1. Let's start with an electrostatic problem. Suppose you are given the potential $\Phi(\mathbf{x})$ along a complete spherical surface of radius R relative to $\Phi(\infty) = 0$, and you know there are no electric charged anywhere outside that sphere. Then the potential at any point $\mathbf{y}$ outside the sphere can be found as

$$\Phi(\mathbf{y}) \;=\; \frac{\mathbf{y}^2 - R^2}{4\pi R} \iint\limits_{\text{sphere}} d^2\text{Area}(\mathbf{x})\, \frac{\Phi(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^3}\,. \tag{1}$$

Your task is to derive this formula in two different ways.

Let's start by solving the Laplace equation $\nabla^2\Phi(\mathbf{x}) = 0$ by separating variables in spherical coordinates: A general solution for the outside of a sphere subject to the asymptotic condition $\Phi(\infty) = 0$ has form

$$\Phi(r,\theta,\phi) \;=\; \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} C_{\ell,m} \times r^{-\ell-1} \times Y_{\ell,m}(\theta,\phi) \tag{2}$$

for some constants $C_{\ell,m}$. The $Y_{\ell,m}(\theta,\phi)$ in this formula are spherical harmonics you should be familiar with from the undergraduate quantum mechanics class.

(0) If you are not familiar with eq. (2) and/or with the spherical harmonics, please read §3.1–3 and §3.5–6 of the Jackson's textbook.

(a) Determine the coefficients $C_{\ell,m}$ in the series (2) from the known potential at $r = R$, then write the potential outside the sphere as

$$\Phi(\mathbf{y}) \;=\; \iint\limits_{\text{sphere}} d^2\text{Area}(\mathbf{x})\, \Phi(\mathbf{x}) \times F(\mathbf{x},\mathbf{y}) \tag{3}$$

$$\text{for } F(\mathbf{x},\mathbf{y}) \;=\; \sum_{\ell,m} (\text{terms you need to calculate}). \tag{4}$$

(b) Sum up the series (4) and show that the integral (3) amounts to eq. (1). Here are some useful formulae:

$$\sum_{m=-\ell}^{+\ell} Y_{\ell,m}(\mathbf{n}_y) Y^*_{\ell,m}(\mathbf{n}_x) \;=\; \frac{2\ell+1}{4\pi} \times P_\ell(\mathbf{n}_x\cdot\mathbf{n}_y), \tag{5}$$

$$\sum_{\ell=0}^{\infty} t^\ell P_\ell(c) = \frac{1}{\sqrt{1-2tc+t^2}} \qquad \text{for } |t|<1 \text{ and } |c|\le 1, \tag{6}$$

$$\sum_{\ell=0}^{\infty} (2\ell+1) t^\ell P_\ell(c) = \text{(find out from the previous formula).} \tag{7}$$

The other method to derive eq. (1) is to use the Green's function $G(\mathbf{x},\mathbf{y})$ for the Laplace equation outside the sphere subject to the Dirichlet boundary condition on the sphere itself, $G(\mathbf{x},\mathbf{y}) \equiv 0$ for $|\mathbf{x}| = R$, as well as $G = 0$ for $\mathbf{x} = \infty$.

(c) Use the image charge method for the outside of a grounded conducting sphere to show that

$$\begin{aligned} G(\mathbf{x},\mathbf{y}) &= \frac{1}{4\pi\,|\mathbf{x}-\mathbf{y}|} - \frac{R/|\mathbf{y}|}{4\pi\,|\mathbf{x} - \frac{R^2}{|\mathbf{y}|^2}\mathbf{y}|} \\ &= \frac{1}{4\pi\sqrt{x^2+y^2-2xyc}} - \frac{R}{4\pi\sqrt{x^2y^2+R^4-2R^2xyc}} \end{aligned} \tag{8}$$

where $x = |\mathbf{x}|$, $y = |\mathbf{y}|$, and $c = \mathbf{n}_x \cdot \mathbf{n}_y$.

(d) Evaluate the normal derivative of this Green's function at the boundary.

(e) Finally, use the Green's function method (*cf.* textbook §1.10) to derive eq. (1).

2. Next, a couple of warm-up exercises on multipole moments.

(a) Consider the quadrupole-like potential

$$\Phi(\mathbf{x}) = \frac{Q_{ij} n_i^x n_j^x}{4\pi\epsilon_0\,|\mathbf{x}|^3} \tag{9}$$

for some would-be quadrupole moment tensor Q_{ij}, which you may assume to be symmetric — $Q_{ij} = Q_{ji}$ — but not necessarily traceless.

Show that the potential (9) obeys the Laplace equation $\nabla^2\Phi = 0$ at $\mathbf{x} \neq 0$ if and only if the Q_{ij} tensor happens to be traceless, $Q_{ii} = 0$.

The point of this exercise is to explain why the quadrupole moment tensor of any charge distribution must be traceless. And it's defined as

$$Q_{ij} \;=\; \iiint d^3\mathbf{y}\,\rho(\mathbf{y})\Big(\tfrac{3}{2}y_iy_j\;-\;\tfrac{1}{2}\delta_{ij}\mathbf{y}^2\Big) \tag{10}$$

— with an extra $-\frac{1}{2}\delta_{ij}\mathbf{y}^2$ term inside the $(\cdots)$ — precisely to make it traceless.

