}\,,\quad \mathbf{n}_2 \;=\; +\sin\beta\,\hat{\mathbf{y}}\,,\quad \mathbf{n}_3 \;=\; -\sin\beta\,\hat{\mathbf{x}}\,,\quad \mathbf{n}_4 \;=\; -\sin\beta\,\hat{\mathbf{y}}\,. \tag{71}$$

Consequently, after summing over contributions of all 4 sides of the square, the horizontal components of the magnetic fields cancel out! On the other hand, the vertical components of the 4 sides' fields add up, thus

$$\mathbf{B}^{\text{net}} \;=\; B^{1\text{ segment}}\big(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3+\mathbf{n}_4\big) \;=\; B^{1\text{ segment}}\times 4\cos\beta\,\hat{\mathbf{z}}\,, \tag{72}$$

hence in light of eq. (67),

$$\mathbf{B}^{\text{net}} \;=\; \frac{\mu_0 I}{4\pi}\times\frac{w}{h\sqrt{h^2+(w/2)^2}}\times\frac{2w}{h}\times\hat{\mathbf{z}} \;=\; \frac{\mu_0 I w^2}{4\pi}\,\frac{2\hat{\mathbf{z}}}{h^2\sqrt{h^2+(w/2)^2}}\,. \tag{73}$$

Note: this is the *exact* magnetic field at height z above the center of the square loop.

Now consider the behavior of the field (73) at large distances from the loop, $z \gg w$. In this limit,

$$h = \sqrt{z^2 + (w/2)^2} \approx z \gg w, \quad \sqrt{h^2 + (w/2)^2} \approx h \approx z, \tag{74}$$

so the magnetic field (73) becomes

$$\mathbf{B}(0,0,z) \approx \frac{\mu_0 I w^2}{4\pi} \frac{2\hat{\mathbf{z}}}{z^3}. \tag{75}$$

Comparing this field to the field of a pure magnetic dipole, we note that the dipole moment of the square loop is

$$\mathbf{m} = I w^2 \, \hat{\mathbf{z}}. \tag{76}$$

hence

$$\mathbf{B}_{\text{dipole}}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}}{r^3} = \frac{\mu_0 I w^2}{4\pi} \frac{3(\hat{\mathbf{r}}\cdot\hat{\mathbf{z}})\hat{\mathbf{r}} - \hat{\mathbf{z}}}{r^3}. \tag{77}$$

For $\mathbf{r}$ in the $+\hat{\mathbf{z}}$ direction — directly above the dipole — we have

$$r^3 = z^3, \quad 3(\hat{\mathbf{r}}\cdot\hat{\mathbf{z}})\hat{\mathbf{r}} - \hat{\mathbf{z}} = 3\hat{\mathbf{z}} - \hat{\mathbf{z}} = 2\hat{\mathbf{z}}, \tag{78}$$

and therefore the dipole field

$$\mathbf{B}_{\text{dipole}}(0,0,z) = \frac{\mu_0 I w^2}{4\pi} \frac{2\hat{\mathbf{z}}}{z^3}. \tag{79}$$

Comparing this formula to eq. (75), we see that at long distances from the square loop, its magnetic field reduces to the pure dipole field of the magnetic moment (76). *Quod erat demonstrandum.*

Problem **5.50**:

Consider two generally-shaped loops of wire carrying respective currents I_1 and I_2. The net magnetic force on loop#1 from loop#2 is

$$\mathbf{F}_{21} = I_1 \oint_{\text{loop}\#1} d\mathbf{r}_1 \times \mathbf{B}_2(\mathbf{r}_1) \tag{80}$$

where $\mathbf{B}_2(\mathbf{r}_1)$ is the magnetic field of the loop#2 at the point $\mathbf{r}_1$ on loop#1. By the Biot–Savart–Laplace Law,

$$\mathbf{B}_2(\mathbf{r}_1) = \frac{\mu_0 I_2}{4\pi} \oint_{\text{loop}\#2} d\mathbf{r}_2 \times \mathbf{G}(\mathbf{r}_1 - \mathbf{r}_2) \tag{81}$$

where I have denoted

$$\mathbf{G}(\mathbf{r}_1 - \mathbf{r}_2) = \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}\,. \tag{82}$$

