ELECTROSTATIC THEOREMS

In these notes I prove several important theorems concerning the electrostatic potential $V(x, y, z)$, namely the Earnshaw theorem, the mean-value theorem, and two uniqueness theorems. Mathematically, all these theorems stem from the Laplace equation

$$\triangle V \stackrel{\text{def}}{=} \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{1}$$

which the potential obeys in the absence of electric charges. Or rather, the potential obeys the Laplace equation in all parts of space which are not occupied by the charges. Thus, if there are no charges inside some volume $\mathcal{V}$ but there are many charges outside that volume, then inside $\mathcal{V}$ the potential obeys the Laplace equation.

Earnshaw Theorem

A scalar field $V(x, y, z)$ obeying the Laplace equation does not have any local maxima or minima; all its stationary points are saddle points. For a potential which obeys the Laplace equation only inside some volume $\mathcal{V}$, the theorem excludes any local maxima or minima strictly inside $\mathcal{V}$ but allows minima and maxima on the boundary of the volume $\mathcal{V}$ in question.

Physically, the Earnshaw theorem means that *It is impossible to keep a charged particle or body in stable static equilibrium by means of electrostatic forces alone.* Indeed, a stable static equilibrium requires a minimum — or at least a local minimum — of the potential energy. For a charged probe particle subject to the electrostatic forces alone, the potential energy is simply $U(x, y, z) = q \times V(x, y, z)$, so a local minimum of U translates into a local minimum of the electric potential V if the charge q is positive, or to a local maximum of the potential if the charge is negative. Alas, neither local minima nor local maxima of the potential are allowed by the Earnshaw theorem, hence a stable static equilibrium due to electric forces alone is quite impossible.

Proof: Any local minimum or maximum of $V(x, y, z)$ that does not lie on the boundary must be a stationary point — *i.e.*, the point where the gradient ∇V vanishes, — but not every stationary point is a maximum or a minimum, some of them are saddle points. In

general, the distinction between a minimum, a maximum, and a saddle points follows from the matrix of second derivatives

$$\mathcal{M} \;=\; \begin{pmatrix} \frac{\partial^2 V}{\partial x\,\partial x} & \frac{\partial^2 V}{\partial x\,\partial y} & \frac{\partial^2 V}{\partial x\,\partial z} \\ \frac{\partial^2 V}{\partial y\,\partial x} & \frac{\partial^2 V}{\partial y\,\partial y} & \frac{\partial^2 V}{\partial y\,\partial z} \\ \frac{\partial^2 V}{\partial z\,\partial x} & \frac{\partial^2 V}{\partial z\,\partial y} & \frac{\partial^2 V}{\partial z\,\partial z} \end{pmatrix} \tag{2}$$

evaluated at the stationary point. This symmetric matrix has 3 real eigenvalues M_1, M_2, and M_3; If all three of these eigenvalues are positive, then the stationary point is a minimum; if all three eigenvalues are negative, then the stationary point is a maximum; and if some eigenvalue(s) are positive while other(s) are negative, than the stationary point in question is a saddle point.

By definition of the Laplacian, it's the sum of the diagonal elements of the matrix (2),

$$\triangle V \;=\; \mathcal{M}_{xx} \;+\; \mathcal{M}_{yy} \;+\; \mathcal{M}_{zz} \;=\; \mathrm{tr}(\mathcal{M}) \tag{3}$$

which is also known as the *trace* of the matrix. Hence, for the potential $V(x, y, z)$ which obeys the Laplace equation $\triangle V = 0$, the matrix (2) has zero trace, $\mathrm{tr}(\mathcal{M}) = 0$. But the trace of a matrix governs the sum of its eigenvalues,

$$M_1 \;+\; M_2 \;+\; M_3 \;=\; \mathrm{tr}(\mathcal{M}) \tag{4}$$

hence zero trace translates to

$$M_1 \;+\; M_2 \;+\; M_3 \;=\; 0. \tag{5}$$

Obviously, this zero sum is incompatible with all 3 eigenvalues being positive — which rules out a local minimum — or all 3 eigenvalues being negative, which rules out a local maximum. Instead, among the three eigenvalues either two must be positive while the third is negative or else two are negative with the third is positive. Either way, this makes the stationary point a saddle point. *Quod erat demonstrandum.*

CAVEAT: There is also the degenerate option in which all three eigenvalues vanish, $M_1 = M_2 = M_3 = 0$, which happens only when all of the second derivatives of $V(x, y, z)$ vanish, thus $\mathcal{M} = 0$ as a matrix. In this case, the distinction between a maximum, a minimum, and a saddle point follow from the higher derivatives. Such higher derivatives form a tensor rather than a matrix, but its form is severely restricted by the requirement of zero Laplacian $\triangle V = 0$, and consequently only the saddle points are allowed but not the local minima or the local maxima.

