MANIFESTLY RELATIVISTIC ELECTRODYNAMICS

In these notes I shall write down Maxwell equations and other important electromagnetic formulae in a manifestly relativistic form which is covariant WRT Lorentz symmetries. I shall focus on the microscopic fields and equations — or equivalently, on EM fields in the vacuum where all the charges and the currents are explicit rather than hiding inside a polarized dielectric or a magnetized material. This way, I would not have to deal with a preferred frame where some dielectric and/or magnetic medium is at rest — in the vacuum, all inertial frames are on equal footing.

As a first step in that direction, let's find out how various EM quantities — the fields $\mathbf{E}$ and $\mathbf{B}$, the potentials Φ and $\mathbf{A}$, the charge and current densities ρ and $\mathbf{J}$, *etc.*, *etc.* — transform under Lorentz symmetries. For simplicity, I shall use the Gauss units rather than the more conventional MKSA units: this way, I would not have to deal with the pesky factors of ϵ_0 and μ_0 all over the place, but just the powers of the speed of light c.

THE 4–CURRENT J^μ.

Let me start by showing that the electric charge density ρ and the electric current density $\mathbf{J}$ form a genuine Lorentz vector

$$J^\mu \;=\; (c\rho, \mathbf{J}). \tag{1}$$

Let me start with a simple model of a charged fluid made of particles having the same charge q. In the rest frame of the fluid — where the particles are moving randomly in all directions, perhaps at high speeds, but the fluid as a whole does not flow — we have

$$\rho_r \;=\; qn_r, \quad \mathbf{J}_r \;=\; 0 \tag{2}$$

where n_r is the particle density in the rest frame. In a different reference frame where the fluid as a whole flows with velocity $\mathbf{v}$, the particle density n would be larger by due to Lorentz contraction of the fluid in the forward direction,

$$n \;=\; \gamma \times n_r, \tag{3}$$

hence the electric charge and current densities

$$\rho\ =\ qn\ =\ qn_r\gamma,\quad \mathbf{J}\ =\ qn\mathbf{v}\ =\ qn_r\gamma\mathbf{v}. \tag{4}$$

Or, in terms of the 4–velocity $u^\mu = (\gamma c, \gamma\mathbf{v})$,

$$c\rho\ =\ qn_r\times u^0,\quad J^i\ =\ qn_r\times U^i\quad\Longrightarrow\quad J^\mu\ \stackrel{\rm def}{=}\ (c\rho,\mathbf{J})\ =\ qn_r\times u^\mu. \tag{5}$$

Under Lorentz transforms q — the charge of a single particle — is invariant, and n_r is also invariant since it's defined relative to the rest frame of the fluid. On the other hand, the 4–velocity u^μ is a Lorentz vector which transforms to $u'^\mu = L^\mu_{\ \nu}u^\nu$, so the 4–current in eq. (5) also transforms as a Lorentz vector,

$$\mathbf{J}'^\mu\ =\ q'n'_r\times u'^\mu\ =\ qn_r\times L^\mu_{\ \nu}u^\nu\ =\ L^\mu_{\ \nu}J^\nu. \tag{6}$$

Now take a more realistic picture of several charged fluids comprised of different particle types, with each fluid moving at its own collective velocity $\mathbf{v}_i$. For example, ion cores and free electrons in a moving metal, or positive ions, negative ions, and free electrons in a plasma. For such a multi-fluid model,

$$\rho\ =\ \sum_i^{\rm fluids} q_i n_i^{\rm rest}\gamma(v_i)\,,\quad \mathbf{J}\ =\ \sum_i^{\rm fluids} q_i n_i^{\rm rest}\gamma(v_i)\mathbf{v}_i \tag{7}$$

where $n_i^{\rm rest}$ is the particle density of the fluid $\#i$ in its own rest frame, hence factors of $\gamma(v_i)$. In 4–vector form eq. (7) becomes

$$J^\mu\ =\ \sum_i^{\rm fluids} q_i n_i^{\rm rest}\times u_i^\mu\,, \tag{8}$$

and since q_i and $n_i^{\rm rest}$ are all Lorentz invariant while the u_i^μ are Lorentz vectors, it follows that the 4–vector current J^μ also transforms like a Lorentz vector.

Finally, consider a different example of a single charged particle moving along a worldline $\mathbf{x}(t)$. In 3D terms, the charge and current densities due to this particle are singular

$$\rho(\mathbf{y},t) \;=\; q\delta^{(3)}(\mathbf{y}-\mathbf{x}(t)), \quad \mathbf{J}(\mathbf{y},t) \;=\; q\mathbf{v}(t)\delta^{(3)}(\mathbf{y}-\mathbf{x}(t)). \tag{9}$$

In 4D terms, these densities can be summarized as

$$J^\mu(\mathbf{y},t) \;=\; qu^\mu(t)\times\frac{\delta^{(3)}(\mathbf{y}-\mathbf{x}(t))}{\gamma(\mathbf{v}(t))}\,. \tag{10}$$

At first blush, the second factor here does not look Lorentz invariant, but we may rewrite it in a manifestly invariant form as

$$\int d\tau\,\delta^{(4)}(y^\mu - x^\mu(\tau)) \;=\; \frac{\delta^{(3)}(\mathbf{y}-\mathbf{x}(t))}{c\gamma(\mathbf{v}(t))}\,, \tag{11}$$

where τ is the proper time along the particle's worldline $x^\mu(\tau)$ and the $1/c\gamma$ factor on the RHS comes from $dx^0/d\tau = c\gamma$. In the context of eq. (10), we put the time-dependent u^μ factor inside the integral, thus

$$J^\mu(y) \;=\; qc\int d\tau\,u^\mu(\tau)\times\delta^{(4)}(y-x(\tau)). \tag{12}$$

Again, the RHS here obviously transforms like a Lorentz vectors, hence so does the 4–current J^μ.

