1. In the Coulomb gauge $\nabla\cdot\mathbf{A}\equiv 0$, the potentials Φ and $\mathbf{A}$ obey

$$-\nabla^2\Phi(\mathbf{x},t) \;=\; \frac{1}{\epsilon_0}\,\rho(\mathbf{x},t), \quad \Box\mathbf{A}(\mathbf{x},t) \;=\; \mu_0\mathbf{J}_T(\mathbf{x},t), \tag{1}$$

where the transverse current $\mathbf{J}_T$ is

$$\mathbf{J}_T \;=\; \mathbf{J} \;+\; \nabla\left(\frac{-1}{\nabla^2}\,(\nabla\cdot\mathbf{J})\right), \quad i.e,, \quad \mathbf{J}_T(\mathbf{x},t) \;=\; \mathbf{J}(\mathbf{x},t) \;+\; \nabla_x \iiint d^3\mathbf{y}\,\frac{\nabla_y\cdot\mathbf{J}(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|}\,. \tag{2}$$

As I've explained in class, the scalar potential Φ and part of the vector potential $\mathbf{A}$ respond instantaneously to the charges and currents, but these instantaneous terms cancel out from the electric and magnetic fields. In this exercise, we shall see how this works for a particularly simple source, namely an electric dipole $\mathbf{p}$ which turns up for just a moment and then turns back off,

$$\rho(\mathbf{x},t) \;=\; -\delta(t)\,(\mathbf{p}\cdot\nabla)\delta^{(3)}(\mathbf{x}), \quad \mathbf{J}(\mathbf{x},t) \;=\; \delta'(t)\,\mathbf{p}\,\delta^{(3)}(\mathbf{x}). \tag{3}$$

Note: derivatives of the delta functions are defined via integration by parts.

(a) As a warm-up trivial exercise, verify the continuity equation for the dipole flash (3) and calculate the scalar potential $\Phi(\mathbf{x},t)$ in the Coulomb gauge.

(b) Calculate the transverse current $\mathbf{J}_T(\mathbf{x},t)$ and show that

$$\mathbf{J}_T(\mathbf{x},t) \;=\; \delta'(t)\left(\mathbf{p}\,\delta^{(3)}(\mathbf{x}) \;+\; \nabla(\mathbf{p}\cdot\nabla)\left(\frac{1}{4\pi r}\right)\right) \tag{4}$$

$$\;=\; \delta'(t)\left(\frac{2}{3}\,\mathbf{p}\,\delta^{(3)}(\mathbf{x}) \;+\; \frac{3(\mathbf{n}\cdot\mathbf{p})\mathbf{n}\;-\;\mathbf{p}}{4\pi r^3}\right). \tag{5}$$

(c) Next, prove a couple of lemmas you would need in the following parts:

$$\underset{\substack{whole\\ space}}{\iiint} d^3\mathbf{z}\,\frac{\delta'(t-|\mathbf{z}|/c)}{|\mathbf{z}|}\,F(\mathbf{z}) \;=\; c^2\Theta(t)\left[\left(1\;+\;r\frac{\partial}{\partial r}\right)\oint\!\!\!\oint d^2\Omega_n\,F(\mathbf{z}=r\mathbf{n})\right]_{@r=ct} \tag{6}$$

where Θ is the step-function, and

$$\frac{1}{4\pi}\oiint d^2\Omega_n \frac{1}{|\mathbf{x}+R\mathbf{n}|} \;=\; \frac{1}{\max(|\mathbf{x}|,R)}\,. \tag{7}$$

(d) Use the retarded Green's function to solve the wave equation $\Box\mathbf{A} = \mu_0\mathbf{J}_T$ for the vector potential under initial condition $\mathbf{A}(\mathbf{x},t<0)\equiv 0$. Show that

$$\begin{aligned}\text{for } t<|\mathbf{x}|/c,\quad \mathbf{A}(\mathbf{x},t) \;&=\; \mu_0 c^2\Theta(t)\,\nabla(\mathbf{p}\cdot\nabla)\left(\frac{1}{4\pi|\mathbf{x}|}\right)\\ &=\; \Theta(t)\,\frac{3(\mathbf{n}\cdot\mathbf{p})\mathbf{n}\,-\,\mathbf{p}}{4\pi\epsilon_0\,|\mathbf{x}|^3}\,.\end{aligned} \tag{8}$$

Hints: use eq. (4) rather that eq. (5) for the transverse current; after applying the retarded Green's function, change integration variable from $\mathbf{y}$ to $\mathbf{z}=\mathbf{y}-\mathbf{x}$, then turn $\partial/\partial y$ derivatives into $\partial/\partial x$ at fixed $\mathbf{z}$; use lemmas (6) and (7).

