1. The first problem is about scattering by a perfectly conducting small sphere of radius $r \ll \lambda$.

   (a) Show that the incident plane EM wave induces in the perfectly conducting small sphere not only an electric dipole moment $\mathbf{p} = +4\pi\epsilon_0 r^3 \mathbf{E}_{\text{inc}}$ but also a magnetic dipole moment $\mathbf{m} = -2\pi r^3 \mathbf{H}_{\text{inc}}$.
   Hint: Because of skin effect, a perfectly conductor acts as a perfect diamagnetic to an oscillating magnetic field.

   (b) Show that the scattered wave has form

$$\mathbf{E}_{\text{sc}} = -k^2 a^3 E_0 \frac{e^{ikr-i\omega t}}{r} \Big(\mathbf{n} \times \big((\mathbf{n} - \tfrac{1}{2}\mathbf{n}_0) \times \mathbf{e}_0\big)\Big) \tag{1}$$

   where $\mathbf{e}_0$ is the unit polarization vector of the incident wave ($\mathbf{e}_0^* \cdot \mathbf{e}_0 = 1$, $\mathbf{n}_0 \cdot \mathbf{e}_0 = 0$), and hence the partial cross-section

$$\frac{d\sigma}{d\Omega} = \frac{k^4 a^6}{4}\Big(5 - 4\mathbf{n}\cdot\mathbf{n}_0 - 4\left|\mathbf{n}\cdot\mathbf{e}_0\right|^2 - \left|(\mathbf{n}\times\mathbf{n}_0)\cdot\mathbf{e}_0\right|^2\Big). \tag{2}$$

   (c) Show that for a linearly polarized incident wave the partial cross section(2) becomes

$$\frac{d\sigma}{d\Omega} = \frac{k^4 a^6}{8}\Big(5 - 8\cos\theta + 5\cos^2\theta - 3\sin^2\theta\cos(2\phi)\Big). \tag{3}$$

   Also, write down a formula for the partial cross-section for the circularly polarized incident wave.

   (d) Finally, suppose the incident wave is un-polarized, *i.e.*, a 50–50 incoherent mixture of two orthogonal linear polarizations. Show that the scattered wave in this case is partially polarized and calculate its degree of polarization Π as a function of the scattering angle θ.

2. The next problem is about the relativistic velocity addition formula: Two velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ *in the same direction* add up according to

$$v_{1+2} = \frac{v_1 + v_2}{1 + v_1 v_2/c^2}\,. \tag{4}$$

   (a) Derive eq. (4) from two successive Lorentz transforms.

   In 1851 Hyppolite Fizeau used interferometry to measure the speed of light in moving water or other liquids. He found that for light traveling in the same direction as the liquid or in the opposite direction, its speed is

$$u = \frac{c}{n} \pm \left(1 - \frac{1}{n^2}\right) \times v \tag{5}$$

   where n is the refraction index of the liquid and v is its velocity.

   (b) Derive eq. (5) from the relativistic velocity addition formula.

   (c) Suppose the refraction index n of the liquid depends on the light frequency ω. Show that in this case, the phase velocity of light in the moving liquid becomes

$$u = \frac{c}{n} \pm \left(1 - \frac{1}{n^2} + \frac{\omega}{n}\frac{dn}{d\omega}\right) \times v. \tag{6}$$

3. The last problem is about the twin paradox. But first, consider a uniformly accelerating spaceship. That is, at any time $t > 0$ it has the same acceleration a relative to an inertial frame which at that moment has the same velocity as the ship.

   (a) Show that the time τ aboard the ship, the time t on the planet where the ship has started from, and the velocity v of the ship relative to that planet are related to each other as

$$\frac{at}{c} = \sinh\left(\frac{a\tau}{c}\right), \quad \frac{v}{c} = \tanh\left(\frac{a\tau}{c}\right) = \frac{at}{\sqrt{c^2 + (at)^2}}\,. \tag{7}$$

   Hint: use the relativistic velocity addition formula (4).

(b) Find the distance of the ship from its starting point as a function of t and as a function of τ. Also, show that a light signal sent from the starting point at any time $t > c/a$ will never catch up with the ship as long as it keeps accelerating.

Now consider a round trip from Earth to (possibly habitable) planet Gliese 667 Cc, about 23.62 light years from Earth. For the astronaut's convenience, the ship accelerates at constant rate $a = g = 9.80\ \mathrm{m/s^2}$ from Earth to the mid-point, then decelerates at the same rate until it stops at the destination. It spends a year at the Gliese 667 Cc planet, then flies back in the same manner: accelerates at constant rate $a = g$ to the midpoint, the decelerates to stop at the Earth.

(c) If a crew member has a twin who stayed on Earth and the trip started on their 21^{st} birthday, how old would be each twin by the time the ship comes back to Earth?