RADIATION BY AN ACCELERATING CHANGE

The electric and the magnetic fields produced by a charged particle moving at constant velocity $\mathbf{v}$ are simply the Lorentz transforms of the Coulomb electric field in the particle's rest frame, but there is no EM radiation. By contrast, an accelerating particle produces EM radiation, in which the electric and the magnetic fields decrease with distance as $1/r$ rather than $1/r^2$, and those fields take away some of the particle's energy. In these notes, we shall see how it works.

Liénard–Wiechert potentials.

Consider a charged particle moving along some curving worldline $\mathbf{x}_c(t)$. At any moment of time t, the charge density and the current density due to this particle are

$$\rho(\mathbf{x},t) \;=\; q\delta^{(3)}(\mathbf{y}-\mathbf{x}_c(t)), \quad \mathbf{J}(\mathbf{y},t) \;=\; q\mathbf{v}(t)\,\delta^{(3)}(\mathbf{y}-\mathbf{x}_c(t)), \tag{1}$$

or in Lorentz-covariant form

$$J^\mu(y) \;=\; q \int\limits_{\text{worldline}} d\tau\, u^\mu(\tau)\,\delta^{(4)}(y-x_c(\tau)). \tag{2}$$

In the Landau gauge, the 4–potential $A^\mu(x)$ due to this current is given by the retarded Green's function

$$\begin{aligned} A^\mu(x) \;&=\; \frac{2}{c}\int d^4y\, J^\mu(y)\times\delta((x-y)^2)\Theta(x^0-y^0) \\ &=\; \frac{2q}{c}\int\limits_{\text{worldline}} d\tau\, u^\mu(\tau)\times\delta((x-x_c(\tau))^2)\Theta(x^0-x_c^0(\tau)). \end{aligned} \tag{3}$$

For any x^ν, the δ–function and the Θ–function select the point $x_c^\nu(\tau)$ where the particle's worldline intersects the past light cone with vertex at the x^ν. For any particle moving slower than light, such intersection is always unique. In 3D terms, it's described by the retarded time condition for the t_c

$$x^0 \;-\; ct_c \;=\; |\mathbf{x}-\mathbf{x}_c(t_c)|, \tag{4}$$

and in the following I shall use the label 'ret' to indicate quantities that should be evaluated at this retarded time rather than at the observer time x^0/c. I shall also use R and $\mathbf{n}$ for the length and the direction of the 3-vector $\mathbf{x}-\mathbf{x}_c(t_c)$, and in 4D notations I'll promote $\mathbf{n}$ to the light-like unit vector $n^\nu = (1,\mathbf{n})$, thus

$$x^\mu \;-\; x_c^\mu(\tau_{\rm ret}) \;=\; Rn^\mu. \tag{5}$$

To evaluate the integral (3) over the δ–function, we need the derivative of the δ–function's argument WRT the integration variable τ,

$$\frac{d}{d\tau}\,(x-x_c(\tau))^2 \;=\; 2(x-x_c)_\nu\left(\frac{d(x-x_c(\tau))^\nu}{d\tau} \;=\; -u^\nu\right) \;=\; -2u_\nu(x-x_c)^\nu \;=\; -2u_\nu n^\nu R, \tag{6}$$

hence

$$A^\mu(x) \;=\; \left[\frac{qu^\mu}{R(u_\nu n^\nu)}\right]_{\rm ret}. \tag{7}$$

In components

$$u_\nu n^\nu \;=\; \gamma(c\;-\;\mathbf{v}\cdot\mathbf{n}) \;=\; \gamma c(1\;-\;\boldsymbol{\beta}\cdot\mathbf{n}), \tag{8}$$

hence

$$\Phi(x) \;=\; \left[\frac{q}{R(1-\boldsymbol{\beta}\cdot\mathbf{n})}\right]_{\rm ret} \quad\text{and}\quad \mathbf{A}(x) \;=\; \left[\frac{q\boldsymbol{\beta}}{R(1-\boldsymbol{\beta}\cdot\mathbf{n})}\right]_{\rm ret}. \tag{9}$$

The potentials (9) are called the Liénard–Wiechert potentials after Alfred–Marie Liénard and Emil Wiechert who derived them in 1898–1900.

The EM tension fields $F^{\mu\nu}$

Given the Liénard–Wiechert potentials (7), the EM tension fields $F^{\mu\nu}(x)$ follow as derivatives

$$F^{\mu\nu}(x) \;=\; \partial^\mu A^\nu(x) \;-\; \partial^\nu A^\mu(x). \tag{10}$$

But when taking these derivatives, one should remember $Rn^\nu = x^\nu - x_c^\nu(\tau)$ depends on the observer position x^ν both explicitly and via τ being constrained by the retarded time condition, and that the u^μ also depends on τ and hence on the x^ν, thus

$$F^{\mu\nu}(x) \;=\; \left[\partial^\mu\left(\frac{qu^\nu}{u\cdot(x-x_c)}\right)\right]_{@\text{fixed}\,\tau} \;+\; \frac{\partial\tau}{\partial x_\mu}\times\frac{\partial}{\partial\tau}\left(\frac{qu^\nu}{u\cdot(x-x_c)}\right) \;-\; (\mu\leftrightarrow\nu). \tag{11}$$

In the explicit derivative term, the x_μ dependence comes only via the $u\cdot(x-x_c)$ factor in the denominator, hence

$$\left[\partial^\mu\left(\frac{qu^\nu}{u\cdot(x-x_c)}\right)\right]_{@\text{fixed}\,\tau} \;=\; -\frac{qu^\nu}{[u\cdot(x-x_c)]^2}\times\bigl(\partial^\mu(u\cdot(x-x_c)) \;=\; u^\mu\bigr) \;=\; -\frac{qu^\mu u^\nu}{[u\cdot(x-x_c)]^2}\,. \tag{12}$$