(e) Use vector potential (8) and the scalar potential you have computed in part (a) to verify that the electric and the magnetic fields do not propagate faster than light,

$$\text{for } t<c|\mathbf{x}|,\quad \mathbf{E}(\mathbf{x},t) \;=\; \mathbf{B}(\mathbf{x},t) \;=\; 0. \tag{9}$$

(f) Now calculate the vector potential $\mathbf{A}(\mathbf{x},t)$ for $t\ge|\mathbf{x}|/c$, including the light front $t=|\mathbf{x}|/c$ itself.

(g) Finally, use the vector potential from part (f) to find the electric and magnetic fields.

2. Consider charge density perturbations $\rho(\mathbf{x},t)$ in food conductors such as metals.

(a) Fourier transform time-dependence to frequency-dependence and show that $\rho(\mathbf{x},\omega)$ obeys

$$\Big(\sigma(\omega)\;-\;i\omega\epsilon(\omega)\epsilon_0\Big)\rho(\omega,\mathbf{x}) \;=\; 0 \tag{10}$$

where $\sigma(\omega)$ is the AC conductance, $\mathbf{J}_{\text{cond}}(\omega,\mathbf{x})=\sigma(\omega)\mathbf{E}(\omega,\mathbf{x})$.
Hint: the net conduction + displacement current has zero divergence.

Drude–Lorentz formula tell us that in metals

$$\sigma(\omega)\;-\;i\omega\epsilon(\omega)\epsilon_0\;=\;\frac{ne^2 f_0}{m_e^*}\,\frac{1}{\gamma_0-i\omega}\;-\;i\omega\epsilon_b\epsilon_0\;\approx\;\epsilon_b\epsilon_0\left(\frac{\omega_p^2}{\gamma_0-i\omega}\;-\;i\omega\right) \tag{11}$$

where $\gamma_0=(1/\tau)$ is the rate at which the conduction electrons lose their average velocity vector to collisions with ions, and ω_p is the plasma frequency of the metal.

(b) Solve eq. (10) for density perturbations in a metal with $\omega_p\gg\gamma_0$. Show that as a function of time rather than frequency, $\rho(t,\mathbf{x})$ oscillates in place with the plasma frequency ω_p while oscillation amplitude decays as $\exp(-\gamma_0 t)$.

3. Consider a 1D wave propagating through a linear and homogeneous but dispersive media with refraction index $n(\omega)$, *i.e.*, the phase velocity of a wave $v(\omega)=c/n(\omega)$. To allow for absorption, $n(\omega)$ may be complex rather than real.

(a) Show that the most general solution of the dispersive wave equation is

$$\psi(x,t)\;=\int\limits_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,e^{-i\omega t}\Big(A(\omega)\times\exp(+i\omega n(\omega)x/c)\;+\;B(\omega)\times\exp(-i\omega n(\omega)x/c)\Big) \tag{12}$$

for some arbitrary complex functions $A(\omega)$ and $B(\omega)$.

(b) Show that a real wave $\psi(x,t)$ requires $n(-\omega)=n^*(+\omega)$ as well as $A(-\omega)=A^*(+\omega)$ and $B(-\omega)=B^*(+\omega)$.

(c) Suppose at $x=0$ we observe ψ and its x derivative as functions of time. Show that in terms of these data

$$\begin{aligned}A(\omega)\;&=\int\limits_{-\infty}^{+\infty}dt\,e^{+i\omega t}\left[\frac{1}{2}\,\psi(0,t)\;-\;\frac{ic}{2\omega n(\omega)}\,\frac{\partial\psi}{\partial x}(0,t)\right],\\ B(\omega)\;&=\int\limits_{-\infty}^{+\infty}dt\,e^{+i\omega t}\left[\frac{1}{2}\,\psi(0,t)\;+\;\frac{ic}{2\omega n(\omega)}\,\frac{\partial\psi}{\partial x}(0,t)\right].\end{aligned} \tag{13}$$