ADDENDUM: Unlike in empty space devoid of charges, in regions with $\rho(x, y, z) \neq 0$ the electric potential may have local maxima or minima. Specifically, *in the positively charged regions V may have local maxima but not minima, while in the negatively charged regions V may have local minima but not maxima.*

Proof: Suppose $V(x, y, z)$ has a stationary point P in the region where $\rho \neq 0$, and consider the matrix (2) of second derivatives of V at that point. By the Poisson equation,

$$\triangle V \;=\; -\frac{\rho}{\epsilon_0}\,, \tag{6}$$

the 3 eigenvalues of the matrix $\mathcal{M}$ add up to

$$M_1 \,+\, M_2 \,+\, M_3 \;=\; \mathrm{tr}(\mathcal{M}) \;=\; \triangle V(P) \;=\; -\frac{\rho(P)}{\epsilon_0}\,. \tag{7}$$

IF the charge density $\rho(P)$ happens to be positive, then all three of the eigenvalues *may* be negative, $M_1, M_2, M_3 < 0$, which makes the stationary point P a local maximum. But note the emphasis on 'may': all 3 eigenvalues do not have to be negative, one or two of them may be positive (as long as the sum is negative), and that would mean that P is a saddle point. However, a local minimum is not allowed — we cannot have 3 positive eigenvalues adding up to a negative $-\rho/\epsilon_0$.

Similarly, if the charge density at P is negative, then M_1, M_2, M_3 add up to a positive number, so its possible for all 3 of them to be positive, — which would make the stationary point P a local minimum. But it is also possible for one or two eigenvalues to be negative, in which case P is a stationary point. However, a local maximum is not allowed for $\rho(P) < 0$ since 3 negative eigenvalues cannot add up to a positive $-\rho/\epsilon_0$.

Physically, the addendum means that one may stabilize a positive probe charge inside a region of negative charge density $\rho < 0$, or a negative probe charge inside a region of positive charge density $\rho > 0$. However, while stabilizing a probe charge, the background charges giving rise to the $\rho \neq 0$ would be de-stabilized, and we would need some non-electrostatic forces to keep them from flying away. So the bottom line of the Earnshaw theorem is: *No system of charged particles or bodies can be kept in stable static equilibrium by the electrostatic forces alone.* We may stabilize some of the charges, but then some other charges would be unstable.

Mean Value Theorem

Take a sphere, or any radius R, centered anywhere you like. If there are no electric charges inside that sphere, then the mean value of the potential over the sphere equals to the potential at the sphere's center,

$$V_{\text{mean}} \stackrel{\text{def}}{=} \frac{1}{4\pi R^2} \iint\limits_{\text{sphere}} V \, d^2\text{Area} \;=\; V(\text{center}). \tag{8}$$

The mean value theorem implies the Earnshaw theorem without using Calculus. Indeed, surround any would-be local maximum or minimum by a small sphere. The highest potential on that sphere must be higher than the sphere's mean, so by the MVT it must be higher than the potential at the center, — which rules out the center being a local maximum. Likewise, the lowest potential on the sphere must be lower than the sphere's mean and hence than the potential at the center, — which rules out the center being a local minimum. Consequently, any point which may be surrounded by a small sphere without any charges in it cannot be a local minimum or maximum of $V(x, y, z)$. Therefore, for any empty volume $\mathcal{V}$, the minimum and the maximum of the potential must lie on the boundary of $\mathcal{V}$ — which is the only place where we cannot surround such a minimum or maximum with an empty sphere.

There are many ways to prove the mean value theorem, so let me show you two: a proof using vector calculus and a proof based on Coulomb Law. First, the **r**vector-calculus based proof: Consider a sphere of radius R centered at a point $\mathbf{c} = (c_x, c_y, c_z)$, so a generic point

on this sphere can be described in vector notations as

$$\mathbf{r} \;=\; \mathbf{c} \;+\; R\mathbf{n} \tag{9}$$

for any unit vector $\mathbf{n}$. Averaging the potential over the sphere's area is equivalent to averaging it over all 4π directions of $\mathbf{n}$ (while $\mathbf{c}$ and R are held constant), thus

$$V_{\rm avg}(\mathbf{c}, R) \;=\; \frac{1}{4\pi}\iint V(\mathbf{c}+R\mathbf{n})\, d^2\Omega(\mathbf{n}). \tag{10}$$

In this formula, the directions of $\mathbf{n}$ may be parametrized by the latitude and longitude angles (θ, ϕ), but we do not need the details here, it's enough that those coordinates (and the ranges in which they vary) do not depend on the sphere's radius R.

Now let's consider the dependence of the average potential (10) on the sphere's radius R. Taking the derivative $\partial V_{\rm avg}/\partial R$ for fixed center location $\mathbf{c}$, we have

$$\frac{\partial}{\partial R}V_{\rm avg}(\mathbf{c}, R) \;=\; \frac{1}{4\pi}\frac{\partial}{\partial R}\iint V(\mathbf{c}+R\mathbf{n})\, d^2\Omega(\mathbf{n}) \;=\; \frac{1}{4\pi}\iint d^2\Omega(\mathbf{n}) \times \left(\frac{\partial V(\mathbf{c}+R\mathbf{n})}{\partial R}\right)^{\text{fixed }\mathbf{c},\mathbf{n}}. \tag{11}$$