Thus expression is manifestly symmetric in the μ and ν indices, so it cancels out from the $F^{\mu\nu}$ after the anti-symmetrization. Thus, we are left with

$$F^{\mu\nu}(x) \;=\; \frac{\partial\tau}{\partial x_\mu}\times\frac{\partial}{\partial\tau}\left(\frac{qu^\nu}{u\cdot(x-x_c)}\right) \;-\; (\mu\leftrightarrow\nu). \tag{13}$$

The x derivative of τ in this formula stems from maintaining the constraint $(x-x_c(\tau))^2\equiv 0$ for all observer points x, hence

$$0 \;=\; d\bigl[(x-x_c(\tau))^2\bigr] \;=\; 2(x-x_c)_\mu\times(dx^\mu-u^\mu d\tau) \;=\; 2(x-x_c)_\mu\times dx^\mu - 2(x-x_c)u^\mu\times d\tau \tag{14}$$

and therefore

$$\frac{\partial\tau}{\partial x_\mu} \;=\; \frac{(x-x_c)^\mu}{(x-x_c)\cdot u}\,. \tag{15}$$

Plugging this formula into eq. (13), we arrive at

$$F^{\mu\nu}(x) \;=\; \frac{(x-x_c)^\mu}{(x-x_c)\cdot u}\times\frac{\partial}{\partial\tau}\left(\frac{qu^\nu}{u\cdot(x-x_c)}\right) \;-\; (\mu\leftrightarrow\nu). \tag{16}$$

Next, let's define the 4–acceleration vector

$$w^\mu \;\stackrel{\text{def}}{=}\; \frac{du^\mu}{d\tau}\,, \tag{17}$$

so in an external EM field $F^{\mu\nu}_{\rm ext}$

$$w^\mu \;=\; \frac{q}{mc}\,F^{\mu\nu}_{\rm ext}u_\nu\,. \tag{18}$$

In components,

$$\begin{aligned} w^0 \;&=\; \gamma\,\frac{d}{dt}\,(c\gamma) \;=\; c\gamma^4\left(\boldsymbol{\beta}\cdot\frac{d\boldsymbol{\beta}}{dt}\right) \;=\; \gamma^4(\boldsymbol{\beta}\cdot\mathbf{a}),\\ \mathbf{w} \;&=\; \gamma\,\frac{d}{dt}\,(\gamma\mathbf{v}) \;=\; \gamma^2\mathbf{a} \;+\; \gamma^4(\boldsymbol{\beta}\cdot\mathbf{a})\boldsymbol{\beta}, \end{aligned} \tag{19}$$

where $\mathbf{a} = d\mathbf{v}/dt$ is the particle's acceleration in 3D terms. Another useful acceleration-related 4–vector is

$$\mathcal{W}^\nu \;\stackrel{\rm def}{=}\; w^\nu \;-\; \frac{w\cdot(x-x_c)}{u\cdot(x-x_c)}\times u^\nu \;=\; w^\nu \;-\; \frac{w_\alpha n^\alpha}{u_\alpha n^\alpha}\times u^\nu. \tag{20}$$

In 3D terms

$$\frac{w_\alpha n^\alpha}{u_\alpha n^\alpha} \;=\; \frac{\gamma^4(\boldsymbol{\beta}\cdot\mathbf{a}) \;-\; \gamma^2(\mathbf{a}\cdot\mathbf{n}) \;-\; \gamma^4(\boldsymbol{\beta}\cdot\mathbf{a})(\boldsymbol{\beta}\cdot\mathbf{n})}{\gamma c(1-\boldsymbol{\beta}\cdot\mathbf{n})} \;=\; \frac{\gamma^3}{c}\,(\boldsymbol{\beta}\cdot\mathbf{a}) \;-\; \frac{\gamma(\mathbf{a}\cdot\mathbf{n})}{c(1-\boldsymbol{\beta}\cdot\mathbf{n})}\,, \tag{21}$$

hence

$$\begin{aligned} \mathcal{W}^0 \;&=\; \gamma^4(\boldsymbol{\beta}\cdot\mathbf{a}) \;-\; \left(\frac{\gamma^3}{c}\,(\boldsymbol{\beta}\cdot\mathbf{a}) \;-\; \frac{\gamma(\mathbf{a}\cdot\mathbf{n})}{c(1-\boldsymbol{\beta}\cdot\mathbf{n})}\right)\gamma c\\ &=\; +\;\frac{\gamma^2(\mathbf{a}\cdot\mathbf{n})}{1-\boldsymbol{\beta}\cdot\mathbf{n}}\,,\\ \vec{\mathcal{W}} \;&=\; \gamma^4(\boldsymbol{\beta}\cdot\mathbf{a})\boldsymbol{\beta} \;+\; \gamma^2\mathbf{a} \;-\; \left(\frac{\gamma^3}{c}\,(\boldsymbol{\beta}\cdot\mathbf{a}) \;-\; \frac{\gamma(\mathbf{a}\cdot\mathbf{n})}{c(1-\boldsymbol{\beta}\cdot\mathbf{n})}\right)\gamma\mathbf{v}\\ &=\; \gamma^2\mathbf{a} \;+\; \frac{\gamma^2(\mathbf{n}\cdot\mathbf{a})\boldsymbol{\beta}}{1-\boldsymbol{\beta}\cdot\mathbf{n}}\,. \end{aligned} \tag{22}$$

