1. Show that in the regime of normal dispersion — *i.e.*, at frequencies not too close to any of the resonances — the group velocity of the EM wave is always less than c. For simplicity, use the low-density approximation

$$\epsilon(\omega) \approx 1 + \frac{Ne^2}{\epsilon_0 m_e} \sum_i^{\text{resonances}} \frac{f_i}{\omega_i^2 - \omega^2 - i\omega\gamma_i} \tag{1}$$

as well as $\mu(\omega) \approx 1$.

2. In conducting materials, the EM waves attenuate with distance. For a specific example, consider a uniform material with dielectric constant ϵ, conductivity σ, and negligible magnetism, $\mu = 1$. The attenuating plane wave has general form

$$\mathbf{E}(x,y,z,t) = \vec{\mathcal{E}} \exp(ikz - \kappa z - i\omega t), \quad \mathbf{H}(x,y,z,t) = \vec{\mathcal{H}} \exp(ikz - \kappa z - i\omega t). \tag{2}$$

(a) Write down formulae for k and κ as functions of ω. Also, relate the electric amplitude $\vec{\mathcal{E}}$ and the magnetic amplitude $\vec{\mathcal{H}}$ to each other.

Now consider a boundary between a conducting material and the vacuum. Suppose an EM wave comes from the vacuum side and hits the boundary head-on.

(b) Calculate the reflectivity $R = |r|^2$ of the boundary.

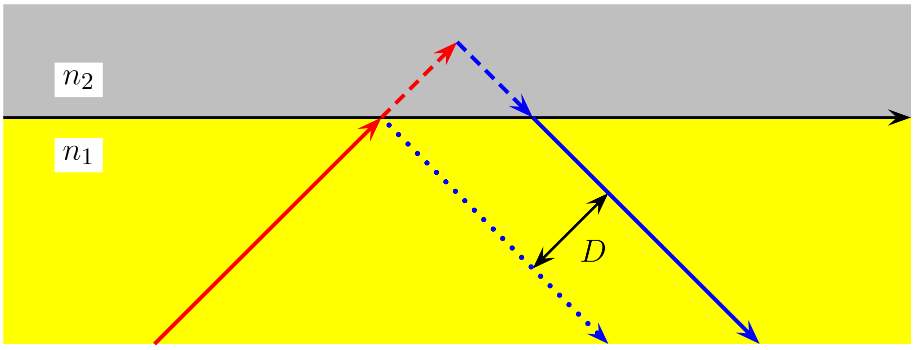
(c) Show that for a good conductor

$$R \approx 1 - \frac{4\pi\delta}{\lambda_0} \tag{3}$$

where λ_0 is the wavelength of the EM wave in the vacuum and δ is the skin-depth of the current of the same frequency in the conductor.

(d) As an example, find the reflectivity of sea water ($\sigma \approx 5\mho/\text{m}$) at an FM radio frequency $\omega = 2\pi \times 100$ MHz.

3. Consider the Goos–Hänchen effect: In a total internal reflection, the reflected ray is displaced sideways relative to the incoming ray as if it's reflected not from the boundary itself but from a small distance behind it.



(4)

The key to the Goos–Hänchen effect is the complex reflection coefficient

$$r(\alpha) \;=\; \exp(i\phi(\alpha)), \tag{5}$$

its magnitude in a total internal reflection is 1, but the phase depends on the incidence angle α.

(a) Suppose the incident wave has a finite but large width in the direction $\perp$ to the wave within the plane of incidence, for example

$$\begin{aligned}\mathbf{E}_i(x,y,z,t) \;&=\; \mathcal{E}_0\,\mathbf{e}_i\,\exp\bigl(ik_0(x\sin\alpha+z\cos\alpha)\;-\;i\omega t\bigr)\times\\ &\quad\times\exp\left(-\frac{(x\cos\alpha-z\sin\alpha)^2}{2a^2}\right).\end{aligned} \tag{6}$$

for $a \gg (1/k_0)$. (In my notations, $\mathcal{E}_0$ is the overall amplitude of the wave and $\mathbf{e}$ its polarization vector.)

Fourier transform this wave to the $\mathbf{k}$ space, calculate the reflected wave (including its overall phase), then Fourier transform that to the coordinate space. Show that

$$\begin{aligned}\mathbf{E}_r(x,y,z,t) \;&=\; \mathcal{E}_0\,\mathbf{e}_r\,\exp\bigl(ik_0(x\sin\alpha-z\cos\alpha)\;-\;i\omega t\bigr)\times\\ &\quad\times\int\frac{d\Delta k}{2\pi}\,A(\Delta k)\times\exp\bigl(i\Delta k((x\cos\alpha+z\sin\alpha)\;+\;i\phi(\mathbf{k}_0+\Delta\mathbf{k})\bigr)\end{aligned} \tag{7}$$

where $A(\Delta k)=\sqrt{2\pi}a\exp(-a^2\Delta k^2/2)$.

(b) Perform the Fourier integral in eq. (7) and show that

$$\mathbf{E}_r(x,y,z,t) \;=\; \mathcal{E}_0\,\mathbf{e}_r\,\exp\bigl(ik_0(x\sin\alpha - z\cos\alpha) \;-\; i\omega t\bigr)\times \\ \times\, e^{i\phi_0}\,\exp\left(-\frac{(x\cos\alpha + z\sin\alpha - D)^2}{2a^2}\right) \tag{8}$$

for the displacement

$$D \;=\; -\frac{\partial\phi}{\partial\Delta\mathbf{k}_\perp} \;=\; -\frac{1}{k_0}\frac{\partial\phi}{\partial\alpha}\,. \tag{9}$$

(c) Analytically continue the Fresnel equations for the reflection coefficient r to the regime of total internal reflection and calculate its phase ϕ as a function of α. Note two different equations for the in-plane and normal-to-the-plane polarizations of the EM wave.

(d) Finally, put it all together and calculate the sideways displacement of the reflected wave. Show that

$$D_\perp \;=\; \frac{2}{k}\,\frac{\sin\alpha}{\sqrt{\sin^2\alpha - (n_2/n_1)^4}}\,, \tag{10}$$

$$D_\parallel \;=\; D_\perp \times \frac{1}{(1+(n_1/n_2)^2)\sin^2\alpha \;-\; 1}\,. \tag{11}$$