The final argument in favor of $J^\mu = (c\rho,\mathbf{J})$ being a genuine Lorentz vector regardless of particular sources of the charges and the currents is the continuity equation

$$\frac{\partial\rho(\mathbf{x},t)}{\partial t} \;+\; \nabla\cdot\mathbf{J}(\mathbf{x},t) \;=\; 0. \tag{13}$$

This equation is a local form of the electric charge conservation, and it must always hold true, come hell or high water, or else the whole body of Electrodynamics would come crashing down. In 4D terms, the continuity equation becomes

$$\partial_\mu J^\mu(x) \;=\; 0. \tag{14}$$

Indeed, since the derivative 4–vector is defined with the sign convention $\partial_\mu = (\frac{1}{c}\frac{\partial}{\partial t}, +\nabla)$, we have

$$\partial_\mu J^\mu(x) \;=\; \partial_0 J^0 \;+\; \nabla\cdot\mathbf{J} \;=\; \frac{1}{c}\frac{\partial}{\partial t}(c\rho) \;+\; \nabla\cdot\mathbf{J} \;=\; 0. \tag{15}$$

The current density and the charge density in eqs. (13) and (14) are the net electric current and charge densities stemming from all possible sources. Therefore, to make sure eqs. (13) and (14) hold true in all inertial frames, the current and charge densities stemming from all kinds of sources must transform in a similar manner under the Lorentz symmetries, so in light of the above examples, $J^\mu(x)$ stemming from whatever source must transform as a Lorentz vector field:

$$\text{for } x'^\mu \;=\; L^\mu_{\ \nu} x^\nu, \quad J'^\mu(x') \;=\; L^\mu_{\ \nu} J^\nu(x).$$

This makes the 4–divergence $\partial_\mu J^\mu(x)$ a Lorentz scalar, so the relativistic continuity equation (14) is manifestly Lorentz invariant.

Let me finish this section with an explicit formula for the charge and current densities in two inertial frames moving at velocity $\mathbf{v}$ relative to each other:

$$\begin{aligned} \rho'(\mathbf{x}',t') \;&=\; \gamma\rho(\mathbf{x},t) \;-\; \frac{\gamma\mathbf{v}}{c^2}\cdot\mathbf{J}(\mathbf{x},t), \\ \mathbf{J}'(\mathbf{x}',t') \;&=\; \mathbf{J}(\mathbf{x},t) \;+\; \frac{\gamma-1}{\mathbf{v}^2}(\mathbf{v}\cdot\mathbf{J}(\mathbf{x},t))\mathbf{v} \;-\; \gamma\rho(\mathbf{x},t)\mathbf{v} \end{aligned} \tag{16}$$

Electromagnetic potentials.

Next, consider the electromagnetic potentials $\Phi(\mathbf{x},t)$ and $\mathbf{A}(\mathbf{x},t)$. In the Landau gauge, these potentials obey the wave equations

$$\Box\,\Phi(\mathbf{x},t) \;=\; 4\pi\rho(\mathbf{x},t), \quad \Box\,\mathbf{A}(\mathbf{x},t) \;=\; \frac{4\pi}{c}\mathbf{J}(\mathbf{x},t). \tag{17}$$

(Note Gauss units, hence factors of 4π and $1/c$.) Under Lorentz transforms, the right hand sides of these equations mix with each other according to eqs. (16), so the left hand sides

— and hence the potentials Φ and $\mathbf{A}$ should also mix with each other. Moreover, the D'Alembert operator $\Box = \partial_\mu \partial^\mu$ is Lorentz invariant, so the potentials Φ and $\mathbf{A}$ should mix with each other *exactly* as $c\rho$ and $\mathbf{J}$, and since the latter are components of a Lorentz vector field $J^\mu(x)$, the potentials should also form a Lorentz vector field

$$A^\mu(x) \;=\; \bigl(\Phi(x), \mathbf{A}(x)\bigr). \tag{18}$$

Consequently, the wave equations (17) can be written in a manifestly covariant form as

$$\Box\, A^\mu(x) \;=\; \frac{4\pi}{c}\, J^\mu(x), \tag{19}$$

or in terms of explicit 4–derivative operators

$$\partial_\nu \partial^\nu A^\mu(x) \;=\; \frac{4\pi}{c}\, J^\mu(x). \tag{20}$$

Note: the wave equations (17) or (19) apply as written only in the Landau gauge,

$$\nabla \cdot \mathbf{A} \;+\; \frac{1}{c}\frac{\partial \Phi}{\partial t} \;=\; 0 \tag{21}$$

(in Gauss units). This gauge condition happens to be Lorentz-invariant and can be written in covariant form as

$$\partial_\mu A^\mu(\mathbf{x}) \;=\; 0. \tag{22}$$

Other gauge conditions — like the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$ — are not Lorentz invariant, which makes for more complicated Lorentz transforms of the vector and scalar potentials. In general, to establish such a rule, we follow a 3-step procedure:

1. First, we gauge transform

$$\mathbf{A}(x) \;\to\; \tilde{\mathbf{A}}(x) \;=\; \mathbf{A}(x) \;+\; \nabla\Lambda(x), \quad \Phi(x) \;\to\; \tilde{\Phi}(x) \;=\; \Phi(x) \;-\; \frac{1}{c}\frac{\partial \Lambda(x)}{\partial t} \tag{23}$$

— or in 4–vector form

$$\tilde{A}^\mu(x) \;=\; A^\mu(x) \;-\; \partial^\mu \Lambda(x) \tag{24}$$

— for some $\Lambda(x)$ which would change the gauge condition of $A^\mu(x)$ to the Landau gauge for the $\tilde{A}^\mu(x)$, namely$\partial_\mu \tilde{A}^\mu = 0$.