Altogether,

$$\mathbf{F}_{21} = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{L1} \oint_{L_2} d\mathbf{r}_1 \times \big(d\mathbf{r}_2 \times \mathbf{G}(\mathbf{r}_1 - \mathbf{r}_2)\big). \tag{83}$$

Now let's work out the double vector product inside the loop integrals here. By the $BAC - CAB$ rule,

$$d\mathbf{r}_1 \times (d\mathbf{r}_2 \times \mathbf{G}) = d\mathbf{r}_2(d\mathbf{r}_1 \cdot \mathbf{G}) - \mathbf{G}(d\mathbf{r}_1 \cdot d\mathbf{r}_2), \tag{84}$$

hence

$$\mathbf{F}_{21} = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{L1} \oint_{L2} \big(d\mathbf{r}_1 \cdot \mathbf{G}(\mathbf{r}_1 - \mathbf{r}_2)\big) d\mathbf{r}_2 - \frac{\mu_0 I_1 I_2}{4\pi} \oint_{L1} \oint_{L2} \big(d\mathbf{r}_1 \cdot d\mathbf{r}_2\big) \mathbf{G}(\mathbf{r}_1 - \mathbf{r}_2). \tag{85}$$

Moreover, the first term on the RHS vanishes for any closed loops $L1$ and $l2$. Indeed, let's rewrite the fist term as

$$X = \frac{\mu_0 I_1 I_2}{4\pi} \oint\limits_{L2} d\mathbf{r}_2 \oint\limits_{L1} d\mathbf{r}_1 \cdot \mathbf{G}(\mathbf{r}_1 - \mathbf{r}_2) \tag{86}$$

where we first integrate over the $d\mathbf{r}_1$ at fixed $\mathbf{r}_2$ and only then integrate over the $d\mathbf{r}_2$. Since

$$\mathbf{G}(\mathbf{r}_1 - \mathbf{r}_2) = \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} = \nabla_{\mathbf{r}_1} \frac{-1}{|\mathbf{r}_1 - \mathbf{r}_2|}, \tag{87}$$

during the integration over $d\mathbf{r}_1$ at fixed $\mathbf{r}_2$ we have

$$d\mathbf{r}_1 \cdot \mathbf{G}(\mathbf{r}_1 - \mathbf{r}_2) = d\mathbf{r}_1 \cdot \nabla_{\mathbf{r}_1} \frac{-1}{|\mathbf{r}_1 - \mathbf{r}_2|} = d\left(\frac{-1}{|\mathbf{r}_1 - \mathbf{r}_2|}\right), \tag{88}$$

a total differential of a single-valued function of $\mathbf{r}_1$. Consequently,

$$\oint\limits_{L1} d\mathbf{r}_1 \cdot \mathbf{G}(\mathbf{r}_1 - \mathbf{r}_2) = \left[\frac{-1}{|\mathbf{r}_1 - \mathbf{r}_2|}\right]^{\text{@the end of } L1} - \left[\frac{-1}{|\mathbf{r}_1 - \mathbf{r}_2|}\right]^{\text{@the beginning of } L1} = 0 \tag{89}$$

because $L1$ is a closed loop — it begins and ends at the same point. Thus, in eq. (86), the inner integral over $d\mathbf{r}_1$ at fixed $\mathbf{r}_2$ yields zero for any $\mathbf{r}_2$, so the outer integral is trivially zero.

Thus, we see that the first term in eq, (85) vanishes, so the entire force between the loops is given by just the second term,

$$\mathbf{F}_{21} = -\frac{\mu_0 I_1 I_2}{4\pi} \oint\limits_{L1} \oint\limits_{L2} \big(d\mathbf{r}_1 \cdot d\mathbf{r}_2\big) \mathbf{G}(\mathbf{r}_1 - \mathbf{r}_2). \tag{90}$$

In this formula, the $\mathbf{G}(\mathbf{r}_1 - \mathbf{r}_2)$ is antisymmetric between the two loops,

$$\mathbf{G}(\mathbf{r}_2 - \mathbf{r}_1) = -\mathbf{G}(\mathbf{r}_1 - \mathbf{r}_2), \tag{91}$$

while everything else is symmetric. Consequently, the force (90) is manifestly antisymmetric,

$$\mathbf{F}_{21} = -\mathbf{F}_{12} \tag{92}$$

in accordance with Newton's Third Law.

