1. An inertial frame of reference K' moves at velocity $\mathbf{u}$ relative to another inertial frame K. A particle has velocity $\mathbf{v}'$ relative to K'.

   (a) As a warm-up exercise, show that the particle velocity vector relative to the K frame is

$$\mathbf{v} = \left(1 + \frac{\mathbf{u}\cdot\mathbf{v}'}{c^2}\right)^{-1}\left(\mathbf{u} + \mathbf{v}_{\|} + \frac{1}{\gamma_u}\mathbf{v}'_{\perp}\right) \qquad (1)$$

   where $\mathbf{v}'_{\|}$ and $\mathbf{v}'_{\perp}$ are the components of the $\mathbf{v}'$ vector parallel and perpendicular to the relative velocity $\mathbf{u}$ of the two frames.
   Note: eq. (1) is not symmetric WRT $\mathbf{v}' \leftrightarrow \mathbf{u}$, unless the two velocity vectors happen to be parallel to each other.

   (b) Verify that eq. (1) is consistent with the speed of light universality. That is, show that if the particle in question is a photon and $|\mathbf{v}'| = c$, then also $|\mathbf{v}| = c$, although the directions of the $\mathbf{v}'$ and $\mathbf{v}$ vectors are generally different.

   Now consider an accelerating particle and the relation between the acceleration vectors $\mathbf{a}'$ and $\mathbf{a}$ relative to the frames K' and K. Please allow for completely general directions of the vectors $\mathbf{a}'$, $\mathbf{v}'$, and $\mathbf{u}$.

   (c) Show that

$$\mathbf{a}_{\|} = \frac{\left(1 - \dfrac{\mathbf{u}^2}{c^2}\right)^{3/2}}{\left(1 + \dfrac{\mathbf{u}\cdot\mathbf{v}'}{c^2}\right)^3}\,\mathbf{a}'_{\|}\,, \qquad (2)$$

$$\mathbf{a}_{\perp} = \frac{\left(1 - \dfrac{\mathbf{u}^2}{c^2}\right)}{\left(1 + \dfrac{\mathbf{u}\cdot\mathbf{v}'}{c^2}\right)^3}\left[\mathbf{a}'_{\perp} + \frac{\mathbf{u}}{c^2}\times(\mathbf{a}'\times\mathbf{v}')\right], \qquad (3)$$

   where $\mathbf{a}_{\|}$ and $\mathbf{a}_{\perp}$ are the components of the acceleration vector $\mathbf{a}$ respectively parallel and perpendicular to the relative velocity $\mathbf{u}$ of the two frames, and ditto for the $\mathbf{a}'_{\|}$ and $\mathbf{a}'_{\perp}$ components of the $\mathbf{a}'$ acceleration vector.

2. In the rest frame of a moving conducting medium the current density obeys Ohm's Law $\mathbf{J}' = \sigma\mathbf{E}'$ where σ is the conductivity and primes denote the rest-frame quantities.

   (a) Write the Ohm's Law for a moving conducting medium in a covariant form as

$$J^\mu \;-\; \frac{J^\nu U_\nu}{c^2}\,U^\mu \;=\; \frac{\sigma}{c}\,F^{\mu\nu}U_\nu \tag{4}$$

   where U^μ is the 4–velocity of the medium,

$$U^\mu \;=\; \frac{dX^\mu}{d\tau}\,, \quad U^0 \;=\; \gamma c, \quad U^i \;=\; \gamma v^i. \tag{5}$$

   (b) Spell out eq. (4) in 3D vector form.

   (c) Suppose we are given the electric charge density ρ in the frame where the medium moves. Show that the electric current density in that frame is (in Gauss units)

$$\mathbf{J} \;=\; \rho\mathbf{v} \;+\; \gamma\sigma\left[\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B} \;-\; \frac{\mathbf{v}(\mathbf{v}\cdot\mathbf{E})}{c^2}\right]. \tag{6}$$

   (d) Now suppose we know that the moving medium is electrically neutral in its rest frame, $\rho' = 0$. Show that in the frame where the medium moves

$$\mathbf{J} \;=\; \gamma\sigma\Big(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\Big), \quad \rho \;=\; \frac{\gamma\sigma}{c^2}\,\big(\mathbf{v}\cdot\mathbf{E}\big). \tag{7}$$

3. Finally, consider a point charge Q moving at a uniform velocity $\mathbf{v}$; in the worldline terms,

$$x^\mu_{\text{charge}}(\tau) \;=\; U^\mu\tau. \tag{8}$$

   Let's measure the electric and magnetic fields of this charge at some spacetime point x^μ.

(a) Use the Lorentz covariant of the EM field tensor $F^{\mu\nu}$ — and only its covariance but no specific EM equations — and the translational invariance of space and time to argue that

$$F^{\mu\nu}(x) \;=\; Q(X^\mu U^\nu - X^\nu U^\mu) \times f\big((X \cdot U), (X \cdot X)\big) \tag{9}$$

where $X^\mu = x^\mu - x^\mu_{\text{charge}}$ and f is some function of the Lorentz scalars $(X \cdot U) = X^\lambda U\lambda$ and $(X \cdot X) = X^\lambda X_\lambda$.
Note: there is another Lorentz scalar at play, namely $(U \cdot U) = U^\lambda U_\lambda$, but its value is a constant $(U \cdot U) = c^2$,

The specific formula for the EM field tensor is

$$F^{\mu\nu}(x) \;=\; \frac{Q}{c} \times \frac{(X^\mu U^\nu - X^\nu U^\mu)}{\left(\frac{1}{c^2}(X \cdot U)^2 \,-\, (X \cdot X)\right)^{3/2}}\,. \tag{10}$$

In this formula $X^\mu = x^\mu - x^\mu_{\text{charge}}(\tau)$, but the τ of the charge can affect the EM field tensor at any given measurement point x^μ. So for simplicity we may set $\tau = 0$ and rewrite eq. (10) as

$$F^{\mu\nu}(x) \;=\; \frac{Q}{c} \times \frac{(x^\mu U^\nu - x^\nu U^\mu)}{\left(\frac{1}{c^2}(x \cdot U)^2 \,-\, (x \cdot x)\right)^{3/2}}\,. \tag{11}$$

(b) Verify this statement.

(c) Show that for a charge at rest, eq. (10) yields the Coulomb electric field and zero magnetic field.

(d) For a moving charge, spell out eq. (10) for the electric and magnetic fields in 3–vector notations.