2. Second, we Lorentz transform the $\tilde{A}^\mu(x)$ fields as Lorentz vectors,

$$\tilde{A}'^\mu(x') \;=\; L^\mu{}_\nu \tilde{A}^\nu. \tag{25}$$

3. Third, we gauge transform again

$$A'^\nu(x') \;=\; \tilde{A}'^\mu(x') \;+\; \partial'^\mu \tilde{\Lambda}(x') \tag{26}$$

for some other $\Lambda'(x)$ chosen so as to reimpose the original gauge condition in the new reference frame.

⋆ However, we may summarize all 3 steps as

$$A'^\nu(x') \;=\; L^\mu{}_\nu A^\nu(x) \;+\; \partial'^\mu\big(\tilde{\Lambda}(x') - \Lambda(x')\big), \tag{27}$$

so all we need is to find the gauge transform parameter $(\tilde{\Lambda}-\Lambda)(x)$ which would impose the desired gauge condition in the new reference frame.

Bottom line: In the Landau gauge — or if no gauge condition is imposed — the 4–potential $A^\mu(x)$ transforms like a Lorentz vector field. But when subject to other gauge conditions, a Lorentz symmetry of the potentials should be accompanied by a gauge transform according to eq. (27) in order to reimpose the gauge condition.

Electric and magnetic fields.

The electric field $\mathbf{E}(\mathbf{x},t)$ and the magnetic field $\mathbf{B}(\mathbf{x},t)$ are related to the scalar and vector potentials $\Phi(\mathbf{x},t)$ and $\mathbf{A}(\mathbf{x},t)$ according to

$$\mathbf{E}(\mathbf{x},t) \;=\; -\nabla\Phi(\mathbf{x},t) \;-\; \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}(\mathbf{x},t), \quad \mathbf{B}(\mathbf{x},t) \;=\; \nabla\times\mathbf{A}(\mathbf{x},t) \tag{28}$$

(Gauss units). In spacetime index notations, the electric field equation becomes

$$\mathbf{E}^i(x) \;=\; -\frac{\partial}{\partial x^i}\,\Phi(x) \;-\; \frac{\partial}{\partial x^0}\,A^i(x) \;=\; -\partial_i A^0(x) \;-\; \partial_0 A^i(x) \;=\; +\partial^i A^0(x) \;-\; \partial^0 A^i(x), \tag{29}$$

while the magnetic field equation becomes

$$B^i(x) \;=\; \epsilon^{ijk}\,\frac{\partial}{\partial x^j}\,A^k(x) \;=\; \epsilon^{ijk}\,\partial_j A^k(x) \;=\; -\epsilon^{ijk}\partial^j A^k(x), \tag{30}$$

or equivalently

$$B^k\epsilon^{k\ell m} \;=\; -\partial^\ell A^m(x) \;+\; \partial^m A^\ell(x). \tag{31}$$

In spacetime terms, the right hand sides of eqs. (29) and (31) are different components of an antisymmetric Lorentz tensor field

$$F^{\mu\nu}(x) \;\stackrel{\text{def}}{=}\; \partial^\mu A^\nu(x) \;-\; \partial^\nu A^\mu(x) \;=\; -F^{\nu\mu}(x). \tag{32}$$

In a 3D space, such an antisymmetric tensor would be equivalent to an axial vector, but in 4D the antisymmetric tensors are not vectors and have different numbers of independent components, namely

$$\frac{4\times 3}{2} \;=\; 6.$$

In particular, the independent components of the tensor (32) are the 3 components of the electric field **E** plus 3 components of the magnetic field **B**. Specifically, in light of eqs. (29) and (31),

$$F^{00} \;=\; 0, \quad F^{i0} \;=\; +E^i, \quad F^{0j} \;=\; -E^j, \quad F^{ij} \;=\; -\epsilon^{ijk}B^k. \tag{33}$$

Note: the signs here are for the $F^{\mu\nu}$ components with upper indices. When we lower both indices, we get

$$F_{00} \;=\; 0, \quad F_{i0} \;=\; -F^{i0} \;=\; -E^i, \quad F_{0i} \;=\; -F^{0i} \;=\; +E^i, \quad F_{ij} \;=\; +F^{ij} \;=\; -\epsilon^{ijk}B^k. \tag{34}$$

The electric and magnetic fields are gauge-invariant, *i.e.* invariant under the gauge transforms of the potentials. In 4D terms, we can see the gauge invariance of the $F^{\mu\nu}$ tensor as

$$A'^\mu(x) \;=\; A^\mu(x) \;-\; \partial^\mu\Lambda(x), \tag{35}$$

$$\begin{aligned} F'^{\mu\nu}(x) \;&=\; \partial^\mu\big(A'^\nu \;=\; A^\nu - \partial^\nu\Lambda\big) \;-\; \partial^\nu\big(A'^\mu \;=\; A^\mu - \partial^\mu\Lambda\big) \\ &=\; \big(\partial^\mu A^\nu \;-\; \partial^\nu A^\mu\big) \;+\; \big(-\partial^\mu\partial^\nu\Lambda \;+\; \partial^\nu\partial^\mu\Lambda\big) \\ &=\; F^{\mu\nu}(x) \;+\; 0. \end{aligned} \tag{36}$$