Problem **5.16**:

The magnetic field of the two solenoids obtains from the superposition principle:

$$\mathbf{B}_{\rm net}(x,y,z) \;=\; \mathbf{B}_1(x,y,z) \;+\; \mathbf{B}_2(x,y,z) \tag{93}$$

where the $\mathbf{B}_1$ field is created by the inner solenoid and the $\mathbf{B}_2$ field by the outer solenoid. As explained in my notes on Ampere Law and its applications, the magnetic field of a standalone infinitely long solenoid is

$$\begin{aligned} \text{outside the solenoid } \mathbf{B} &= 0. \\ \text{inside the solenoid } \mathbf{B} &= \mu_0 In(\pm\hat{\mathbf{z}}), \end{aligned} \tag{94}$$

where $\pm\hat{\mathbf{z}}$ is the direction of the solenoid's axis whose sign depends on the direction of the current in the solenoid's winding. For the problem at hand, we take the $+\hat{\mathbf{z}}$ direction of the two solenoid's common axis on figure 5.42 to run from left to right, so relative to this positive direction the current I_1 in the inner solenoid flows counterclockwise while the current I_2 in the outer solenoid flows clockwise. By the right hand rule, this means that the $\mathbf{B}_1$ field of the inner solenoid points left (the $-\hat{\mathbf{z}}$ direction) while the $\mathbf{B}_2$ field of the outer solenoid points right (the $+\hat{\mathbf{z}}$ direction). Thus, for each solenoid we have

$$\mathbf{B}_2 \;=\; \begin{cases} +\mu_0 I_2 n_2 & \text{inside the outer solenoid,} \\ 0 & \text{outside the outer solenoid,} \end{cases} \tag{95}$$

while

$$\mathbf{B}_1 \;=\; \begin{cases} -\mu_0 I_1 n_1 & \text{inside the inner solenoid,} \\ 0 & \text{outside the inner solenoid.} \end{cases} \tag{96}$$

Finally, by the superposition principle, the combined field of the two solenoids is:

$$\begin{aligned} \text{outside the outer solenoid } \mathbf{B}_{\rm net} &= 0, \\ \text{between the two solenoids } \mathbf{B}_{\rm net} &= +\mu_0 I_2 n_2\,, \\ \text{inside the inner solenoid } \mathbf{B}_{\rm net} &= +\mu_0 I_2 n_2 \;-\; \mu_0 I_1 n_1\,. \end{aligned} \tag{97}$$

Problem **5.18**:

(a) Regardless of the solenoid's cross-sectional shape, as long as the solenoid is infinitely long and both the cross-section and the density N/L of wiring are uniform along its length, there is translational symmetry in z direction of the solenoid's axis. Consequently, the magnetic field $\mathbf{B}(x,y,z)$ may depend on the transverse coordinates z and y but not on the z.

Another symmetry of the infinitely long uniform solenoid is the mirror reflection of the z axis,

$$(x,y,z) \;\to\; (+x,+y,-z), \tag{98}$$

and since the magnetic field is an axial vector, it transforms like

$$B_x \;\to\; -B_x\,, \quad B_y \;\to\; -B_y\,, \quad B_z \;\to\; +B_z\,. \tag{99}$$

This symmetry requires

$$\begin{aligned} B_x(x,y,-z) &= -B_x(x,y,+z), \\ B_y(x,y,-z) &= -B_y(x,y,+z), \\ B_z(x,y,-z) &= +B_z(x,y,+z), \end{aligned} \tag{100}$$

which for a z–independent field immediate leads to

$$B_x(x,y,z) \;=\; 0, \quad B_y(x,y,z) \;=\; 0 \quad \text{for any } (x,y,z). \tag{101}$$

Thus, *the symmetries of the system require the magnetic field everywhere — inside or outside the solenoid — to be parallel to the z axis.*