The w^μ and $\mathcal{W}^\mu$ vectors are useful for calculating the τ-derivatives in eq. (16) for the EM fields generated by the moving charge. Indeed, in terms of the w^μ,

$$\begin{aligned} \frac{\partial}{\partial\tau}\left(u^\nu(x-x_c)_\nu\right) \;&=\; w^\nu(x-x_c)_\nu \;+u^\nu\left(\frac{\partial(x-x_c)_\nu}{\partial\tau} \;=\; -u_\nu\right)\\ &=\; w^\nu(x-x_c)_\nu \;-\; u^\nu u_\nu \;=\; w\cdot(x-c_c) \;-\; c^2, \end{aligned} \tag{23}$$

hence

$$\frac{\partial}{\partial\tau}\left(\frac{u^\nu}{u\cdot(x-x_c)}\right) = \frac{w^\nu}{u\cdot(x-x_c)} - \frac{u^\nu(w\cdot(x-x_c))}{(u\cdot(x-x_c))^2} + \frac{c^2u^\nu}{(u\cdot(x-x_c))^2}$$
$$= \frac{\mathcal{W}^\nu}{u\cdot(x-x_c)} + \frac{c^2u^\nu}{(u\cdot(x-x_c))^2} \tag{24}$$

and that's where the $\mathcal{W}^\mu$ vector becomes useful. Plugging eq. (24) into eq. (16) for the $F^{\mu\nu}$ fields, we arrive at

$$F^{\mu\nu}(x) = \left[\frac{q[(x-x_c)^\mu\mathcal{W}^\nu - (x-x_c)^\nu\mathcal{W}^\mu]}{[(x-x_c)\cdot u]^2} + \frac{qc^2[(x-x_c)^\mu u^\nu - (x-x_c)^\nu u^\mu]}{[(x-x_c)\cdot u]^3}\right]_{\text{ret}} \tag{25}$$

or in terms of the distance $R=|\mathbf{x}-\mathbf{x}_c|$ and the lightlike unit vector n^μ,

$$F^{\mu\nu}(x) = \left[\frac{q[n^\mu\mathcal{W}^\nu - n^\nu\mathcal{W}^\mu]}{R[n^\alpha u_\alpha]^2} + \frac{qc^2[n^\mu u^\nu - n^\nu u^\mu]}{R^2[n^\alpha u_\alpha]^3}\right]_{\text{ret}}. \tag{26}$$

The two terms here have different physical origins — the first term is due to the particle's acceleration, while the second terms follows just from its charge and velocity, — and have different distance dependences, $1/R$ versus $1/R^2$. At large distances, the first term dominates, and the $1/R$ scaling of the fields translates to the $1/R^2$ scaling of the Poynting vector and therefore finite EM power per unit of solid angle. Thus, the first term

$$F^{\mu\nu}_{\text{rad}} = \left[\frac{q[(x-x_c)^\mu\mathcal{W}^\nu - (x-x_c)^\nu\mathcal{W}^\mu]}{[(x-x_c)\cdot u]^2}\right]_{\text{ret}} \tag{27}$$

can be identified as the EM wave radiated by the accelerating charge.

On the other hand, the second term

$$F^{\mu\nu}_C = \left[\frac{qc^2[n^\mu u^\nu - n^\nu u^\mu]}{R^2[n^\alpha u_\alpha]^3}\right]_{\text{ret}} \tag{28}$$

which scales with distance as $1/R^2$ is basically the Lorentz-transformed Coulomb field of a point charge. To see how it works, consider a charged particle which never accelerates but simply keeps moving at a constant velocity $\mathbf{v}$. As we saw in the homework set#11 (problem 3), the the Coulomb electric field Lorentz-boosted to the lab frame can be written in the covariant form as

$$F_C^{\mu\nu}(x) \;=\; \frac{q}{c}\,\frac{(x-x_c)^\mu u^\nu \;-\; (x-x_x)^\nu u^\mu}{\big[(1/c^2)((x-x_c)\cdot u)^2 \;-\; (x-x_c)^2\big]^{3/2}}\,, \tag{29}$$

where x_c^ν can be evaluated at any point τ on the particle's worldline. In that homework, we have evaluated it at the current time (same as the observer's time), but now let's evaluated it at the retarded time so that $(x-x_c)^2 = 0$. Consequently, eq. (29) becomes

$$F_C^{\mu\nu}(x) \;=\; \left[\frac{qc^2\big[(x-x_c)^\mu u^\nu \;-\; (x-x_x)^\nu u^\mu\big]}{[(x-x_c)\cdot u]^3}\right]_{\rm ret} \;=\; \left[\frac{qc^2[n^\mu u^\nu \;-\; n^\nu u^\mu]}{R^2[n^\alpha u_\alpha]^3}\right]_{\rm ret}, \tag{30}$$

in perfect agreement with eq. (28).

If the particle happened to have zero acceleration at the retarded time but had changed its velocity afterward, then we would need eq. (28) rather than (29)to find its EM fields. Let's spell out this formula in 3D terms. First we spell out

$$n^\alpha u_\alpha \;=\; \gamma c(1-\mathbf{n}\cdot\boldsymbol{\beta}), \tag{31}$$

$$(n^i u^0 \;-\; n^0 u^i) \;=\; \gamma c(n^i - \beta^i), \tag{32}$$

$$(n^i u^j \;-\; n^j u^i) \;=\; \gamma c\epsilon^{ijk}(\mathbf{n}\times\boldsymbol{\beta})^k, \tag{33}$$

and then we get

$$\begin{aligned} \mathbf{E}(x) \;&=\; \left[\frac{q(\mathbf{n}-\boldsymbol{\beta})}{R^2\gamma^2(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}\right]_{\rm ret}, \\ \mathbf{B}(x) \;&=\; \left[\frac{q(-\mathbf{n}\times\boldsymbol{\beta})}{R^2\gamma^2(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}\right]_{\rm ret} \;=\; \mathbf{n}_{\rm ret}\times\mathbf{E}(x). \end{aligned} \tag{34}$$