On the RHS here we have moved the derivative inside the integral, which means we first differentiate WRT R for fixed $\mathbf{c}$ and $\mathbf{n}$, and only then integrate over the $\mathbf{n}$. Let's evaluate this derivative using the chain rule:

$$\left(\frac{\partial V(\mathbf{c}+R\mathbf{n})}{\partial R}\right)^{\text{fixed}}_{\mathbf{c},\mathbf{n}} \;=\; \big(\nabla V(\mathbf{c}+R\mathbf{n})\big)\cdot\frac{d(\mathbf{c}+R\mathbf{n})}{dR} \;=\; \big(\nabla V(\mathbf{c}+R\mathbf{n})\big)\cdot\mathbf{n} \;=\; -\mathbf{E}(\mathbf{c}+R\mathbf{n})\cdot\mathbf{n}. \tag{12}$$

Consequently,

$$\frac{\partial}{\partial R}V_{\rm avg}(\mathbf{c}, R) \;=\; -\frac{1}{4\pi}\iint \mathbf{E}(\mathbf{c}+R\mathbf{n})\cdot\mathbf{n}\, d^2\Omega(\mathbf{n}) \;=\; -\frac{1}{4\pi R^2}\iint\limits_{\text{sphere}} \mathbf{E}\cdot d^2\mathbf{A} \tag{13}$$

where the second equality follows from

$$d^2\mathbf{A} \;=\; \mathbf{n}\, d^2A \;=\; \mathbf{n}\, R^2\, d^2\Omega(\mathbf{n}) \quad \text{for the spherical surface.} \tag{14}$$

Now, the integral on the RHS of eq. (13) is the electric flux through the spherical surface,

and by the Gauss Law Ir's related to the net electric charge inside the sphere. Thus,

$$\frac{\partial}{\partial R} V_{\rm avg}(\mathbf{c}, R) \;=\; -\frac{1}{4\pi R^2} \times \frac{Q^{\rm net}[\text{inside the sphere}]}{\epsilon_0}\,. \tag{15}$$

In particular, if there are no charges inside the sphere then

$$\frac{\partial}{\partial R} V_{\rm avg}(\mathbf{c}, R) \;=\; 0. \tag{16}$$

Moreover, there would be no charges inside any smaller sphere with the same center $\mathbf{c}$, thus

$$\text{for any } R' \le R, \quad \frac{\partial}{\partial R'} V_{\rm avg}(\mathbf{c}, R') \;=\; 0, \tag{17}$$

which means that *the mean potential over the sphere does not depend on the radius* (as long as there no charges inside it). In particular, the mean potential over the original sphere is the same as the mean potential over the infinitesimally small sphere around the center — which is obviously equal to the potential at the center,

$$V_{\rm avg}(\mathbf{c}, R) \;=\; \lim_{R' \to 0} V_{\rm avg}(\mathbf{c}, R') \;=\; V(\mathbf{c}. \tag{18}$$

In other words, *the mean potential over any sphere which does not contain any electric charges is equal to the potential at the sphere's center, quod erat demonstrandum.*

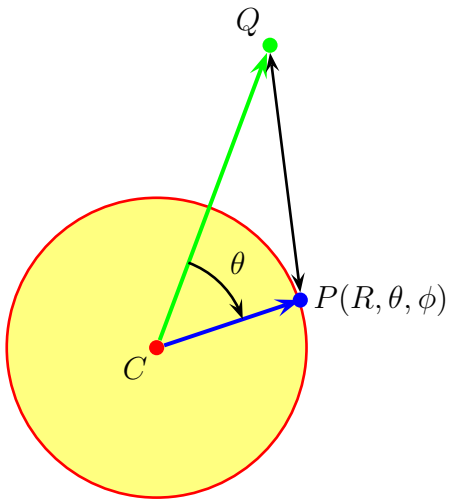
Alternative proof from Coulomb Law: Instead of using the Laplace equation, let's presume that the potential $V(x, y, z)$ is due to some charges (discrete or continuous) *outside the sphere,* and then use the Coulomb law to prove the mean value theorem.

Lemma: *the mean value theorem* (8) *holds true for the Coulomb potential of a point charge outside the sphere in question,*

$$V(\mathbf{r}) \;=\; \frac{Q}{4\pi\epsilon_0} \times \frac{1}{|\mathbf{r} - \mathbf{r}_{\rm charge}|}\,. \tag{19}$$

Before we average this potential over the surface of the sphere, let's set up our notations: $r_q = |\mathbf{c} - \mathbf{r}_{\rm charge}|$ is the distance from the charge Q to the sphere's center C; $R < r_q$ is the

sphere's radius; and the sphere itself is paramerized by the latitude and longitude angles (θ, ϕ) relative to the "North pole" $\theta = 0$ pointing towards the charge. Consequently, the angle between the radius-vector CP of some point $P = (R, \theta, \phi)$ on the sphere and the direction CQ from the center to the charge is simply the latitude angle θ,



(20)

By the cosine theorem, the distance^2 between point P and the charge Q is

$$|QP|^2 \;=\; r_q^2 \;+\; R^2 \;-\; 2r_qR \times \cos\theta, \tag{21}$$

hence the potential at the point P is

$$V(P) \;=\; \frac{Q}{4\pi\epsilon_0} \times \frac{1}{\sqrt{r_q^2 + R^2 - 2r_qR\cos\theta}}\,. \tag{22}$$