(b) Proceeding exactly as in the textbook example **5.9** and using the Ampere loop as on textbook figure **5.37**, we find that the magnetic field outside the solenoid must be uniform, the magnetic field outside the solenoid is also uniform, and the difference between them is

$$\mathbf{B}^{\text{inside}} \;-\; \mathbf{B}^{\text{outside}} \;=\; \mu_0 I(N/L)\,\hat{\mathbf{z}}. \tag{102}$$

This works for a solenoid of any cross-sectional shape, circular, square, star-shaped, whatever. In addition, the magnetic field far outside the solenoid should diminish to zero at infinitely long distance, hence

$$\mathbf{B}^{\text{outside}} = 0, \quad \mathbf{B}^{\text{inside}} = \mu_0 I(N/L)\,\hat{\mathbf{z}}. \tag{103}$$

(c) The magnetic field of a toroid of any cross-sectional shape is

$$\mathbf{B}^{\text{outside}} \equiv 0, \quad \mathbf{B}^{\text{inside}}(\mathbf{r}) = \frac{\mu_0 NI}{2\pi s(\mathbf{r})}\,\hat{\phi}\,, \tag{104}$$

where $s(\mathbf{r})$ is the distance between the point $\mathbf{r}$ inside the toroid and the rotational axis of symmetry.

Now let's make the overall radius of the toroid very large while keeping its cross-sectional shape and size fixed. In this limit, a finite piece of the toroid looks like a slightly bend solenoid, and in the strict $R \to \infty$ limit the curvature goes away, and the toroid becomes locally indistinguishable from an infinitely long straight solenoid. Also, the $\hat{\phi}$ direction of the toroid becomes the $\hat{\mathbf{z}}$ direction of the solenoid, so the magnetic field (104) points along the solenoid's axis, just as it should for a long solenoid.

As to the magnitude of the magnetic field inside the toroid, in the $R \to \infty$ limit, $s = R + \text{finite} \approx R$ and hence

$$2\pi s \approx 2\pi R = L, \quad \text{the length of the toroid.} \tag{105}$$

Consequently, the magnetic field (104) inside the toroid is approximately uniform — as it should be inside the long solenoid — and its magnitude is

$$B = \frac{\mu_0 IN}{L}\,, \tag{106}$$

precisely as the field inside the long solenoid with winding density N/L.

Note: in order to keep the magnetic field (106) fixed as we make the toroid larger and larger, we should increase the number of windings so that their density $n = N/L = N/(2\pi R)$ remains constant.

Problem **5.24**:

Given the vector potential $\mathbf{A}(x,y,z)$, the current density follows from the Ampere's Law,

$$\mu_0\mathbf{J} \;=\; \nabla\times\mathbf{B} \;=\; \nabla\times(\nabla\times\mathbf{A}) \;=\; \nabla(\nabla\cdot\mathbf{A}) \;-\; \nabla^2\mathbf{A}. \tag{107}$$

For the vector potential $\mathbf{A}=k\hat{\phi}$ (with $k=$ const) in the cylindrical ϕ direction, the divergence $\nabla\cdot\mathbf{A}$ vanishes, so the current density is simply

$$\mathbf{J} \;=\; \frac{1}{\mu_0}\nabla^2\mathbf{A} \;=\; \frac{k}{\mu_0}\nabla^2\hat{\phi}\,, \tag{108}$$

but it's actually easier to take the double curl that the Laplacian here. Indeed, using the formulae from the front cover of the textbook for the curl in cylindrical coordinates, we find

$$\nabla\times(k\hat{\phi}) \;=\; \frac{1}{s}\frac{\partial ks}{\partial s}\,\hat{\mathbf{z}} \;=\; \frac{k}{s}\,\hat{\mathbf{z}}\,, \tag{109}$$

$$\nabla\times\left(\frac{k}{s}\,\hat{\mathbf{z}}\right) \;=\; -\frac{\partial k/s}{\partial s}\,\hat{\phi} \;=\; +\frac{k}{s^2}\,\hat{\phi}\,, \tag{110}$$

and therefore

$$\mathbf{B} \;=\; \frac{k}{s}\,\hat{\mathbf{z}}\,,\quad \mathbf{J} \;=\; \frac{k}{\mu_0}\,\frac{\hat{\phi}}{s^2}\,. \tag{111}$$