(b) Now generalize this result to higher multipoles: Consider a would-be 2^ℓ–pole potential

$$\Phi(\mathbf{x}) \;=\; \frac{\mathcal{M}^{(\ell)}_{i_1,\ldots,i_\ell}\, n^x_{i_1}\cdots n^x_{i_\ell}}{4\pi\epsilon_0\,|\mathbf{x}|^{\ell+1}} \tag{11}$$

for some would-be 2^ℓ–pole moment tensor $\mathcal{M}^{(\ell)}_{i_1,\ldots,i_\ell}$. Assume this ℓ–index tensor to be totally symmetric under all possible permutations of its indices, but do not assume its tracelessness.

Show that the potential (11) obeys the Laplace equation if and only if the tensor $\mathcal{M}^{(\ell)}$ happens to be traceless, $\mathcal{M}^{(\ell)}_{i_1,\ldots,i_{\ell-2},j,j} = 0$ for all $i_1,\ldots,i_{\ell-2} = 1,2,3$.

Let me clarify the meaning of the trace of a tensor with more that 2 indices. Choose any two indices of the tensor, and fix all the remaining $\ell-2$ indices. Restrict the chosen indices equal values, and sum over all allowed valued of an index — just as you would do for a two-index tensor. Repeat this procedure for all other values of the $\ell-2$ indices you are not tracing over — and this makes the trace into another tensor, but with $\ell-2$ indices instead of ℓ. For example, take a 3-index tensor T_{ijk} and take the trace over the first and the third index, thus $t_j = T_{iji}$ (implicit sum over $i=1,2,3$). This trace t_j itself is a one-index tensor (*i.e.*, a vector).

For a non-symmetric tensor, we should specify which two of its ℓ indices we are tracing over. For example, for a non-symmetric 3–index tensor T_{ijk} we can trace over the first two indices and get $t^{(12)}_k = T_{iik}$, or over the last two indices and get $t^{(23)}_i = T_{ijj}$, or over the first and the third index and get $t^{(13)}_j = T_{iji}$. And in general, $t^{(12)}$, $t^{(23)}$, and $t^{(13)}$ would be three different 1-index tensors.

However, for a totally symmetric ℓ-index tensor — such as the would-be multipole moment in eq. (11) — it does not matter which two indices we are tracing over. We may pick any two indices we like, and we would get exactly the same trace. Moreover, the trace itself would be a totally-symmetric $(\ell - 2)$–index tensor.

3. And here is another problem about the multipole expansion.

   (a) Consider a spherical body or radius R with a highly nonuniform charge density

$$\rho(r,\theta,\phi) = \frac{Q_0}{R^5} \times (3R^2 - 5r^2) \times \sin\theta \quad (\text{for } r \leq R \text{ only}). \tag{12}$$

   Find the leading multipole moment of the body and the electric potential $\Phi(r,\theta,\phi)$ it creates far away from the body.

   (b) Now consider a generic compact body of net charge q, dipole moment $\mathbf{p}$, quadrupole moment Q_{ij}, octupole moment $\mathcal{O}_{ijk}$, *etc.*, *etc.* This body is subject to a slowly varying *external* potential $\Phi_e(\mathbf{x})$. Show that the potential energy of the body in this external potential is

$$\begin{aligned} U &= q\Phi(0) + p_i\nabla_i\Phi(0) + \tfrac{1}{3}Q_{ij}\nabla_i\nabla_j\Phi(0) + \tfrac{1}{15}\mathcal{O}_{ijk}\nabla_i\nabla_j\nabla_k\Phi(0) + \cdots \\ &= \sum_{\ell=0}^{\infty} K_\ell \mathcal{M}^{(\ell)}_{i_1\cdots i_\ell} \times \nabla_{i_1}\cdots\nabla_{i_\ell}\Phi(0) \end{aligned} \tag{13}$$

   where

$$K_\ell = \frac{1}{(2\ell-1)!!} = \frac{1}{(2\ell-1)(2\ell-3)\cdots(3)(1)}\,. \tag{14}$$

   (Deriving eq. (14) for the coefficients K_n is an optional exercise. If you are short on time, skip it.)

   (c) Use eq. (13) to derive a similar formula for the net force on the body and the net torque on the body (relative to the pivot point $\mathbf{x}_0 = 0$) in terms of the external electric field $\mathbf{E} = -\nabla\Phi$ and its derivative at $\mathbf{x}_0 = 0$.

   (d) Finally, let's go back to the spherical body from part (a) and calculate the leading terms on the net force and net torque on this body in external potential

$$\Phi_e(x_1,x_2,x_3) = \Phi_0\,\sin(\kappa x_1)\,\cos(\kappa x_2)\,\exp(\sqrt{2}\kappa x_3), \quad \kappa R \ll 1. \tag{15}$$