Now consider the Lorentz transformation properties of the $F^{\mu\nu}(x)$ fields. Without a gauge constraint (or in the Landau gauge), the potential 4-vector $A^\mu(x)$ transforms like a Lorentz vector field, and the derivative 4–vector ∂^μ also transforms like a Lorentz vector,

$$x'^\mu \;=\; L^\mu_{\ \nu} x^\nu, \quad A'^\mu(x') \;=\; L^\mu_{\ \nu} A^\nu(x), \partial'^\mu \;=\; L^\mu_{\ \nu}\partial^\nu, \tag{37}$$

so the antisymmetric tensor field

$$F^{\mu\nu}(x) \;=\; \partial^\mu A^\nu(x) \;-\; \partial^\nu A^\mu(x) \tag{38}$$

transforms like a genuine Lorentz tensor field

$$F'^{\mu\nu}(x') \;=\; L^\mu_{\ \kappa} L^\nu_{\ \lambda} F^{\kappa\lambda}(x). \tag{39}$$

Let's spell out this transformation rule in components for a Lorentz boost of velocity v in x direction:

$$L^0_{\ 0} \;=\; L^x_{\ x} \;=\; \gamma, \quad L^0_{\ x} \;=\; L^x_{\ 0} \;=\; -\beta\gamma, \quad L^y_{\ y} \;=\; L^z_{\ z} \;=\; 1, \quad \text{other } L^\mu_{\ \nu} \;=\; 0, \tag{40}$$

hence

$$\begin{aligned} E'^x \;=\; F'^{x0} \;&=\; \gamma\times\gamma\times F^{x0} \;-\; \beta\gamma\times\gamma\times F^{00} \;-\; \beta\gamma\times\gamma\times F^{xx} \;+\; \beta\gamma\times\beta\gamma\times F^{0x} \\ &=\; \gamma^2\times E^x \;-\; \beta\gamma^2\times 0 \;-\; \beta\gamma^2\times 0 \;+\; \beta^2\gamma^2\times(-E^x) \\ &=\; \gamma^2(1-\beta^2)\times E^x \;=\; E^x, \end{aligned} \tag{41}$$

$$\begin{aligned} E'^y \;=\; F'^{y0} \;&=\; 1\times\gamma\times F^{y0} \;+\; 1\times(-\beta\gamma)\times F^{yx} \\ &=\; \gamma\times E^y \;-\; \beta\gamma\times B^z \;=\; \gamma(E^y - \beta B^z), \end{aligned} \tag{42}$$

$$\begin{aligned} E'^z \;=\; F'^{z0} \;&=\; 1\times\gamma\times F^{z0} \;+\; 1\times(-\beta\gamma)\times F^{zx} \\ &=\; \gamma\times E^z \;-\; \beta\gamma\times(-B^y) \;=\; \gamma(E^z+\beta B^y), \end{aligned} \tag{43}$$

$$B'^x \;=\; F'^{zy} \;=\; 1\times 1\times F^{zy} \;=\; B^x, \tag{44}$$

$$\begin{aligned} B'^y \;=\; F'^{xz} \;&=\; \gamma\times 1\times F^{xz} \;-\; \beta\gamma\times 1\times F^{0z} \\ &=\; \gamma\times B^y \;-\; \beta\gamma\times(-E^z) \;=\; \gamma(B^y+\beta E^z), \end{aligned} \tag{45}$$

$$\begin{aligned} B'^z \;=\; F'^{yx} \;&=\; 1\times\gamma\times F^{yx} \;+\; 1\times(-\beta\gamma)\times F^{y0} \\ &=\; \gamma\times B^z \;-\; \beta\gamma\times(+E^y) \;=\; \gamma(B^z-\beta E^y). \end{aligned} \tag{46}$$

Or in 3-vector notations

$$\begin{aligned} \mathbf{E}'_{\|} \;=\; \mathbf{E}_{\|}\,, \quad \mathbf{E}'_{\perp} \;&=\; \gamma\big(\mathbf{E}_{\perp} + \vec{\beta}\times\mathbf{B}_{\perp}\big), \\ \mathbf{B}'_{\|} \;=\; \mathbf{B}_{\|}\,, \quad \mathbf{B}'_{\perp} \;&=\; \gamma\big(\mathbf{B}_{\perp} - \vec{\beta}\times\mathbf{E}_{\perp}\big). \end{aligned} \tag{47}$$

For an example, consider the EM fields of a point charge moving at constant velocity $\mathbf{v}$. In the rest frame of the charge, the electric field is Coulomb and the magnetic field is absent,

$$\mathbf{E}'(\mathbf{x}') \;=\; \frac{Q\,\mathbf{x}}{R'^3}\,, \quad \mathbf{B}(\mathbf{x}') \;=\; 0. \tag{48}$$

Transforming these fields back to the lab frame, we get

$$\begin{aligned} \mathbf{E}_{\|}(\mathbf{x},t) \;&=\; \mathbf{E}'_{\|}(\mathbf{x}') \;=\; \frac{Q\mathbf{x}'_{\|}}{R'^3}\,, \\ \mathbf{E}_{\perp}(\mathbf{x},t) \;&=\; \gamma\mathbf{E}_{\perp}(\mathbf{x}') \;=\; \frac{\gamma Q\mathbf{x}'_{\perp}}{R'^3}\,, \\ \mathbf{B}_{\|}(\mathbf{x},t) \;&=\; 0, \\ \mathbf{B}_{\perp}(\mathbf{x},t) \;&=\; -\gamma\beta\times\mathbf{E}_{\perp}(\mathbf{x}') \;=\; +\frac{\gamma Q(\mathbf{x}'_{\perp}\times\beta)}{R'^3}\,, \end{aligned} \tag{49}$$

where

$$\mathbf{x}'_{\perp} \;=\; \mathbf{x}_{\perp}\,, \quad x'_{\|} \;=\; \gamma(x_{\|}-vt), \tag{50}$$