Problem **5.25**:

(a) Let $\mathbf{A}=-\frac{1}{2}\mathbf{r}\times\mathbf{B}$ for a *uniform* magnetic field $\mathbf{B}$. By the Leibniz for divergence and curl of a vector product — see eqs. (6) and (8) inside the textbook cover —

$$\nabla\cdot\mathbf{A} \;=\; -\tfrac{1}{2}\mathbf{B}\cdot(\nabla\times\mathbf{r}) \;+\; \tfrac{1}{2}\mathbf{r}(\nabla\times\mathbf{B}), \tag{112}$$

$$\nabla\times\mathbf{A} = \;-\tfrac{1}{2}(\mathbf{B}\cdot\nabla)\mathbf{r} \;+\; \tfrac{1}{2}(\mathbf{r}\cdot\nabla)\mathbf{B} \;-\; \tfrac{1}{2}\mathbf{r}(\nabla\cdot\mathbf{B}) \;+\; \tfrac{1}{2}\mathbf{B}(\nabla\cdot\mathbf{r}). \tag{113}$$

However, since $\mathbf{B}$ is uniform, all of its space derivatives vanish, so the above formulae simplify to

$$\begin{aligned} \nabla\cdot\mathbf{A} &= -\tfrac{1}{2}\mathbf{B}(\nabla\times\mathbf{r}), \\ \nabla\times\mathbf{A} &= -\tfrac{1}{2}(\mathbf{B}\cdot\nabla)\mathbf{r} + \tfrac{1}{2}\mathbf{B}(\nabla\cdot\mathbf{r}). \end{aligned} \tag{114}$$

On the right hand sides here,

$$\nabla\times\mathbf{r} = 0, \quad \nabla\cdot\mathbf{r} = 3, \quad (\mathbf{B}\cdot\nabla)\mathbf{r} = \mathbf{B}, \tag{115}$$

which gives us

$$\nabla\cdot\mathbf{A} = 0 \tag{116}$$

and

$$\nabla\times\mathbf{A} = -\tfrac{1}{2}\mathbf{B} + \tfrac{3}{2}\mathbf{B} = \mathbf{B}. \tag{117}$$

Quod erat demonstrandum.

(b) Without any boundary condition, there other vector potentials for the same uniform $\mathbf{B}$, even in the transverse gauge. Indeed, let

$$\mathbf{A}(\mathbf{r}) = -\tfrac{1}{2}\mathbf{r}\times\mathbf{B} + \nabla\Lambda(\mathbf{r}) \tag{118}$$

for some scalar function $\Lambda(x,y,z)$. Any such vector potential has the same curl $\nabla\times\mathbf{A} = \mathbf{B}$, regardless of $\Lambda(\mathbf{r})$, but the divergence is Λ dependent, namely

$$\nabla\cdot\mathbf{A} = \nabla^2\Lambda. \tag{119}$$

Thus, in order to preserve the transverse gauge condition we are limited to Λ' which obey the Laplace equation $\nabla^2\Lambda(x,y,z)\equiv 0$.

Given enough boundary conditions for the vector potential $\mathbf{A}(\mathbf{r})$, the Laplace equation for the $\Lambda(x,y,z)$ would have no solutions except for the trivial $\Lambda = \text{const}$ — which would mean unique $\mathbf{A}(\mathbf{r}) = -\frac{1}{2}\mathbf{r}\times\mathbf{B}$. But in the absence of any boundary conditions — or asymptotic conditions at $\mathbf{r}\to\infty$ — the Laplace equation $\nabla^2\Lambda=0$ has an infinite series of independent solutions. In tensor notations, such solutions have general form

$$\Lambda(\mathbf{r}) \;=\; \sum_{i=x,y,z} T^{(1)}_i r_i \;+\; \sum_{i,j} T^{(2)}_{ij} r_i r_j \;+\; \sum_{i,j,k} T^{(3)}_{ijk} r_i r_j r_k \;+\; \cdots \tag{120}$$

where each $T(n)$ is an n-index totally symmetric tensor obeying a zero-trace condition,

$$\sum_{i=x,y,z} T^{(2)}_{ii} \;=\; 0, \qquad \sum_{i=x,y,z} T^{(3)}_{iik} \;=\; 0 \text{ for each } k, \quad \textit{etc.} \tag{121}$$

Consequently, there is an infinite family of vector potentials which all have the same curl and zero divergence.

