1. In three space dimension, the retarded solution of the wave equation with an instant point source for example, a light flash at $\mathbf{x}=0$ and $t=0$ — is a spherical shell disturbance of radius $R=ct$ and zero thickness,

$$\Box\,\Psi(\mathbf{x},t) \;=\; \delta(t)\delta^{(3)}(\mathbf{x}) \quad\Longrightarrow\quad \Psi(\mathbf{x},t) \;=\; \frac{\delta(t-|\mathbf{x}|/c)}{4\pi|\mathbf{x}|}\,. \tag{1}$$

In spaces of other odd dimensions $d=3,5,7,\ldots$ except $d=1$, we have similar behavior, bun in even space dimensions $d=2,4,6,\ldots$, the wave of an instant point source has a *wake* behind the light front. For example, in two space dimensions

$$\Box\,\Psi(\mathbf{x},t) \;=\; \delta(t)\delta^{(2)}(\mathbf{x}) \quad\Longrightarrow\quad \Psi(\mathbf{x},t) \;=\; \frac{2c\Theta(ct-|\mathbf{x}|)}{\sqrt{c^2t^2-\mathbf{x}^2}} \tag{2}$$

where Θ is the step function.

(a) Derive the 2D wave (2) from the 3D wave generated by an instant *line* source.

On one space dimension, the disturbance spreads out at light speed, but then does not go away; instead,

$$\Box\,\Psi(x,t) \;=\; \delta(t)\delta(x) \quad\Longrightarrow\quad \Psi(x,t) \;=\; \frac{c}{2}\,\Theta(ct-|x|). \tag{3}$$

(b) Again, derive this 1D wave from the 3D wave generated by an instant source on an infinite plane.

2. A thin toroidal coil of N turns has mean radius R, cross-section $a \ll R^2$, and carries steady current I, A point charge Q is placed at the center of the toroid. The whole system — the change, the coil, the battery and the wires providing the current — is at rest.

(a) Calculate the net momentum of the electromagnetic fields in the system.

(b) Pick realistic values of the input parameters and calculate the resulting momentum. What kind of a mechanical system would have a similar momentum? A speeding bullet? A crawling ant? Something else?

(c) Let's turn off the current in the coil. The transient electric field induced by the dropping magnetic field imparts an impulse on the change. Show that the net impulse given to the charge in this process is precisely the former momentum of the EM fields.

3. Consider the angular momentum of the electromagnetic field,

$$\mathbf{L}_{\mathrm{EM}} \;=\; \iiint d^3\mathbf{x}\,\mathbf{x}\times\frac{1}{c^2}\mathbf{S}(\mathbf{x}) \tag{4}$$

where $S = \mathbf{E}\times\mathbf{H}$ is the Poynting vector.

(a) Show that for the static electric and magnetic fields, the angular momentum (4) can be written as

$$\mathbf{L}_{\mathrm{EM}} \;=\; \frac{1}{c^2}\iiint d^3\mathbf{x}\,\Phi(\mathbf{x})\Big(\mathbf{x}\times\mathbf{J}(\mathbf{x}) \;-\; 2\mathbf{H}(\mathbf{x})\Big). \tag{5}$$

(b) Now suppose magnetic monopoles exist. Put a static point electric charge Q at some distance a from a static monopole of magnetic charge M. Show that the angular momentum created by these electric and magnetic charges is

$$\mathbf{L}_{\mathrm{EM}} \;=\; -\frac{\mu_0 MQ}{4\pi}\,\mathbf{n} \tag{6}$$

where $\mathbf{n}$ is the unit vector pointing from the monopole towards the electric charge. Note: the magnitude of this angular momentum does not depend on the distance a between the electric charge and the monopole!

(c) To verify eq. (6), let an electrically charge particle fly by a static monopole. For simplicity, assume that the only force acting on the particle is the Lorentz force due to the monopole's magnetic field, and that its motion is slow enough that we may use the quasistatic approximation to its electric field. Consequently, the net angular momentum of the system is the mechanical angular momentum of the particle plus the EM angular momentum (6),

$$\mathbf{L}_{\mathrm{net}} \;=\; \mathbf{x}\times m\mathbf{v} \;-\; \frac{\mu_0 MQ}{4\pi}\,\mathbf{n}. \tag{7}$$

Verify the conservation of this angular momentum.

4. In class I have discussed the electromagnetic energy, momentum and related quantities for the vacuum. In a uniform linear medium we have similar formulae:

$$\text{power density:} \qquad P = \mathbf{J}\cdot\mathbf{E}, \tag{8}$$

$$\text{force density:} \qquad \mathbf{f} = \rho\mathbf{E} + \mathbf{J}\times\mathbf{B}, \tag{9}$$

$$\text{energy density:} \qquad u = \tfrac{1}{2}\mathbf{E}\cdot\mathbf{D} + \tfrac{1}{2}\mathbf{H}\cdot\mathbf{D}, \tag{10}$$

$$\text{energy flux density:} \qquad \mathbf{S} = \mathbf{E}\times\mathbf{H}, \tag{11}$$

$$\text{momentum density:} \qquad \mathbf{g} = \mathbf{D}\times\mathbf{B} = \frac{\epsilon\mu}{c^2}\mathbf{S}, \tag{12}$$

$$\text{stress tensor:} \qquad T_{ij} = E_iD_j + H_iB_j - \tfrac{1}{2}\delta_{ij}(\mathbf{E}\cdot\mathbf{D}+\mathbf{H}\cdot\mathbf{B}). \tag{13}$$

(a) Verify the local energy and momentum conservation laws

$$\frac{\partial u}{\partial t} + \nabla\cdot\mathbf{s} + P = 0, \tag{14}$$

$$\frac{\partial g_i}{\partial t} - \nabla_j T_{ij} + f_i = 0. \tag{15}$$

Hint: for a linear medium $E_iD_j = D_iE_j$ and $H_iB_j = B_iJ_j$.

(b) Which of the equations (8) through (13) — if any — should be modified in order to keep the conservation laws (14) and (15) in a linear but non-uniform media where ϵ and μ vary with $\mathbf{x}$ (but not with time),

$$\mathbf{D}(\mathbf{x},t) = \epsilon(\mathbf{x})\epsilon_0\mathbf{E}(\mathbf{x},t), \quad \mathbf{B}(\mathbf{x},t) = \mu(\mathbf{x})\mu_0\mathbf{H}(\mathbf{x},t), \tag{16}$$

What is the physical meaning of this modification?