Now consider the acceleration-dependent EM radiation

$$F^{\mu\nu}_{\rm rad} \;=\; \left[\frac{q[(x-x_c)^\mu\mathcal{W}^\nu \;-\; (x-x_c)^\nu\mathcal{W}^\mu]}{[(x-x_c)\cdot u]^2}\right]_{\rm ret} \tag{27}$$

and let's spell it out in 3D terms. This time, the numerator algebra is a bit more involved:

$$\begin{aligned}
\mathbf{n}\mathcal{W}^0 - n^0\vec{\mathcal{W}} &= \mathbf{n}\left(\frac{\gamma^2(\mathbf{a}\cdot\mathbf{n})}{1-\mathbf{n}\cdot\boldsymbol{\beta}}\right) - (1)\left(\gamma^2\mathbf{a} + \frac{\gamma^2(\mathbf{a}\cdot\mathbf{n})\boldsymbol{\beta}}{1-\mathbf{n}\cdot\boldsymbol{\beta}}\right) \\
&= \frac{\gamma^2}{1-\mathbf{n}\cdot\boldsymbol{\beta}}\Big((\mathbf{a}\cdot\mathbf{n})\mathbf{n} - \mathbf{a} + (\mathbf{n}\cdot\boldsymbol{\beta})\mathbf{a} - (\mathbf{a}\cdot\mathbf{n})\boldsymbol{\beta}\Big) \\
&= \frac{\gamma^2}{1-\mathbf{n}\cdot\boldsymbol{\beta}}\Big(\mathbf{n}\times(\mathbf{n}\times\mathbf{a}) - \mathbf{n}\times(\boldsymbol{\beta}\times\mathbf{a}) = \mathbf{n}\times((\mathbf{n}-\boldsymbol{\beta})\times\mathbf{a})\Big)
\end{aligned} \tag{35}$$

for the electric field, and

$$\begin{aligned}
-\mathbf{n}\times\vec{\mathcal{W}} &= -\mathbf{n}\times\left(\gamma^2\mathbf{a} + \frac{\gamma^2(\mathbf{a}\cdot\mathbf{n})\boldsymbol{\beta}}{1-\mathbf{n}\cdot\boldsymbol{\beta}}\right) \\
&= -\frac{\gamma^2}{1-\mathbf{n}\cdot\boldsymbol{\beta}}\,\mathbf{n}\times\Big(\mathbf{a} - (\mathbf{n}\cdot\boldsymbol{\beta})\mathbf{a} + (\mathbf{a}\cdot\mathbf{n})\boldsymbol{\beta} = \mathbf{a} + \mathbf{n}\times(\boldsymbol{\beta}\times\mathbf{a})\Big) \\
&= \frac{\gamma^2}{1-\mathbf{n}\cdot\boldsymbol{\beta}}\Big(-\mathbf{n}\times\mathbf{a} - \mathbf{n}\times(\mathbf{n}\times(\boldsymbol{\beta}\times\mathbf{a}))\Big) \\
&= \frac{\gamma^2}{1-\mathbf{n}\cdot\boldsymbol{\beta}}\Big(\mathbf{n}\times\big(\mathbf{n}\times((\mathbf{n}-\boldsymbol{\beta})\times\mathbf{a})\big)\Big)
\end{aligned} \tag{36}$$

for the magnetic field. But at the end of this calculation, we get fairly simple formulae for the radiated fields:

$$\mathbf{E}_{\text{rad}}(x) = \left[\frac{q}{c^2R}\,\frac{\mathbf{n}\times((\mathbf{n}-\boldsymbol{\beta})\times\mathbf{a})}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}\right]_{\text{ret}}, \tag{37}$$

$$\mathbf{B}_{\text{rad}}(x) = \left[\frac{q}{c^2R}\,\frac{\mathbf{n}(\mathbf{n}\times((\mathbf{n}-\boldsymbol{\beta})\times\mathbf{a}))}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}\right]_{\text{ret}} = \mathbf{n}_{\text{ret}}\times\mathbf{E}_{\text{rad}}(x). \tag{38}$$

Note that both of these fields are $\perp$ to the direction $\mathbf{n}$ from the charge to the observation point, and the $\mathbf{E}$ and $\mathbf{B}$ fields are $\perp$ to each other. This — as well as the $1/R$ behavior of the fields — is the typical behavior or the EM wave radiated by the accelerating charge, and that's why we call the fields (37) and (38) the *radiated fields*.

EM power radiated by an accelerated charge

The energy flux density of the EM fields (37) and (38) radiated by an accelerated charges given by the Poynting vector

$$\mathbf{S} \;=\; \frac{c}{4\pi}\,\mathbf{E}\times\mathbf{B} \tag{39}$$

(in Gauss units). For $\mathbf{E}\perp\mathbf{n}$ and $\mathbf{B}=\mathbf{n}\times\mathbf{E}$, the Poynting vector points in the radial direction,

$$\mathbf{S} \;=\; \frac{c|\mathbf{E}|^2}{4\pi}\,\mathbf{n}\,; \tag{40}$$

in particular, for the fields (37) and (38),

$$\mathbf{S} \;=\; \frac{q^2}{4\pi c^3}\left[\frac{\bigl|\mathbf{n}\times((\mathbf{n}-\boldsymbol{\beta})\times\mathbf{a})\bigr|^2}{R^2(1-\mathbf{n}\cdot\boldsymbol{\beta})^6}\,\mathbf{n}\right]_{\text{ret}}. \tag{41}$$