For example, for the uniform $\mathbf{B}$ field in $+\hat{\mathbf{z}}$ direction, we may use $\mathbf{A}=-\frac{1}{2}\mathbf{r}\times\mathbf{B}=(B/2)(x\hat{\mathbf{y}}-y\hat{\mathbf{x}})$, but we may also use $\mathbf{A}=Bx\hat{\mathbf{y}}$ or $\mathbf{A}=-By\hat{\mathbf{x}}$ — as well as an infinite choice of more complicated vector potentials.

Problem **5.26**:

According to textbook equation (5.65), for a thick conductor

$$\mathbf{A}(\mathbf{r}) \;=\; \frac{\mu_0}{4\pi}\iiint \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d^3\mathrm{Vol}'. \tag{122}$$

The current in a long straight wire has uniform direction $\hat{\mathbf{z}}$, so the vector potential (122) at any point $\mathbf{r}$ — inside or outside the wire — must also point in the same direction. Furthermore, by the cylindrical symmetry of the wire, we should have

$$\mathbf{A}(s,\phi,z) \;=\; A(s\text{ only})\,\hat{\mathbf{z}}, \tag{123}$$

where the integral for the $A(s)$ looks exactly like the electrostatic potential of a uniformly charged wire. Indeed, by Gauss Law for the charged wire,

$$E(s) = \begin{cases} \dfrac{\lambda}{2\pi\epsilon_0}\,\dfrac{1}{s} & \text{outside the wire, for } s > R, \\[2ex] \dfrac{\lambda}{2\pi\epsilon_0}\,\dfrac{s}{R^2} & \text{inside the wire, for } s < R, \end{cases} \tag{124}$$

hence

$$V(s) = \begin{cases} -\dfrac{\lambda}{2\pi\epsilon_0} \times \ln\dfrac{s}{R} & \text{outside the wire, for } s > R, \\[2ex] +\dfrac{\lambda}{2\pi\epsilon_0}\,\dfrac{R^2 - s^2}{2R^2} & \text{inside the wire, for } s < R. \end{cases} \tag{125}$$

By analogy, the vector potential of the infinitely long straight wire is

$$A(s) = \begin{cases} -\dfrac{\mu_0 I}{2\pi} \times \ln\dfrac{s}{R} & \text{outside the wire, for } s > R, \\[2ex] +\dfrac{\mu_0 I}{2\pi}\,\dfrac{R^2 - s^2}{2R^2} & \text{inside the wire, for } s < R. \end{cases} \tag{126}$$

Alternative solution:

Instead of working by analogy with the electric potential of a charged wire, we may first use the Ampere's law to find the **B** field, and then solve for the $\mathbf{A} = A(s)\hat{\mathbf{z}}$ which has the right curl. As explained in my my notes on the Ampere's Law (example#1), the magnetic field inside and outside the wire is

$$\mathbf{B} = B(s)\hat{\phi} \quad \text{where} \quad B(s) = \begin{cases} \dfrac{\mu_0 I}{2\pi}\,\dfrac{1}{s} & \text{outside the wire, for } s > R, \\[2ex] \dfrac{\mu_0 I}{2\pi}\,\dfrac{s}{R^2} & \text{inside the wire, for } s < R. \end{cases} \tag{127}$$

We need to match that to

$$\mathbf{B} = \nabla \times \big(A(s)\hat{\mathbf{z}}\big) = -\frac{dA}{ds}\,\hat{\phi} \implies B(s) = -\frac{dA}{ds}\,, \tag{128}$$

hence

$$A(s) = \int_s^{\text{const}} B(s')\,ds'. \tag{129}$$

Setting the constant reference point here to the surface of the wire and integrating the wire's

