1. Consider the magnetic field inside a tightly wound solenoid of finite length L and finite radius R.

   (a) Using nothing but the rotational symmetry of the solenoid and the analyticity of the magnetic field as a function of the position $\mathbf{x}$, argue that in the cylindrical coordinates (z, s, ϕ),

$$B_z(z,s,\phi) = \sum_{n=0}^{\infty} \alpha_n(z) \times s^{2n} = \alpha_0(z) + \alpha_1(z)\,s^2 + \alpha_2(z)\,s^4 + \cdots, \tag{1}$$

$$B_s(z,s,\phi) = \sum_{n=0}^{\infty} \beta_n(z) \times s^{2n+1} = \beta_0(z)\,s + \beta_1(z)\,s^3 + \beta_2(z)\,s^5 + \cdots, \tag{2}$$

$$B_\phi(z,s,\phi) = 0, \tag{3}$$

   for some analytic functions $\alpha_n(z)$ and $\beta_n(z)$.

   (b) Next, use $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{B} = 0$ (inside the solenoid) to derive recursive relations between the functions $\alpha_n(z)$, $\beta_n(z)$ and their derivatives, and show that

$$\alpha_n(z) = \frac{(-1)^n}{2^{2n}\,(n!)^2}\,\frac{\partial^{2n}}{\partial z^{2n}}\,\alpha_0(z), \qquad \beta_n(z) = \frac{(-1)^{n+1}}{2^{2n+1}\,(n+1)!\,n!}\,\frac{\partial^{2n+1}}{\partial z^{2n+1}}\,\alpha_0(z). \tag{4}$$

In light of parts (a) and (b), given the magnetic field $B_z(z,0) = \alpha_0(z)$ on the axis of the solenoid as a function of z, the field off the axis obtains from it as a series

$$B_z(z,s) = \alpha_0(z) - \frac{s^2}{4}\,\alpha_0''(z) + \frac{s^4}{64}\,\alpha_0''''(z) + \cdots, \tag{5}$$

$$B_s(z,s) = -\frac{s}{2}\,\alpha_0'(z) + \frac{s^3}{16}\,\alpha_0'''(z) - \frac{s^5}{384}\,\alpha_0'''''(s) + \cdots. \tag{6}$$

   (c) Now approximate the winding of the solenoid as a cylindrical current sheet of density $K = IN/L$ and use the Biot–Savart–Laplace formula to show that the field on the cylinder's axis is

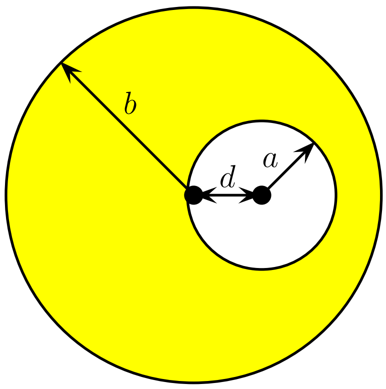
$$B_z(z,0) \;=\; \frac{\mu_0 I N}{L}\times\frac{1}{2}\left(\frac{(L/2)+z}{\sqrt{((L/2)+z)^2+R^2}}+\frac{(L/2)-z}{\sqrt{((L/2)-z)^2+R^2}}\right). \tag{7}$$

(d) Finally, consider a solenoid that's much longer than its radius and focus on the central region of $|z| = O(R) \ll L$. Estimate the derivatives of the on-axis field in this region and show that for $z = O(R)$ and any s between 0 and R,

$$B_z(z,s) \;=\; B_z(0,0)\times\left(1\,+\,O\left(\frac{R^4}{L^4}\right)\right), \quad B_s(z,s) \;=\; B_z(0,0)\times O\left(\frac{R^4}{L^4}\right). \tag{8}$$

Also, calculate the leading $O(R^4/L^4)$ terms in these formulae.

2. Consider a conductor in the shape of a long cylinder of radius b with a hole drilled in it. The hole is also cylindrical, or radius a; the hole's axis is parallel to the axis of the conductor itself, but is shifted from that axis by some distance d ($a + d < b$). Here is the cross-section:



A current of uniform density $\mathbf{J}$ flows along this conductor.

Use Ampere's Law and the linear superposition principle to find the magnetic field $\mathbf{B}$ — both the magnitude and the direction — inside the hole.

3. Consider a thin spherical shell of radius R with a uniform surface charge density σ. The sphere rotates about an axis through its center with angular velocity $\vec{\omega}$.

   (a) Calculate the vector potential $\mathbf{A}$ and the magnetic field $\mathbf{B}$ inside and outside the rotating sphere.

   (b) Now consider two such rotating charged spheres, one inside the other. The two spheres are concentric, but they rotate around different, non-parallel axes, $\vec{\omega}_1 \nparallel \vec{\omega}_2$.
   Calculate the torque between the two spheres.