To find the net power radiated by the charge into a unit of solid angle, we surround it by sphere of some large radius R. To be precise, we center this sphere at the location $\mathbf{x}_c(t_{\text{ret}})$ where the charge was at the retarded time $t_{\text{ret}}=t-R/c$, so that the radiation emitted by the charge at that time would reach all points of the sphere at the same moment t. Consequently, the EM power crossing the sphere at time t — and hence emitted by the charge at the retarded time — is

$$\frac{dP}{d\Omega} \;=\; R^2\mathbf{n}\cdot\mathbf{S} \;=\; \frac{q^2}{4\pi c^3}\times\frac{\bigl|\mathbf{n}\times((\mathbf{n}-\boldsymbol{\beta})\times\mathbf{a})\bigr|^2}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^6} \tag{42}$$

in Gauss units, or

$$\frac{dP}{d\Omega} \;=\; \frac{q^2}{16\pi^2\epsilon_0c^3}\times\frac{\bigl|\mathbf{n}\times((\mathbf{n}-\boldsymbol{\beta})\times\mathbf{a})\bigr|^2}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^6} \tag{43}$$

in MKSA units.

Larmor Formula

Before we explore eq. (42) in all its relativistic glory, let's take a look at its non-relativistic limit. So let's assume that the charge's velocity $\mathbf{v}(t)$ is time-dependent but always stays much slower than the speed of light, $|\mathbf{v}| \ll c$. In this limit, $\beta \ll 1$ so eq. (42) becomes a drastically simpler formula

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3} \times \left|\mathbf{n} \times (\mathbf{n} \times \mathbf{a})\right|^2. \tag{44}$$

Or in terms of the acceleration magnitude a and the angle θ between the acceleration and the direction $\mathbf{n}$ towards the observer,

$$\frac{dP}{d\Omega} = \frac{q^2 a^2}{4\pi c^3} \times \sin^2\theta. \tag{45}$$

The angular distribution of this radiated power is similar to the radiation by a linear dipole: no power is radiated in the direction of the acceleration $\mathbf{a}$ or in the opposite direction, while the directions $\perp \mathbf{a}$ receive the maximal power. Note that it's the direction of the acceleration which determines the power distribution, while the velocity direction has no effect in the non-relativistic limit. Thus, a particle accelerating along a straight line radiates mostly sideways but not in the forward or backward direction, while the particle moving at a constant speed in a horizontal circle radiates mostly upward, downward, and in the tangent direction to the circle, but not in the radial direction towards or away from the center.

The total power radiated in all directions by the accelerating charge is given by the *Larmor formula*

$$P = \frac{q^2 a^2}{4\pi c^3} \times \frac{8\pi}{3} = \frac{2q^2 a^2}{3c^3} \tag{46}$$

in Gauss units, or

$$P = \frac{q^2 a^2}{6\pi\epsilon_0 c^3} \tag{47}$$

in MKSA units. For example, an electron in an electric field $\mathbf{E}$ accelerates at the rate

$$\mathbf{a} = \frac{-e}{m_e}\mathbf{E} \tag{48}$$

so the net power of EM waves it radiates is

$$P[\text{in eV/s}] \;=\; \frac{e^4E^2}{6\pi\epsilon_0\,c^3m_e^2} \;\approx\; 1.1\cdot 10^{-12}\times(E[\text{in V/m}])^2. \tag{49}$$

Radiation by relativistic particles

For relativistic particles, the angular distribution of radiation is given by eq. (42), which is much more complicated than its non-relativistic limit (45). However, the net power emitted by a relativistic accelerating charge can be easily obtained through the magic of Lorentz transforms.

Consider two frames of reference, the lab frame K and the frame K' which happens to be co-moving with the charged particle at the time we measure the radiation, or rather at the retarded time t_r when that radiation was emitted. Note: K' is an inertial frame which moves at constant velocity $\mathbf{u}$ relative to the lab frame K, we just choose $\mathbf{u}$ to be equal to the particle's velocity $\mathbf{v}(t_r)$ at a specific moment of the retarded time t_r. Consequently, at the time t_r the particle has zero velocity $\mathbf{v}'$ relative to the K' frame, but its acceleration $\mathbf{a}'$ has no reason to vanish. Instead, $\mathbf{a}'$ is related to the acceleration $\mathbf{a}$ in the lab frame as

$$\mathbf{a}'_{\|} \;=\; \gamma^3\mathbf{a}_{\|}\,,\quad \mathbf{a}'_{\perp} \;=\; \gamma^2\mathbf{a}_{\perp}\,, \tag{50}$$

cf. homework set#11, eqs. (2–3) for $\mathbf{v}'=0$. Alternatively, we may write the acceleration^2 in the K' frame in the covariant form as

$$\mathbf{a}'^2 \;=\; \frac{1}{m^2}\left(\frac{d\mathbf{p}'}{dt'}\right)^2 \;=\; \frac{1}{m^2}\left(-\frac{dp'^\mu}{d\tau}\,\frac{dp'_\mu}{d\tau}\right) \;=\; \frac{1}{m^2}\left(-\frac{dp^\mu}{d\tau}\,\frac{dp_\mu}{d\tau}\right) \tag{51}$$

and then me may use this covariant formula in the lab frame or any other frame of reference.