Now let's average this potential over the surface of the sphere.

$$\begin{aligned} V_{\text{mean}} \;&=\; \frac{1}{4\pi R^2} \iint\limits_{\text{sphere}} \frac{Q}{4\pi\epsilon_0} \times \frac{1}{\sqrt{r_q^2 + R^2 - 2r_qR\cos\theta}} \times \Big(d^2A \;=\; R^2\,\sin\theta\,d\theta\,d\phi\Big) \\ &=\; \frac{Q}{4\pi\epsilon_0} \times \frac{R^2}{4\pi R^2} \int\limits_0^{\pi} \frac{\sin\theta\,d\theta}{\sqrt{r_q^2 + R^2 - 2r_qR\cos\theta}} \times \int\limits_0^{2\pi} d\phi \\ &=\; \frac{Q}{4\pi\epsilon_0} \times \frac{2\pi}{4\pi} \int\limits_0^{\pi} \frac{\sin\theta\,d\theta}{\sqrt{r_q^2 + R^2 - 2r_qR\cos\theta}}\,. \end{aligned} \tag{23}$$

In the remaining integral over the latitude angle θ, we may use

$$\sin\theta\, d\theta \;=\; -d(\cos\theta) \tag{24}$$

and hence

$$\frac{\sin\theta\, d\theta}{\sqrt{r_q^2+R^2-2r_qR\cos\theta}} \;=\; -\frac{d\cos\theta}{\sqrt{r_q^2+R^2-2r_qR\cos\theta}} \;=\; +\frac{1}{r_qR}\, d\Big(\sqrt{r_q^2+R^2-2r_qR\cos\theta}\Big). \tag{25}$$

Therefore, the integral over θ evaluates to

$$\begin{aligned}\int_0^\pi \frac{\sin\theta\, d\theta}{\sqrt{r_q^2+R^2-2r_qR\cos\theta}} \;&=\; \frac{1}{r_qR}\Big[\sqrt{r_q^2+R^2-2r_qR\cos\theta}\Big]_{\theta=0}^{\theta=\pi} \\ &=\; \frac{1}{r_qR}\Big[\sqrt{r_q^2+R^2+2r_qR}\;-\;\sqrt{r_q^2+R^2-2r_qR}\Big] \\ &=\; \frac{1}{r_qR}\Big[(r_q+R)\;-\;|r_q-R|\Big].\end{aligned} \tag{26}$$

Note that the square root here is the distance between points P and Q, so it must be the *positive* square root, that's why the second term on the second line becomes the absolute value

$$\sqrt{r_q^2+R^2-2r_qR} \;=\; \sqrt{(r_q-R)^2} \;=\; |r_q-R|. \tag{27}$$

For the charge outside the sphere we have

$$r_q > R \quad\Longrightarrow\quad (r_q+R)\;-\;|r_q-R| \;=\; (r_q+R)\;-\;(r_q-R) \;=\; 2R \tag{28}$$

while for the charge inside the sphere

$$r_q > R \quad\Longrightarrow\quad (r_q+R)\;-\;|r_q-R| \;=\; (r_q+R)\;-\;(R-r_q) \;=\; 2r_q\,, \tag{29}$$

hence in general, the integral (26) evaluates to

$$\int_0^\pi \frac{\sin\theta\, d\theta}{\sqrt{r_q^2+R^2-2r_qR\cos\theta}} \;=\; \begin{cases} \dfrac{2}{r_q} & \text{when } r_q > R, \\[2ex] \dfrac{2}{R} & \text{when } r_q < R, \end{cases} \tag{30}$$

hence the mean potential from eq. (23) is

$$\text{for } r_q > R, \quad V_{\text{mean}} \;=\; \frac{Q}{4\pi\epsilon_0} \times \frac{1}{r_q}\,, \tag{31}$$

$$\text{for } r_q < R, \quad V_{\text{mean}} \;=\; \frac{Q}{4\pi\epsilon_0} \times \frac{1}{R}\,. \tag{32}$$

Note: both of these formulae would be very useful for the homework problem **3.1** (set#4). But for the lemma at hand we assume the charge Q outside the sphere, $r_q > R$, so let's focus on eq. (32). The RHS of this formula is manifestly equal to the Coulomb potential at the sphere's center — which is at distance r_q from the charge —

$$V(C) \;=\; \frac{Q}{4\pi\epsilon_0} \times \frac{1}{r_q} \;=\; V_{\text{mean}}\,. \tag{33}$$

so *this completes the proof of the Lemma.*

The theorem for multiple charges — discrete or continuous, but all outside the sphere — follows from the lemma by the superposition principle. For example consider several point charges Q_i, all outside the sphere. At any point P on the sphere, their potentials add up to the total potential,

$$V_{\text{net}}(P) \;=\; \sum_i V(P)[\text{due to } Q_i], \tag{34}$$

hence after averaging over the sphere

$$V_{\text{net}}^{\text{mean}} \;=\; \sum_i V^{\text{mean}}[\text{due to } Q_i]. \tag{35}$$

By the Lemma, each term on the RHS here equals to the potential at the center C due to the same charge Q_i, hence

$$V_{\text{net}}^{\text{mean}} \;=\; \sum_i V(C)[\text{due to } Q_i] \;=\; V_{\text{net}}(C). \tag{36}$$

Thus, the net potential averaged over the sphere equals to the net potential at the center.