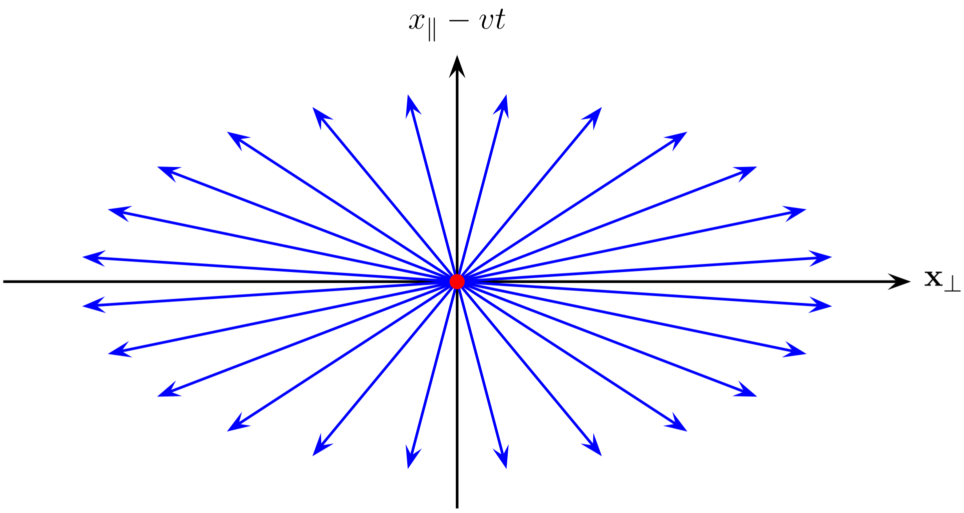
hence

$$R' \;=\; \sqrt{\gamma^2(x_\| - vt)^2 + x_\perp^2}, \tag{51}$$

and therefore

$$\begin{aligned}
E_\|(\mathbf{x},t) \;&=\; \frac{\gamma Q(x_\| - vt)}{\left[\gamma^2(x_\| - vt)^2 + x_\perp^2\right]^{3/2}}\,,\\
\mathbf{E}_\perp(\mathbf{x},t) \;&=\; \frac{\gamma Q\mathbf{x}_\perp}{\left[\gamma^2(x_\| - vt)^2 + x_\perp^2\right]^{3/2}}\,,\\
\mathbf{B}_\|(\mathbf{x},t) \;&=\; 0,\\
\mathbf{B}_\perp(\mathbf{x},t) \;&=\; \mathbf{E}(\mathbf{x},t)\times\vec{\beta} \;=\; \frac{\gamma Q(\mathbf{x}_\perp\times\vec{\beta})}{\left[\gamma^2(x_\| - vt)^2 + x_\perp^2\right]^{3/2}}\,.
\end{aligned} \tag{52}$$

At high charge speeds v, the denominators in these formulae are more sensitive to the longitudinal distance $x_\| - vt$ from the charge than to the transverse distance $\mathbf{x}_\perp$, so the electric and the magnetic fields become squashed in the longitudinal direction. To illustrate this point, here is the picture of the electric field lines for $\gamma = 2$:



In the limit of ultra-relativistic charge velocity — v so close to c that $\gamma \gg 1$, — this picture becomes so squashed that the fields are pretty much limited to the transverse plane containing the charge. Specifically,

$$\text{for } \gamma \to \infty, \quad \frac{\gamma}{\left[\gamma^2(x_\parallel - vt)^2 + x_\perp^2\right]^{3/2}} \longrightarrow \frac{\delta(vt - x_\parallel)}{\mathbf{x}_\perp^2} \approx \frac{\delta(ct - x_\parallel)}{\mathbf{x}_\perp^2}. \tag{53}$$

and therefore

$$\begin{aligned} \mathbf{E}_\parallel &\to 0, \\ \mathbf{E}_\perp &\to \frac{Q\mathbf{x}_\perp}{\mathbf{x}_\perp^2}\,\delta(ct - x_\parallel), \\ \mathbf{B}_\parallel &= 0, \\ \mathbf{B}_\parallel &= \mathbf{E} \times \vec{\beta} \to \mathbf{E} \times \mathbf{n}, \end{aligned} \tag{54}$$

where $\mathbf{n}$ is the direction of the charges motion. These fields look like a δ–pulse of a plane EM wave; indeed, both $\mathbf{E}$ and $\mathbf{B}$ fields are transverse to the wave's direction, and $\mathbf{B} = \mathbf{E} \times \mathbf{n}$.

Lorentz covariant Maxwell equations.

The microscopic Maxwell equation

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \tag{55}$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0, \tag{56}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{57}$$

$$\nabla \times \mathbf{B} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{J}, \tag{58}$$

(Gauss units) can be written in a Lorentz-covariant form. Let's start by writing the induction law (56) in terms of the $F^{\mu\nu}(x)$ tensor fields and it's components:

$$(\nabla \times \mathbf{E})^i = \epsilon^{ijk}\nabla^j E^k = \epsilon_{ijk}(-\partial^j)(+F^{k0}), \tag{59}$$

$$(\nabla \times \mathbf{E})^i \times e^{ijk} \; - \partial^j F^{k0} + \partial^k F^{j0}, \tag{60}$$

$$\frac{1}{c}\frac{\partial B^i}{\partial t} \times \epsilon^{ijk} = \partial_0(-F^{jk}), \tag{61}$$

so induction law (56) becomes

$$-\partial^j F^{k0} + \partial^k F^{j0} - \partial^0 F^{jk} = 0. \tag{62}$$

We can give all terms here by letting $F^{j0} = -F^{0j}$ in the middle term, then after flipping the overall sign we get a more symmetric formula

$$\partial^j F^{k0} \;+\; \partial^k F^{0j} \;+\; \partial^0 F^{jk} \;=\; 0. \tag{63}$$

Let's define a 3-index Lorentz tensor

$$H^{\lambda\mu\nu} \;\stackrel{\rm def}{=}\; \partial^\lambda F^{\mu\nu} \;+\; \partial^\mu F^{\nu\lambda} \;+\; \partial^\nu F^{\lambda\mu}. \tag{64}$$