B field from eq. (127), we arrive at

$$A(s) \;=\; \begin{cases} -\dfrac{\mu_0 I}{2\pi} \times \ln\dfrac{s}{R} & \text{outside the wire, for } s > R, \\[2ex] +\dfrac{\mu_0 I}{2\pi}\,\dfrac{R^2 - s^2}{2R^2} & \text{inside the wire, for } s < R. \end{cases} \tag{126}$$

Problem **5.30**:

Textbook equation (5.68) gives the vector potential of a rotating charged spherical shell of surface charge density σ,

$$\mathbf{A}^{\text{sphere}}(\mathbf{r}) \;=\; \frac{\mu_0\sigma}{3}\, f(r,R)\,(\vec{\omega}\times\mathbf{r}) \quad \text{where} \quad f(r,R) \;=\; \begin{cases} R & \text{for } r < R, \\[1ex] \dfrac{R^4}{r^3} & \text{for } r > R. \end{cases} \tag{130}$$

A solid ball of uniform volume charge density ρ can be thought as a sum of co-rotating concentric shells of radii r' ranging from 0 to R of surface density $\sigma = \rho\,dr'$. Hence, by the superposition principle,

$$\mathbf{A}^{\text{ball}}(\mathbf{r}) \;=\; \frac{\mu_0\rho}{3} F(r,R)\,(\vec{\omega}\times\mathbf{r}) \quad \text{where} \quad F(r,R) \;=\; \int\limits_0^R f(r,r')\,dr'. \tag{131}$$

All we need to do is to perform the integral here.

For $r < R$ — *i.e.*, for calculating the $\mathbf{A}$ inside the ball — we need to include shells with both $r' < r$ and $r' > r$. Consequently,

$$\begin{aligned} F(r<R) \;&=\; \int\limits_0^r f(r<r')\,dr' \;+\; \int\limits_r f(r>r')\,dr \\ &=\; \int\limits_0^r r'\,dr' \;+\; \int\limits_r^R \frac{r'^4}{r^3}\,dr' \\ &=\; \frac{r^2}{2} \;+\; \frac{R^5 - r^5}{5r^3} \\ &=\; \frac{3r^5 + 2R^5}{10r^3}\,, \end{aligned} \tag{132}$$

and therefore

$$\text{inside the ball,}\quad \mathbf{A}(\mathbf{r}) \;=\; \frac{\mu_0\rho}{30}\,\frac{3r^5+2R^5}{r^3}\,(\vec{\omega}\times\mathbf{r}) \tag{133}$$

Our last task is to take the curl of this vector potential to get the magnetic field **B**. In general,

$$\begin{aligned}
\nabla\times\big(F(r)\,(\vec{\omega}\times\mathbf{r})\big) \;&=\; F(r)\,\nabla\times(\vec{\omega}\times\mathbf{r}) \;+\; \nabla F(r)\times(\vec{\omega}\times\mathbf{r})\\
&=\; F(r)\,(2\vec{\omega}) \;+\; \frac{dF}{dr}\times\hat{\mathbf{r}}(\vec{\omega}\times\mathbf{r})\\
&=\; 2F\,\vec{\omega} \;+\; rF'(R)\Big(\hat{\mathbf{r}}\times(\vec{\omega}\times\hat{\mathbf{r}}) \;=\; \vec{\omega} \;-\; (\hat{\mathbf{r}}\cdot\vec{\omega})\hat{\mathbf{r}}\Big)\\
&=\; (2F+rF')\vec{\omega} \;-\; rF'\,(\hat{\mathbf{r}}\cdot\vec{\omega})\hat{\mathbf{r}}.
\end{aligned} \tag{134}$$

so for the $F(r)$ as in eq. (133),

$$\mathbf{B}(\mathbf{r}) \;=\; \frac{\mu_0\rho}{30}\left(\frac{12r^5-2R^5}{r^3}\,\vec{\omega} \;+\; \frac{6r^5-6R^5}{r^3}\,(\hat{\mathbf{r}}\cdot\vec{\omega})\hat{\mathbf{r}}\right). \tag{135}$$