Generalizing this formula to continuous charges — or to a mix of discrete and continuous charges — as long as all of them lie outside the sphere — is completely straightforward. And this completes the proof of the mean value theorem for the electric potential.

Uniqueness Theorem

The solution to the Laplace equation $\triangle V(x,y,z) \equiv 0$ inside a volume $\mathcal{V}$ is uniquely determined by the boundary condition specifying V on the entire boundary $\mathcal{S}$ of the volume $\mathcal{V}$.

In other words, consider some volume $\mathcal{V}$ without any electric charges except maybe on the surface $\mathcal{S}$, so that *inside* $\mathcal{V}$ the potential $V(x,y,z)$ obeys the Laplace equation $\triangle V = 0$. Suppose we know V(surface coordinates) everywhere on the surface $\mathcal{S}$. Than this knowledge — plus the Laplace equation inside $\mathcal{V}$ — completely determines the potential $V(x,y,z)$ everywhere inside the volume $\mathcal{V}$.

The uniqueness theorem allows us to use all kinds of heuristic tools for finding the potential, even if those tools are not reliable. Once we get a would-be solution for the $V(x,y,z)$ from any such tool — or just from a lucky guess — all we need to do is to check that it obeys the Laplace equation and has the right boundary values. If it does, than it is THE solution — by the uniqueness theorem, there are no other solutions, so once we find one, this is it!

Proof: Suppose we have two solutions, $V_1(x,y,z)$ and $V_2(x,y,z)$, with the same boundary values but different values in the interior. Let

$$V_3(x,y,z) \;=\; V_1(x,y,z) \;-\; V_2(x,y,z) \tag{37}$$

be the difference between the two solutions. The V_3 also obeys the Laplace equation $\triangle V_3 = 0$, but its boundary conditions are much simple,

$$V_3 \;\equiv\; 0 \quad \text{over the entire boundary } \mathcal{S}. \tag{38}$$

Next, consider the vector field $V_3 \nabla V_3$. By the boundary condition (38) it must vanish

everywhere on the boundary, hence

$$\iint_{\mathcal{S}} V_3 \, \nabla V_3 \cdot d^2\mathbf{A} \;=\; 0. \tag{39}$$

OOH, by the Gauss Theorem, this surface integral equals to the volume integral of the divergence,

$$0 \;=\; \iint_{\mathcal{S}} V_3 \, \nabla V_3 \cdot d^2\mathbf{A} \;=\; \iiint_{\mathcal{V}} \big(\nabla \cdot (V_3 \nabla V_3)\big) d^3\mathrm{Vol}. \tag{40}$$

In the integrand of this volume integral

$$\nabla \cdot (V_3 \nabla V_3) \;=\; (\nabla V_3) \cdot (\nabla V_3) \;+\; V_3 (\nabla^2 V_3) \;=\; (\nabla V_3)^2 \;+\; 0 \tag{41}$$

where the second equality stems from the Laplace equation $\nabla^2 V_3 = 0$. Plugging this formula back into the volume integral (40), we arrive at

$$\iiint_{\mathcal{V}} (\nabla V_3)^2 \, d^3\mathrm{Vol} \;=\; 0. \tag{42}$$

The integrand here is non-negative — it's the square of the vector ∇V_3 — so the only way this integral can vanish is if the integrand vanishes *everywhere*. Thus, eq. (42) requires

$$(\nabla V_3)^2 \;\equiv\; 0 \text{ everywhere} \quad\Longrightarrow\quad \nabla V_3 \;\equiv\; 0 \text{ everywhere} \quad\Longrightarrow\quad V_3(x,y,z) \;\equiv\; \text{const.} \tag{43}$$

The specific value of this constant — namely zero — obtains from the boundary conditions (38): A constant which vanishes at the boundary vanishes everywhere.

Altogether, the Laplace equation and the boundary conditions require $V_3(x,y,z) \equiv 0$ and therefore

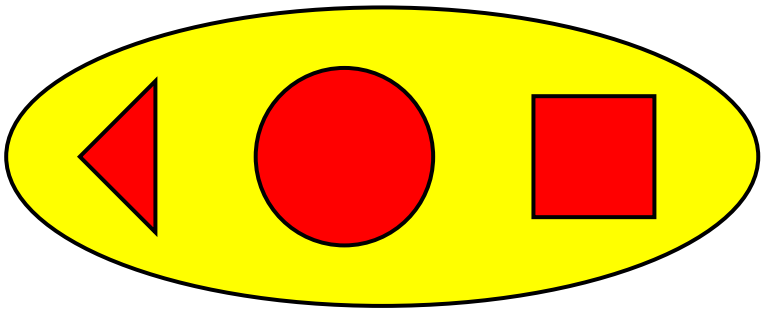
$$V_1(x,y,z) \;\equiv\; V_2(x,y,z). \tag{44}$$

In other words, there is only one solution to the Laplace equation plus the boundary conditions, *quod erat demonstrandum.*