In the K' frame, the particle radiates EM energy according to the Larmor formula (46),

$$P' \;=\; \frac{dE'}{dt'} \;=\; \frac{2q^2}{3c^3}\,\mathbf{a}'^2 \;=\; \frac{2q^2}{3c^3m^2}\left(-\frac{dp^\mu}{d\tau}\,\frac{dp_\mu}{d\tau}\right). \tag{52}$$

Moreover, in the K' frame the radiated power is symmetric WRT reflections of space,

$$\frac{dP'}{d\Omega'}(\mathbf{n}') \;=\; \frac{q^2a'^2}{4\pi c^3}\times\sin^2\theta' \;=\; \frac{dP'}{d\Omega'}(-\mathbf{n}'), \tag{53}$$

so the EM radiation has zero net momentum. Consequently, when we translate the net EM energy-momentum from the K' frame to the lab frame K, we get

$$dE = \gamma \times dE', \quad d\mathbf{P} = \frac{\gamma\mathbf{u}}{c^2}\,dE'. \tag{54}$$

Also, along the particle's worldline $dt = \gamma dt'$, so the rate at which the particle emit EM energy is the same in both K and K' frames,

$$P = \frac{dE}{dt} = \frac{dE'}{dt'} = P'. \tag{55}$$

Thus, the energy emission rate by the accelerating particle in the lab frame is

$$P = \frac{2q^2}{3c^3m^2}\left(-\frac{dp^\mu}{d\tau}\,\frac{dp_\mu}{d\tau}\right) \tag{56}$$

in covariant form, or in terms of the acceleration in the lab frame,

$$P = \frac{2q^2}{3c^3} \times \left(\gamma^6\mathbf{a}_\parallel^2 + \gamma^4\mathbf{a}_\perp^2\right). \tag{57}$$

According to this formula, acceleration of the same magnitude produces stronger radiation radiation if its directed forward or backward than sideways. But in practice, the forward or backward acceleration requires electric fields while the sideways acceleration can be done by magnetic fields, and making strong magnetic fields is easier than making equally strong electric fields (in Gauss units). Thus, relativistic electrons in linear accelerators (linacs) loose much less energy to EM radiation than electrons in circular accelerators (synchrotrons).

For example, the biggest electron linac — the SLAC at Stanford — accelerates electrons to $E = 50$ GeV ($\gamma = 100\,000$) using electric fields $E \sim 17$ MV/m. Consequently, the forward acceleration of SLAC electrons (in a lab frame) is given by

$$\frac{d(\gamma mv)}{dt} = \gamma^3 ma = eE \quad\Longrightarrow\quad a = \frac{1}{\gamma^3} \times \frac{eE}{m_e} \approx \frac{3\cdot 10^{18}\ \mathrm{m/s^2}}{\gamma^3}\ : \tag{58}$$

humongous acceleration while the electrons are still slow, but drops to only 3000 $\mathrm{m/s^2}$ when they reach $\gamma = 100\,000$. But due to the γ^6 factor in eq. (57), the power loss to EM radiation is uniform throughout the acceleration process,

$$P = \frac{e^4 e^2}{6\pi\epsilon_0\, c^3 m_e^2} \approx 5 \cdot 10^{-17}\ \mathrm{W}. \tag{59}$$

This power may be remarkable for a single electron, but it's completely negligible compared to the power $eE \times v \approx 8 \cdot 10^{-4}$ W it gains from the accelerating field.

In plasma wake-field accelerators (currently not quite ready for particle physics, but hopefully soon), the accelerating electric fields are much higher — up to 50 GV/m in current experiments, and hopefully even higher in future accelerators. In such fields, the accelerating electrons radiate EM power at much higher rate — about $5 \cdot 10^{-8}$ W per electron — but it's still much smaller than the power gain from the accelerating E field.

But now consider a synchrotron in which electrons are made to move in a circle by a transverse magnetic field B while occasional forward electric fields slowly increase their energies. The magnetic force $q\mathbf{v} \times \mathbf{B}$ gives electrons centripetal acceleration

$$a_c = \omega v = \frac{evB}{\gamma m} \tag{60}$$

(MKSA units) hence rotation frequency

$$\omega = \frac{eB}{\gamma m}, \tag{61}$$

and turning radius

$$R = \frac{v}{\omega} = \frac{\gamma m v}{eB} = \frac{p}{eB}. \tag{62}$$

Note that this formula applies to electrons and protons alike, for the same relativistic momentum p we get the same turning radius in the same magnetic field. In a synchrotron, the turning radius is fixed by the beam pipe geometry, so the magnetic field has to increase as the particles accelerate to higher and higher momenta.

In terms of the turning radius R, the centripetal acceleration in the lab frame is given by the good old non-relativistic formula

$$a_c = \frac{v^2}{R}, \tag{63}$$

hence the net power of the *synchrotron radiation* emitted by the particle is

$$P = \frac{e^2}{6\pi\epsilon_0 c^3}\, a_c^2 \gamma^4 = \frac{e^2}{6\pi\epsilon_0 c^3} \times \frac{\gamma^4\beta^4 c^4}{R^2}, \tag{64}$$

which for ultra-relativistic particles becomes

$$P = \frac{e^2 c}{6\pi\epsilon_0\, R^2} \times \gamma^4. \tag{65}$$

Note that this power increases as the fourth power of γ, so it becomes a major problem for the electron synchrotrons which achieve much larger γ's than the proton accelerators. For example, the protons at the LHC are accelerated to $E = 6500$ GeV or $\gamma = 7000$ in a tunnel of turning radius $R = 2800$ m, so they lose energy to the synchrotron radiation at the rate $P = 1.4 \cdot 10^{-11}$ W per proton or about 90 MeV/s. By comparison, the LEP II accelerator — which used to occupy the same tunnel as the LHS is using now — had accelerated electrons to a lower energy $E = 100$ Gev but much higher γ factor $\gamma = 200\,000$. Consequently, the LEP II electrons lost energy to the synchrotron radiation at the much higher rate of $P = 10^{-5}$ W per electron or about $6 \cdot 10^5$ GeV/s. In other words, in 5.9 μs it took an electron to make a complete circle around the accelerator, it lost 3.5 GeV worth of energy to the synchrotron radiation, or about 3.5% of its net kinetic energy. This lost energy had to be replenished by the electric fields of the accelerating RF cavities for each electron each time it made a complete circle, and this lead to spectacularly high electric bills.