In terms of this tensor, the Induction Law (56) and hence (63) becomes

$$H^{jk0} \;=\; 0, \tag{65}$$

which suggest that $H^{\lambda\mu\nu}$ should also vanish for the other index combinations. But before we check all those combinations, let's take a look at the index symmetries of the $H^{\lambda\mu\nu}$ tensor regardless of any Maxwell equations. By construction (64), this tensor is symmetric under cyclic permutations of its 3 indices,

$$H^{\lambda\mu\nu} \;=\; H^{\mu\nu\lambda} \;=\; H^{\nu\lambda\mu} \tag{66}$$

and thanks to the antisymmetry of the tension field tensor $F^{\mu\nu} = -F^{\nu\mu}$, H is antisymmetric under pairwise index permutations,

$$\begin{aligned} H^{\lambda\nu\mu} \;&=\; \partial^\lambda F^{\nu\mu} \;+\; \partial^\nu F^{\mu\lambda} \;+\; \partial^\mu F^{\lambda\nu} \\ &=\; -\partial^\lambda F^{\mu\nu} \;-\; \partial^\nu F^{\lambda\mu} \;-\; \partial^\mu F^{\nu\lambda} \\ &=\; -H^{\lambda\mu\nu} \end{aligned} \tag{67}$$

and likewise

$$H^{\nu\mu\lambda} \;=\; H^{\mu\lambda\nu} \;=\; -H^{\lambda\mu\nu}. \tag{68}$$

In other words, the tensor $H^{\lambda\mu\nu}$ is totally antisymmetric in its 3 indices. Consequently, it has only

$$\frac{4\times 3\times 2}{3!} \;=\; 4$$

independent components, so there are only 4 independent equations $H^{\lambda\mu\nu} = 0$. the rest are trivially true and do not constrain the EM fields in any fashion. Specifically, the indices

λ, μ, ν must be all different from each other, and their order does not matter. In other words, the 4 independent equations $H^{\lambda\mu\nu} = 0$ correspond to

$$H^{120} = 0, \quad H^{230} = 0, \quad H^{310} = 0, \quad \text{and} \quad H^{123} = 0. \tag{69}$$

The first 3 of these equations comprise the Induction Law (56) in the form of eq. (65), while the fourth equation is the Magnetic Gauss Law (57). Indeed,

$$\begin{aligned} H^{123} &= \partial^1 F^{23} + \partial^2 F^{31} + \partial^3 F^{12} \\ &= (-\partial_x)(-B^x) + (-\partial_y)(-B^y) + (-\partial_z)(-B_z) \\ &= +\nabla \cdot \mathbf{B}, \end{aligned} \tag{70}$$

thus

$$H^{123} = 0 \quad \Longleftrightarrow \quad \nabla \cdot \mathbf{B} = 0. \tag{71}$$

Bottom Line: the *homogeneous Maxwell equations* (56) (the induction law) and (57) (the magnetic Gauss law) can be written in Lorentz-covariant form as

$$\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0. \tag{72}$$

Now consider the inhomogeneous Maxwell equations (55) (the Gauss law) and (58) (Maxwell–Ampere law). In terms of the $F^{\mu\nu}$ tensor,

$$(\nabla \times \mathbf{B})^i = \epsilon^{ijk} \frac{\partial}{\partial x^j} B^k = \partial_j \big(B^k \epsilon^{ijk} = -F^{ij} = +F^{ji} \big) = +\partial_j F^{ji}, \tag{73}$$

$$-\frac{1}{c} \frac{\partial E^i}{\partial t} = -\partial_0 \big(E^i = F^{i0} = -F^{0i} \big) = +\partial_0 F^{0i}, \tag{74}$$

hence the LHS of the Maxwell–Ampere equation becomes

$$\left(\nabla \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \right)^i = \partial_j F^{ji} + \partial_0 F^{0i} = \partial_\mu F^{\mu i}. \tag{75}$$

Likewise, we may write the LHS of the Gauss law as

$$\nabla \cdot \mathbf{E} = \partial_j E^j = \partial_j F^{j0} = \partial_j F^{j0} + \partial_0 \big(F^{00} = 0 \big) = \partial_\mu F^{\mu 0}. \tag{76}$$

Consequently, the inhomogeneous Maxwell equations become

$$\partial_\mu F^{\mu 0} \;=\; 4\pi\rho \;=\; \frac{4\pi}{c}\,J^0 \quad \text{(Gauss Law)}, \tag{77}$$

$$\partial_\mu F^{\mu i} \;=\; \frac{4\pi}{c}\,J^i \quad \text{(Maxwell–Ampere Law)}, \tag{78}$$

so we may combine them in a Lorentz-covariant form as

$$\partial_\mu F^{\mu\nu}(x) \;=\; \frac{4\pi}{c}\,J^\nu(x). \tag{79}$$

Wave Equations for the EM fields

A while ago, we have learned that in the absence of charges and currents, all components of the electric and magnetic fields obey the wave equation $\Box(\mathbf{E}\,\text{or}\,\mathbf{B})=0$. Let's see how this works in the 4D language.

Taking the ∂_λ derivative of the homogeneous Maxwell equation (72) and contracting the index λ, we get

$$\begin{aligned} 0 \;&=\; \partial_\lambda\Big(\partial^\lambda F^{\mu\nu} \,+\, \partial^\mu F^{\nu\lambda} \,+\, \partial^\nu F^{\lambda\mu}\Big) \\ &=\; \Box F^{\mu\nu} \,+\, \partial^\mu\big(\partial_\lambda F^{\nu\lambda}\big) \,+\, \partial^\nu\big(\partial_\lambda F^{\lambda\mu}\big) \\ &\qquad \langle\!\langle \text{ using the inhomogeneous Maxwell equation (79) for the last 2 terms} \rangle\!\rangle \\ &=\; \Box F^{\mu\nu} \,+\, \partial^\mu\left(-\frac{4\pi}{c}\,J^\nu\right) + \,\partial^\nu\left(+\frac{4\pi}{c}J^\mu\right) \end{aligned} \tag{80}$$

and therefore

$$\Box F^{\mu\nu} \;=\; \frac{4\pi}{c}\Big(\partial^\mu J^\nu \,-\, \partial^\nu J^\mu\Big). \tag{81}$$

In particular, in the absence of any electric charges or currents,

$$\Box F^{\mu\nu}(x) \;=\; 0. \tag{82}$$

Covariant Maxwell equations for the 4–potential A^μ.

Expressing the EM tension field tensor $F^{\mu\nu}(x)$ in terms of the potential 4–vector $A^\mu(x)$ as

$$F^{\mu\nu}(x) \;=\; \partial^\mu A^\nu(x) \;-\; \partial^\nu A^\mu(x) \tag{83}$$

automagically solves the homogeneous Maxwell equation (72). Indeed, thanks to commutativity of the spacetime derivatives $\partial^\mu\partial^\nu = \partial^\nu\partial^\mu$, we have a total cancellation

$$\begin{aligned} \partial^\lambda F^{\mu\nu} \;+\; \partial^\mu F^{\nu\lambda} \;+\; \partial^\nu F^{\lambda\mu} \;=& \\ =\; \partial^\lambda\Big(\partial^\mu A^\nu - \partial^\nu A^\mu\Big) \;+\; \partial^\mu\Big(\partial^\nu A^\lambda - \partial^\lambda A^\nu\Big) \;+\; \partial^\nu\Big(\partial^\lambda A^\mu - \partial^\mu A^\lambda)& \\ =\; \Big(\partial^\lambda\partial^\mu - \partial^\mu\partial^\lambda\Big)A^\nu \;+\; \Big(\partial^\mu\partial^\nu - \partial^\nu\partial^\mu\Big)A^\lambda \;+\; \Big(\partial^\nu\partial^\lambda - \partial^\lambda\partial^\nu\Big)A^\mu& \\ =\; 0 \;+\; 0 \;+\; 0 \;=\; 0.& \end{aligned} \tag{84}$$

Conversely, if the homogeneous Maxwell equation (72) holds throughout the 4D spacetime, then all its solutions have form (83) for some potentials (83). This is a theorem of differential geometry which generalizes $\mathbf{B}(\mathbf{x}) = \nabla\times\mathbf{A}(\mathbf{x})$ for some $\mathbf{A}(\mathbf{x})$ for any divergence-less $\mathbf{B}(\mathbf{x})$.

Now consider the inhomogeneous Maxwell equation (79) in terms of the 4–potential A^μ:

$$\frac{4\pi}{c}\,J^\nu(x) \;=\; \partial_\mu F^{\mu\nu}(x) \;=\; \partial_\mu\Big(\partial^\mu A^\nu(x) \,-\, \partial^\nu A^\mu(x)\Big) \;=\; \Box A^\nu(x) \;-\; \partial^\nu\big(\partial_\mu A^\mu(x)\big). \tag{85}$$

Note: the RHS here is gauge invariant, but the individual terms $\Box A^\nu$ and $\partial^\nu(\partial_\mu A^\mu)$ are not invariant, so we may impose a gauge condition to simplify them. From the Lorentz-covariant point of view, the best gauge condition is the Lorentz-invariant Landau gauge

$$\partial_\mu A^\mu(x) \;=\; 0 \quad (\text{at all } x). \tag{86}$$

In this gauge, the the Maxwell equation (85) becomes simply

$$\Box A^\mu(x) \;=\; \frac{4\pi}{c}\,J^\mu(x). \tag{87}$$

We may formally solve this wave equation using a Green's function. Of particular interest is the retarded Green's function

$$G_R(x-y) \;=\; \frac{1}{2\pi}\,\delta\bigl((x-y)^2\bigr)\times\Theta(x^0-y^0) \tag{88}$$

(where $(x-y)^2=(x-y)_\mu(x-y)^\mu$) which vanishes outside the future light cone, so the potential

$$A^\mu(x) \;=\; \frac{2}{c}\int\!d^4y\,J^\mu(y)\times\delta\bigl((x-y)^2\bigr)\times\Theta(x^0-y^0) \tag{89}$$

generated by some current $J^\mu(y)$ exists only in the absolute future of that current. In particular, for a point charge moving along some worldline $x_c(\tau)$, the 4-current is

$$J^\mu(y) \;=\; qc\int\limits_{\text{worldline}} d\tau\,u^\mu(\tau)\times\delta^{(4)}(y-x_c(\tau)), \tag{90}$$

so the 4-potential generated by this moving charge is

$$A^\mu(x) \;=\; 2q\int\limits_{\text{worldline}} d\tau\,u^\mu(\tau)\times\delta\bigl((x-x_x(\tau)^2\bigr)\times\Theta\bigl(x^0-x_c^0(\tau)\bigr). \tag{91}$$

I shall return to this Liénard–Wiechert potential — and the EM radiation it represents — in the last week of classes. Meanwhile, you can read the Wikipedia article on the subject.