Let me conclude with a note on volumes and boundaries in this uniqueness theorem. In the simplest case, $\mathcal{V}$ is some finite volume without holes, and $\mathcal{S}$ is its complete outer boundary. But the uniqueness theorem works just as well for the more complicated volumes and surfaces. In particular, $\mathcal{V}$ may have holes like the donut or even cavities. Indeed, consider some larger volume $\widetilde{\mathcal{V}}$ which is mostly empty but contains a few charged bodies. Inside a charged body, $\triangle V(x,y,z) \propto \rho(x,y,z) \neq 0$, so the Laplace equation does not hold, but it holds for the empty part of the volume,

$$\mathcal{V} \;=\; \widetilde{\mathcal{V}} \;-\; \sum_i (\text{body})_i \,, \tag{45}$$

for example



$$\mathcal{V} \;=\; \text{outer ellipse} \;-\; \text{triangle} \;-\; \text{circle} \;-\; \text{square}. \tag{46}$$

However, if the volume $\mathcal{V}$ in questions have cavities — *i.e.*, excludes islands occupied by charged bodies — than we must specify the boundary values of V over the complete surface $\mathcal{S}$ which includes both the outer boundary and the inner boundaries of the excluded islands.

Finally, it is OK for the volume $\mathcal{V}$ to be infinite in some directions. In this case, the solution is unique if we specify both the values of V over the actual boundaries and the asymptotic values of V for $R \to \infty$ in the infinite directions. Usually, the asymptotic value of the potential at $R \to \infty$ is set to zero.

For example, take $\mathcal{V}$ to be the upper half of the 3D space:

$$\mathcal{V} \text{ spans } (x,y,z): \;\; \text{any } x, \;\; \text{any } y, \;\; \text{but } z > 0 \text{ only}. \tag{47}$$

To get a unique solution for the $\triangle V(x,y,z) \equiv 0$ in this half-space, we must specify **both**

$$@z = 0, \quad V(x,y,0) \;=\; \text{given } V_{\text{boundary}}(x,y), \tag{48}$$

and

$$\lim_{R\to\infty} V(R,\theta\,\phi) \;=\; 0 \quad \text{in any open direction } (\theta,\phi), \quad \theta < \tfrac{\pi}{2}\,. \tag{49}$$

For a more specific example, let

$$V_{\text{boundary}}(x,y) \;=\; \frac{A}{\sqrt{x^2+y^2+D^2}} \tag{50}$$

for some constants A and D. This boundary potential looks like the Coulomb potential of a point charge hiding below the boundary at $(x_c = 0, y_c = 0, z_c = -D)$, which immediately suggest the solution

$$V(x,y,z) \;=\; \frac{A}{\sqrt{x^2+y^2+(D+z)^2}} \quad \text{for } z \geq 0 \text{ only!} \tag{51}$$

Indeed, this solution obeys both boundary conditions — for $z = 0$ and for $R \to \infty$ — and the Laplace equation for $z > 0$,

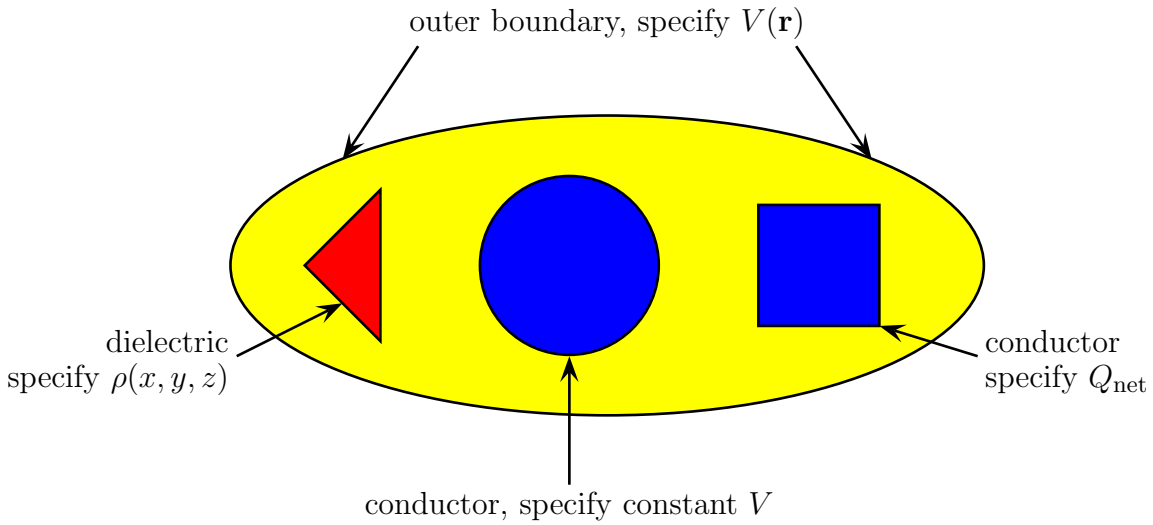
$$\triangle V(x,y,z) \;=\; -4\pi A \times \delta(x)\delta(y)\delta(z+D) \;=\; 0 \quad \text{for } z > 0, \tag{52}$$

so it must be the unique solution to this problem.