In general, the fraction of electron's energy lost to the synchrotron radiation while the electron makes a single turn around the accelerator is

$$\frac{\Delta E}{E} = \frac{P \times (2\pi R/c)}{\gamma m c^2} = \frac{e^2}{3\epsilon_0\, m_e c^2} \times \frac{\gamma^3}{R} = \frac{e^2}{3\epsilon_0\, (m_e c^2)^4} \times \frac{E^3}{R} \approx 8.8 \cdot 10^{-5} \times \frac{(E[\mathrm{GeV}])^3}{R[\mathrm{m}]}. \tag{66}$$

Since the main advantage of synchrotrons over linear accelerators is that you can accelerate the particles over many turns around the circle instead of in a single pass through the linac, the energy loss fraction should be as small as possible, certainly no more than a few percent. Consequently, the sizes of future electron synchrotrons — if they are ever built — must increase with energy as E^3. For example, the CEPC which China is planning to build in the next decade would accelerate electrons and positrons to 125 GeV, only 25% higher energy than the LEP II, but it would have twice the length of LEP or LHS, about 54 km.

By comparison, in proton synchrotrons energy losses to the synchrotron radiation are not a problem — note the $1/m^4$ factor in eq. (66). Instead, they need large sizes simply because it's hard to make big magnets stronger than a few Tesla, so the turning radius (62) grows with the particle's momentum. Thus, given the magnet technology, you need $R \propto E$; numerically

$$R[\mathrm{m}] \;=\; 3.35 \times \frac{p[\mathrm{Gev}/c]}{B[\mathrm{T}]}\,. \tag{67}$$

Angular distribution of radiation

Let's go back to the formula

$$\frac{dP}{d\Omega} \;=\; \frac{2e^2}{3c^3} \times \frac{|\mathbf{n}\times((\mathbf{n}-\boldsymbol{\beta})\times\mathbf{a}))|^2}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^6} \tag{68}$$

for the EM power emitted by a relativistic accelerating charge and consider the angular distribution of this power. The power in eq. (68),

$$\frac{dP}{d\Omega} \;=\; \frac{dE_{\mathrm{EM}}}{dt\,d\Omega} \tag{69}$$

is the EM energy crossing a distant sphere surrounding the particle (or rather it's retarded position at time $t - R/c$) per unit of the lab-frame time t. This is a useful quantity to measure when the particle is continuously accelerating and hence continuously radiating.

But what if the particle suddenly changes its velocity by notable amount in a very brief time, for example by colliding with another particle, or a fast electron suddenly hitting a dense target and rapidly coming to stop. In this case, we have a very brief pulse of very intense radiation (due to very large acceleration) called *bremsstrahlung* — which is German for "breaking radiation", — and the most useful quantitative feature of this pulse is not its power but rather its net energy $E_{\rm EM}$, and also its angular distribution

$$\frac{dE_{\rm EM}}{d\Omega} = \int dt\,\frac{dP(t)}{d\Omega}\,. \tag{70}$$

Due to Doppler effect, observers in different directions from the charge (or rather its retarded position) see different durations of the bremsstrahlung pulse: if the pulse was *emitted* by the moving particle during time interval $\Delta t_c = \gamma\Delta\tau$, then the observer at rest sees its duration as

$$\Delta t = \gamma(1-\mathbf{n}\cdot\boldsymbol{\beta})\,\Delta\tau = (1-\mathbf{n}\cdot\boldsymbol{\beta})\,\Delta t_c. \tag{71}$$

Consequently, the net bremsstrahlung energy per unit of solid angle emitted in different directions is

$$\frac{dE_{\rm EM}}{d\Omega} = \frac{2e^2}{3c^3}\int dt_c \frac{|\mathbf{n}\times((\mathbf{n}-\boldsymbol{\beta})\times\mathbf{a})|^2}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^{6-1}}\,. \tag{72}$$

More generally, the EM energy emitted by a relativistic accelerating charge per unit of *emission time* dt_c per unit of solid angle is

$$\frac{dE_{\rm EM}}{dt_c\,d\Omega} = \frac{2e^2}{3c^3}\times\frac{|\mathbf{n}\times((\mathbf{n}-\boldsymbol{\beta})\times\mathbf{a})|^2}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^5} \tag{73}$$

and its angular distribution is

$$\frac{dE_{\rm EM}}{dt_c\,d\Omega} \propto \frac{|\mathbf{n}\times((\mathbf{n}-\boldsymbol{\beta})\times\mathbf{a})|^2}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^5}\,. \tag{74}$$

Note the $5^{\rm th}$ power of the denominator here instead of the $6^{\rm th}$ power in eq. (68).