Macroscopic Maxwell Equations

Let me conclude these notes by writing the macroscopic Maxwell equations

$$\begin{aligned}\nabla\cdot\mathbf{D} &= 4\pi\rho,\\ \nabla\times\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} &= 0,\\ \nabla\cdot\mathbf{B} &= 0,\\ \nabla\times\mathbf{H}+\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t} &= \frac{4\pi}{c}\mathbf{J},\end{aligned} \tag{92}$$

(Gauss units) in a Lorentz covariant form, although the relation between the **E**, **B**, **D**, and **H** form would have to depend on the macroscopic medium's velocity. Let me start by introducing two antisymmetric Lorentz tensors, $F^{\mu\nu}(x)$ comprising the macroscopic **E** and **B** fields, and $G^{\mu\nu}$ comprising the **D** and **H** fields. The two tensors are constructed in a similar way:

$$F^{ij} \;=\; -\epsilon^{ijk}B^k, \quad F^{i0} \;=\; -F^{0i} \;=\; E^i, \quad F^{00} \;=\; 0, \tag{93}$$

exactly like in the microscopic theory, and

$$G^{ij} \;=\; -\epsilon^{ijk}H^k, \quad G^{i0} \;=\; -G^{0i} \;=\; D^i, \quad G^{00} \;=\; 0. \tag{94}$$

Then, exactly like in the microscopic theory, the two homogeneous macroscopic Maxwell equations become

$$\partial^\lambda F^{\mu\nu} \;+\; \partial^\mu F^{\nu\lambda} \;+\; \partial^\nu F^{\lambda\mu} \;=\; 0, \tag{95}$$

while the two inhomogeneous macroscopic Maxwell equations become

$$\partial_\mu G^{\mu\nu} \;=\; \frac{4\pi}{c}\, J^\nu. \tag{96}$$

The real issue here is writing a Lorentz-covariant relation between the $F^{\mu\nu}$ and $G^{\mu\nu}$ tensor fields and the 4–velocity u^μ of the macroscopic medium. For simplicity, let's assume that *in the rest frame* the medium is linear and isotropic, and there is no cross-coupling between the electric and the magnetic fields. Instead, *in the rest frame*

$$\mathbf{D} \;=\; \epsilon\mathbf{E} \quad \text{and} \quad \mathbf{H} \;=\; \frac{1}{\mu}\mathbf{B} \tag{97}$$

for some dielectric constant ϵ and magnetic permeability μ (note Gauss units). In terms of tensors $F^{\alpha\beta}$ and $G^{\alpha\beta}$, eqs. (97) become

$$G^{ij} \;=\; \frac{1}{\mu}\, F^{ij} \quad \text{but} \quad G^{i0} \;=\; \epsilon F^{i0} \quad \text{and} \quad G^{0i} \;=\; \epsilon F^{0i}. \tag{98}$$

We see a difference between space and time indices here, and to make it look covariant, let's use the medium's 4–velocity which in the rest frame is simply $u^\mu = (c, 0, 0, 0)$. Consequently, eqs. (96) become

$$G^{\alpha\beta} \;=\; \frac{1}{\mu}\,F^{\alpha\beta} \;+\; \frac{\epsilon\mu-1}{\mu c^2}\Big(u^\alpha u_\gamma F^{\gamma\beta} \;-\; u^\beta u_\gamma F^{\gamma\alpha}\Big). \tag{99}$$

Indeed, in the rest frame

$$u_\gamma F^{\gamma\beta} \;=\; cF^{0\beta} \;=\; \begin{cases} -cE^j & \text{for } \beta=j=1,2,3,\\ 0 & \text{for } \beta=0, \end{cases} \tag{100}$$

hence

$$u^\alpha u_\gamma F^{\gamma\beta} \;=\; \begin{cases} -c^2E^j & \text{for } \alpha=0 \text{ and } \beta=j=1,2,3,\\ 0 & \text{for all other index combinations,} \end{cases} \tag{101}$$

and therefore $G^{\alpha\beta}$ in eq. (99) becomes

$$\begin{aligned}
G^{ij} \;&=\; \frac{F^{ij}}{\mu} \;+\; \frac{\epsilon\mu-1}{\mu c^2}\times(0-0) \;=\; \frac{F^{ij}}{\mu}\\
G^{0i} \;&=\; \frac{F^{0i}=-E^i}{\mu} \;+\; \frac{\epsilon\mu-1}{\mu c^2}\times(-c^2E^i-0)\\
&=\; \left(\frac{1}{\mu}+\frac{\epsilon\mu-1}{\mu}=\epsilon\right)\times(-E^i) \;=\; \epsilon\times F^{0i},\\
G^{i0} \;&=\; \frac{F^{i0}=+E^i}{\mu} \;+\; \frac{\epsilon\mu-1}{\mu c^2}\times(0+c^2E^i)\\
&=\; \left(\frac{1}{\mu}+\frac{\epsilon\mu-1}{\mu}=\epsilon\right)\times(+E^i) \;=\; \epsilon\times F^{i0},\\
G^{00} \;&=\; \frac{F^{00}=0}{\mu} \;+\; \frac{\epsilon\mu-1}{\mu c^2}\times(0-0) \;=\; 0,
\end{aligned} \tag{102}$$

in perfect agreement with eqs. (98) for the rest frame.

Since eq. (99) is Lorentz covariant, once we have established it in the rest frame of the macroscopic medium, it should be valid in any other frame of reference, as long as we set u^μ to the medium's 4–velocity. In 3D terms, eq. (99) in a moving medium becomes

$$\begin{aligned}
\mathbf{D} \;&=\; \epsilon\mathbf{E} \;+\; \left(\epsilon-\frac{1}{\mu}\right)\gamma^2\Big(\vec\beta\times\mathbf{B} \;-\; \vec\beta(\vec\beta\cdot\mathbf{E})\Big),\\
\mathbf{H} \;&=\; \frac{\mathbf{B}}{\mu} \;+\; \left(\epsilon-\frac{1}{\mu}\right)\gamma^2\Big(\vec\beta\times\mathbf{E} \;+\; \beta\times(\beta\times\mathbf{B})\Big).
\end{aligned} \tag{103}$$