Note that the solution is limited to the potential above the boundary, and we do not know what happens below the boundary. Maybe there is an actual point charge at $(0, 0, -D)$, or maybe there are some surface charges $\sigma(x, y)$ right at the boundary, or maybe something more complicated, — we do not know and we don't care: By the uniqueness theorem, the potential above the boundary will be exactly the same (51) in any set up which produces the same potential (50) on the boundary itself.

Second Uniqueness Theorem

The second uniqueness theorem allows for a more general setup than the first theorem. *Suppose some volume $\mathcal{V}$ is not empty but contains several charged bodies, dielectrics or conductors. For each conducting body, we specify either the net electric charge or the potential (which is constant over a conductor). For each dielectric body, we specify the entire charge distribution, i.e. $\rho(x,y,z)$ as a function of position within the body. Finally, we specify the potential $V(\mathbf{r})$ along the outer boundary $\mathcal{S}$ of the volume $\mathcal{V}$. Under these conditions, there is a unique solution for the potential $V(x,y,z)$ inside $\mathcal{V}$ — including the dielectric and the conducting bodies themselves.*



Proof: Let me start similar to the first uniqueness theorem. Suppose we have two solutions, $V_1(x,y,z)$ and $V_2(x,y,z)$ which obey the same boundary conditions and the same Poisson equation

$$\triangle V_1(x,y,z) \;=\; \triangle V_2(x,y,z) \;=\; -\frac{\rho(x,y,z)}{\epsilon_0}\,. \tag{53}$$

Consider the difference between the two solutions,

$$\begin{aligned} V_3(x,y,z) \;&=\; V_1(x,y,z) \;-\; V_2(x,y,z), \\ \mathbf{E}_3(x,y,z) \;&=\; \mathbf{E}_1(x,y,z) \;-\; \mathbf{E}_2(x,y,z) \;=\; -\nabla V_3(x,y,z). \end{aligned} \tag{54}$$

In the empty part of the volume $\mathcal{V}$, both solutions V_1 and V_2 obey the Laplace equation, so the difference V_3 also obeys $\triangle V_3 = 0$. Inside a dielectric body, the solutions V_1 and V_2 obey

the Poisson equation (53) for the same charge density $\rho(x,y,z)$, so the difference V_3 obeys the Laplace equation $\triangle V_3 = 0$ regardless of $\rho(x,y,z)$. Indeed,

$$\triangle V_3(x,y,z) \;=\; \triangle V_1(x,y,z) \;-\; \triangle V_2(x,y,z) \;=\; \frac{-1}{\epsilon_0}\rho(x,y,z) \;-\; \frac{-1}{\epsilon_0}\rho(x,y,z) \;=\; 0. \quad (55)$$

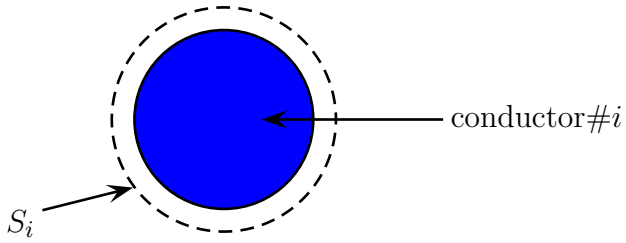
Thus, the difference obeys the Laplace equation everywhere outside the conductors.

Now consider the boundary conditions for the $V_3(x,y,z)$. On the outer boundary $\mathcal{S}$ of the volume $\mathcal{V}$, both V_1 and V_2 have the same specified boundary values $V_{\text{boundary}}(\mathbf{r})$, so the difference $V_3(\mathbf{r})$ must vanish. The same is true for the surface of a conducting body for which we have specified the potential V — on any such surface $V_3 = 0$. But the surfaces of conductors for which we have specified the net charge instead of the potential are trickier: All we know that on such a surface $V_1 = \text{const}$ and $V_2 = \text{const}'$, but the two constants might be different, hence $V_3 = \text{const}''$ but not necessarily $V_3 = 0$.

To see that the V_3 has to vanish over all the boundaries — and hence throughout the volume $\mathcal{V}$ — let's consider the integrals

$$I_0 \;=\; \iint\limits_{\mathcal{S}} V_3\mathbf{E}_3\cdot d^2\mathbf{A}, \quad \text{and} \quad I_i \;=\; \iint\limits_{S_i} V_3\mathbf{E}_3\cdot d^2\mathbf{A} \qquad (56)$$

where $\mathcal{S}$ is the outer surface of $\mathcal{V}$ while the S_i is the complete surface of the conductor$\#i$. Or rather, the S_i follows the conductor's surface just a tiny bit outside it.



This way, we may identify the $\mathbf{E}$ on S_i as the normal electric field just outside the conductor's surface, but at the same time, the potential $V = V_{\text{conductor}} = \text{const}$.

Going back to the integrals (55), for the outer surface $\mathcal{S}$, the integral $I_0 = 0$ because $V_3 = 0$ on $\mathcal{S}$. For the same reason, $I_i = 0$ for the conductors for which we have specified the

potential. For the remaining conductors, we use $V_3 = \text{const}$, thus

$$I_i \;=\; \iint\limits_{S_i} V_3 \mathbf{E}_3 \cdot d^2\mathbf{A} \;=\; V_3 \times \iint\limits_{S_i} \mathbf{E}_3 \cdot d^2\mathbf{A} \tag{57}$$

where thanks to the Gauss Law

$$\begin{aligned} \iint\limits_{S_i} \mathbf{E}_3 \cdot d^2\mathbf{A} \;&=\; \iint\limits_{S_i} \mathbf{E}_1 \cdot d^2\mathbf{A} \;-\; \iint\limits_{S_2} \mathbf{E}_2 \cdot d^2\mathbf{A} \\ &=\; \frac{1}{\epsilon_0}\, Q_1(\text{conductor}\#i) \;-\; \frac{1}{\epsilon_0}\, Q_2(\text{conductor}\#i) \\ &=\; 0 \end{aligned} \tag{58}$$

since we specify the same net conductor charge $Q(\text{conductor}\#i)$ for both solutions. Consequently, $I_i = 0$.

The bottom line is that for all the conducting surfaces as well as the outer surface the integrals (56) vanish. Together, the conductor surfaces S_i and the outer surface $\mathcal{S}$ act as the net surface of the volume which is either empty or occupied by the dielectrics but not the conductors,

$$\widetilde{\mathcal{V}} \;=\; \mathcal{V} \;-\; \sum_i \text{Vol}(\text{conductor}\#i). \tag{59}$$

Consequently, by the Gauss theorem, for any vector field $\mathbf{C}(x, y, z)$ in this volume,

$$\iiint\limits_{\widetilde{\mathcal{V}}} (\nabla \cdot \mathbf{C}) d^3\text{Vol} \;=\; \iint\limits_{\mathcal{S}} \mathbf{C} \cdot d^2\mathbf{A} \;-\; \sum_i \iint\limits_{S_i} \mathbf{C} \cdot d^2\mathbf{A}\,. \tag{60}$$

Note the negative sign for the integrals over the conductor surfaces S_i since they act as *inner* surfaces of cavities in $\widetilde{\mathcal{V}}$. In particular, for $\mathbf{C} = V_3\mathbf{E}_3$

$$\iiint\limits_{\widetilde{\mathcal{V}}} \big(\nabla \cdot (V_3\mathbf{E}_3)\big) d^3\text{Vol} \;=\; \iint\limits_{\mathcal{S}} V_3\mathbf{E}_3 \cdot d^2\mathbf{A} \;-\; \sum_i \iint\limits_{S_i} V_3\mathbf{E}_3 \cdot d^2\mathbf{A} \;=\; I_0 \;-\; \sum_i I_i \;=\; 0, \tag{61}$$

where the last equality follows from the $I_0 = 0$ and all the $I_i = 0$.

Now, on the LHS of (60), the divergence under the integral evaluates to

$$\nabla \cdot (V_3 \mathbf{E}_3) \;=\; (\nabla V_3) \cdot \mathbf{E}_3 \;+\; V_3 (\nabla \cdot \mathbf{E}_3) \;=\; -\mathbf{E}_3^2 \;+\; 0 \tag{62}$$

thanks to the Laplace equation $\nabla \cdot \mathbf{E}_3 = -\triangle V_3 = 0$. Consequently, eq. (61) becomes

$$0 \;=\; \iiint\limits_{\widetilde{\mathcal{V}}} \big(\nabla \cdot (V_3 \mathbf{E}_3)\big) d^3\text{Vol} \;=\; -\iiint\limits_{\widetilde{\mathcal{V}}} \mathbf{E}_3^2 \, d^3\text{Vol}. \tag{63}$$

At this point we do not know much about the $\mathbf{E}_3^2(x,y,z)$, but as a square of a real vector it has to be non-negative. Consequently, the only way the integral of such non-negative quantity may vanish as in eq. (63) is for the integrand to vanish everywhere. Thus

$$\mathbf{E}_3^2(x,y,z) \;=\; 0 \quad \text{everywhere in } \widetilde{\mathcal{V}} \quad\Longrightarrow\quad \mathbf{E}_3 \;=\; -\nabla V_3(x,y,z) \;=\; 0 \quad \text{everywhere in } \widetilde{\mathcal{V}} \tag{64}$$

and therefore $V_3(x,y,z) = \text{const}$. Finally, at the outer boundary $V_3 = 0$, so if it's a constant then it must vanish everywhere, $V_3(x,y,z) \equiv 0$.

To be precise, we just proved that $V_3 \equiv 0$ everywhere between the conductors and the outer surface $\mathcal{S}$. But over each conductor $V_3 = \text{const}$, so $V_3 = 0$ at the conductor's surface S_i means that $V_3 = 0$ over the whole conductor. Thus, $V_3(x,y,z) \equiv 0$ over the entire volume $\mathcal{V}$ in question — in the empty space, inside the dielectrics, and even inside the conductors, — which means that the two solutions $V_1(x,y,z)$ and $V_2(x,y,z)$ are identically equal to each other. In other words, *the electrostatic problem in question has a unique solution, quod erat demonstrandum.*