For the ultra-relativistic particles, the denominator factor in eq. (74) strongly skews the energy distribution in the forward direction. Indeed, for $\gamma \gg 1$ and hence $\beta \approx 1$, the denominator factor becomes

$$\frac{1}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^5} = \frac{1}{(1-\beta\cos\theta)^5} \approx \frac{1}{(1-\cos\theta)^5} = \frac{1}{32\sin^{10}(\theta/2)}\,, \tag{75}$$

except for very small angles $\theta \lesssim (1/\gamma)$ from the velocity's direction for which

$$1 - \cos\theta \approx \frac{\theta^2}{2} \ll 1, \quad 1-\beta \approx \frac{1}{2\gamma^2} \ll 1 \quad \Longrightarrow \quad 1-\beta\cos\theta \approx \frac{\theta^2}{2} + \frac{1}{2\gamma^2} \tag{76}$$

and hence

$$\frac{1}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^5} = \frac{1}{(1-\beta\cos\theta)^5} \approx \left(\frac{2\gamma^2}{1+(\gamma\theta)^2}\right)^5. \tag{77}$$

To get the full picture of the radiation's angular distribution, we must modulate this forward peak by the numerator of eq. (74). For the general directions of the acceleration and the velocity, this is a rather painful exercise in trigonometry, so let's limit our analysis to the two special cases: (A) $\mathbf{a} \parallel \mathbf{v}$ and (B) $\mathbf{a} \perp \mathbf{v}$.

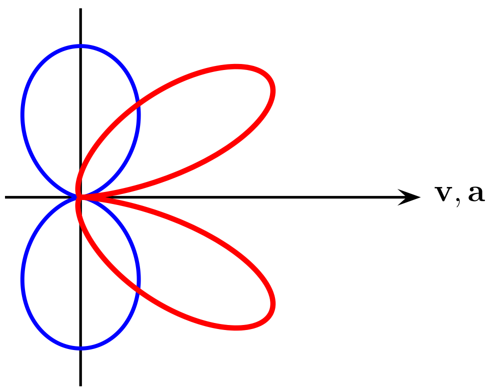
(**A**) Forward or backward acceleration, for example a relativistic electron slamming into a solid target and rapidly coming to stop. Thanks to $\mathbf{a} \parallel \mathbf{v}$, we have $\boldsymbol{\beta} \times \mathbf{a} = 0$ and hence

$$\mathbf{n} \times ((\mathbf{n}-\boldsymbol{\beta}) \times \mathbf{a}) = \mathbf{n} \times (\mathbf{n} \times \mathbf{a}) \quad \Longrightarrow \quad \left|\mathbf{n} \times ((\mathbf{n}-\boldsymbol{\beta}) \times \mathbf{a})\right|^2 = a^2 \times \sin^2\theta. \tag{78}$$

Combining this numerator factor of the angular distribution (74) with the denominator (75), we get

$$\frac{dE}{dt_c\,d\Omega} \propto \frac{\sin^2\theta}{(1-\beta\cos\theta)^5}\,. \tag{79}$$

Let me graphically compare this distribution for a moderately relativistic particle with $\beta = 0.6$ (red line on the diagram below) to that of a non-relativistic particle (blue line):



For the ultra-relativistic particles, this angular distribution becomes

$$\frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \approx \frac{\sin^2\theta}{(1-\cos\theta)^5} = \frac{\cos^2(\theta/2)}{8\sin^8(\theta/2)} \tag{80}$$

except at very small angles from the forward direction for which

$$\frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \approx \frac{32\gamma^8(\gamma\theta)^2}{[1+(\gamma\theta)^2]^5}\,. \tag{81}$$

This distribution peaks at $\theta = 1/(2\gamma)$, and 95% of the total energy is emitted into the forward cone $\theta \leq (2/\gamma)$; here is the plot of the power density as a function of $\gamma\theta$ for small θ:

(**B**) Sideways acceleration — for example in synchrotron radiation, — $\mathbf{a} \perp \mathbf{v}$. In this case, there is no axial symmetry, and the radiation intensity depends on both spherical angles θ and ϕ. Let's use the coordinate system where the velocity points in z direction while the acceleration points in x direction, thus in Cartesian coordinates

$$\mathbf{a} = (a,0,0), \quad \boldsymbol{\beta} = (0,0,\beta), \quad \text{and} \quad \mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta). \tag{82}$$

Consequently,

$$\begin{aligned}
\big[\mathbf{n}\times((\mathbf{n}-\boldsymbol{\beta})\times\mathbf{a})\big]_x &= a\Big(\sin^2\theta\cos^2\phi - 1 + \beta\cos\theta\Big),\\
\big[\mathbf{n}\times((\mathbf{n}-\boldsymbol{\beta})\times\mathbf{a})\big]_y &= a\times\sin^2\theta\cos\phi\sin\phi,\\
\big[\mathbf{n}\times((\mathbf{n}-\boldsymbol{\beta})\times\mathbf{a})\big]_z &= a\Big(\cos\theta\sin\theta\cos\phi - \beta\sin\theta\cos\phi\Big),
\end{aligned} \tag{83}$$

hence in the numerator of the distribution (74) we get (after some algebra)

$$\left|\mathbf{n}\times((\mathbf{n}-\boldsymbol{\beta})\times\mathbf{a})\right|^2 \;=\; a^2\Big((1-\beta\cos\theta)^2 \;-\; (1-\beta)^2\sin^2\theta\cos^2\phi\Big). \tag{84}$$

Altogether, the angular distribution of the emitted power is

$$\frac{dE}{dt_c\,d\Omega} \;\propto\; \left[\frac{1}{(1-\beta\cos\theta)^3} \;-\; \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^5}\right]. \tag{85}$$

Again, for ultra-relativistic particles with $\gamma \gg 1$ most of this power is emitted into the forward cone of $\theta \lesssim (1/\gamma)$. Within that forward cone, we may approximate

$$\frac{dE}{dt_c\,d\Omega} \;\propto\; \frac{1 \;-\; 2(\gamma\theta)^2\cos(2\phi) \;+\; (\gamma\phi)^4}{(1+(\gamma\theta)^2)^5}\,; \tag{86}$$

here is the plot of this power density as a function of $\gamma\theta$ for small θ and for several values of ϕ: